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ENERGY-BASED ROBUST CONTROLLER DESIGN FOR MULTI-LINK FLEXIBLE ROBOTS

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Abstract—A class of robust stable controllers to control the tip position of a multi-link flexible robot is presented. In contrast to traditional model-based methods, the controllers are derived by using a basic relationship of system energy and are *independent* of the system dynamics. The approach allows controller design in the absence of a system model (which is very complicated in the case of the multi-link flexible robot) and provides great freedom in feedback control design. Further, the controllers are easy to implement in practice because, depending on actual instrumentation, all the signals can be chosen to be measurable. Simulation results of a two-link flexible robot are provided to show the effectiveness of the presented approach. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Controller design for flexible robots is one of the most challenging problems in control system design. Modelling of flexible link manipulators is very complicated because the system is actually infinite-dimensional. The inherent nonminimum phase behavior of the flexible manipulator system makes it very difficult to achieve high level performance and robustness simultaneously [1].

Collocating the sensors and actuators at the joint of a flexible manipulator, for example the joint PD controller, can guarantee a certain degree of robustness of the system. Actually, as mentioned in [1, 2], the robustness of collocated controllers comes directly from the energy dissipating configuration of the resulting system. However, the performance of the flexible system with only a collocated controller, for example the joint PD controller, is often not very satisfactory because the elastic modes of the flexible beam are seriously excited and not effectively suppressed. For this, various kinds of control techniques have been investigated to improve the performance of flexible systems.

A large number of controllers for flexible robots have been designed based on truncated models from Modal Analysis (MA) or Finite Element Method (FEM).

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Generally speaking, the desired performance of a flexible manipulator in position control can be described as: (i) the joint motion converges to the final position fast, and simultaneously (ii) the elastic vibrations are effectively suppressed. Obviously there is a tradeoff between the two requirements. The above description directly leads to the application of the singular perturbation method to the control of flexible robots [3–6]. In this method, the dynamics of the system is divided into two parts, i.e. a slow sub-system (corresponding to the joint motion) and a fast sub-system (related to the flexible vibrations), and two sub-controllers are designed accordingly. Since what we care about most in tip position control is the tip motion of the flexible beam, inverse dynamics (computed torque) method, which has been shown to be effective in controlling rigid-link manipulators, is also investigated for the flexible-link case [7, 8]. This method seems to result in better tip motions over other techniques. However, the successful application of this method heavily depends on highly accurate models and efficient computational algorithms/powerful computing facilities. Other approaches to improve position control performance include linear control [9], robust control [10], sliding mode method [11], bounded input LQG control [12], feedback control [13, 14] and input pre-shaping approach [15, 16], among others. The control approaches mentioned above are all based on the truncated models. It is the truncation of the original infinite dimensional system to a finite dimensional model that makes the above mentioned control techniques applicable. However, the following problems inevitably arise: (i) a relatively high order controller (corresponding to a model with a relatively large number of flexible modes in MA or elements in FEM) is often necessary to achieve high accuracy of performance; (ii) control and observation spillovers may occur due to the ignored high frequency dynamics [2]; and (iii) the controllers may be difficult to implement from the engineering point of view since full state measurements/observers are often required.

An alternative model-based approach is to derive controllers directly from the Partial Differential Equations (PDEs) of flexible robot systems and thus avoid the undesired model truncation [2, 17, 18]. All these three papers deal with a single-link flexible robot only (in [18], the robot is two-link but only the upper link is flexible). In [2], *linear* direct strain feedback (DSFB) at the base of the bending beam was considered to enhance the performance of the joint PID controller. Tracking control of an Euler–Bernoulli beam is discussed in [17], in which an assumption is made that there is no tip payload and no motor hub inertia. A collocated joint PD controller was designed for a two-link unloaded rigid-flexible manipulator in [18]. In the above three papers, the stability of the closed-loop systems is proven directly based on the PDEs which govern the motion of the flexible systems considered. As a result, their controllers can avoid the above mentioned problems associated with model truncation. However, because the dynamics of flexible link robots, described by PDEs, are very complicated, especially for the multi-link case, closed-form solutions are not currently available. Thus, this controller design approach, though possessing many advantages over the traditional truncated-model-based methods, is generally very complicated as well.

In this paper, a non-model-based controller design method is presented for multi-link flexible robots. The method only makes use of a very basic energy relationship of the system and does not need any information on the system dynamics (subsequently the drawbacks/problems of model-based methods are avoided). Furthermore, the corresponding derivation is very easy. With this method, a class of tip position controllers is developed for multi-link flexible robots without gravitational influence. This is the case for space applications, and for robots moving in a horizontal plane on the ground. The controllers are very robust in terms of system parameter variations and can guarantee the stability of the closed-loop system. Further, the controllers allow great freedom in feedback design according to actual instrumentation, and are readily usable for practical applications.

The paper is organized as follows: in section 2, the multi-link flexible robot is briefly introduced; the energy-based controller design approach is presented in section 3; computer simulations are carried out on a two-link flexible robot to verify the effectiveness of the controllers in section 4, followed by the conclusion in section 5.

2. MULTI-LINK FLEXIBLE ROBOT

We shall consider flexible robots (i) deployed in space, or (ii) moving in the horizontal plane. In both cases, the effect of gravity is ignored. Figure 1 shows an N-link flexible robot which moves on a horizontal operation platform.

The N links are connected using N motors. Motor 1 is fixed in position, which is at the origin of the fixed base frame $X_1O_1Y_1$. The remaining motors, each being supported by a roller, are movable on the platform. The free tip of the last link has a payload attached.

For clarity, the geometry of the robot is shown in Fig. 2. There are a total of $2N$ frames being used to describe the system, i.e. $X_iO_iY_i$ and $x_iO_iy_i$, $i = 1, 2, \ldots, N$. Frame $X_1O_1Y_1$, as stated above, is the fixed base frame. Other frames are all local reference frames attached to the corresponding motors; specifically axis O_iX_i ($i = 2, 3, \ldots, N$) is defined as the tangent to the end tip of link $i - 1$, and axis O_ix_i ($i = 1, 2, \ldots, N$) is tangent to link i at its base. The angular position of the ith link is denoted by θ_i measured in frame $X_iO_iY_i$. θ_i is actually the angular difference between frames $x_iO_iy_i$ and $X_iO_iY_i$.

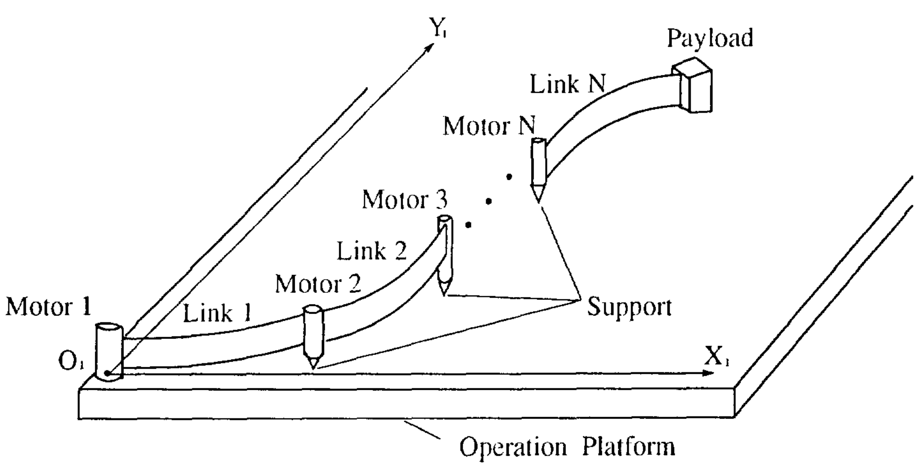

Fig. 1. Multi-link flexible robot in the horizontal plane.

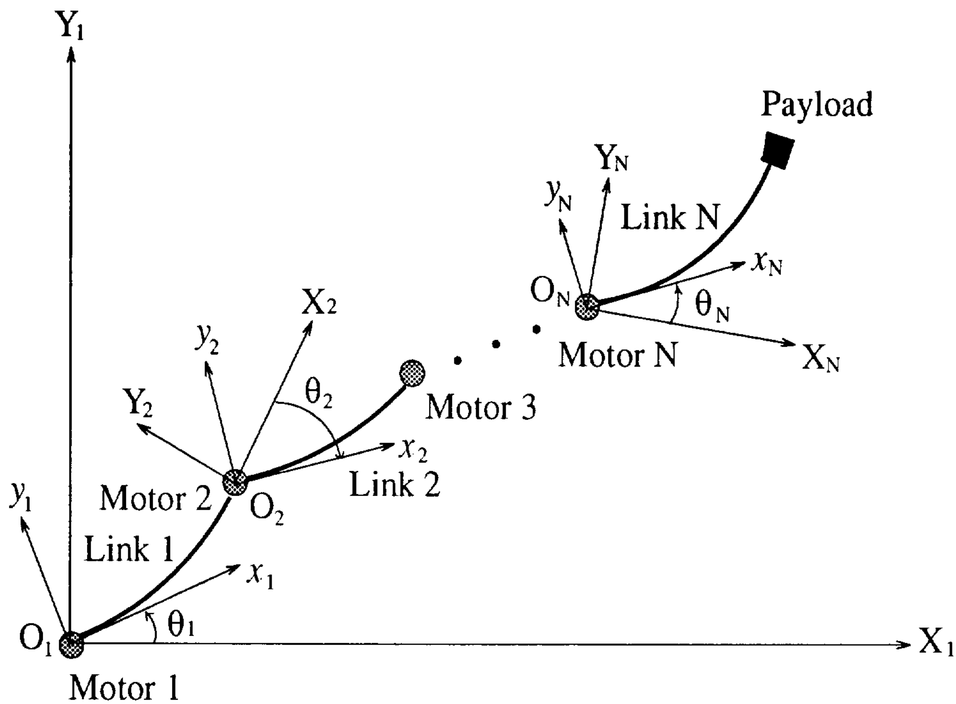

Fig. 2. Geometry of the multi-link flexible robot.

3. ENERGY-BASED ROBUST CONTROLLER DESIGN

In this section, we present an energy-based controller design approach for the multi-link flexible robot described in section 2. The control objective is to simultaneously drive the N motors such that the tip payload is moved to a pre-defined position quickly, smoothly and accurately. Although the joint PD control can stabilize the flexible robot system, generally the system performance is not satisfactory because the elastic vibrations cannot be effectively suppressed. In this paper, in addition to the joint PD control, some feedbacks related to the bending of the flexible links will be introduced into the controller, and thus provide direct control effort on the elastic vibrations.

We begin with the following positive definite energy (Lyapunov) function:

$$V = E_{\mathrm{k}} + E_{\mathrm{p}} + \frac{1}{2}\sum_{i=1}^{N} k_{\mathrm{p}i}[\theta_i(t) - \theta_{\mathrm{f}i}]^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M_i} k_{ij}\left[\int_0^t \dot{\theta}_i(s) f_{ij}(s)\,\mathrm{d}s\right]^2, \tag{1}$$

where E_{k} and E_{p} are the total kinetic energy and total potential energy of the system; constants $k_{\mathrm{p}i} > 0$, $k_{ij} \geq 0$; $\theta_{\mathrm{f}i}$ is the constant final position of the ith link; $\dot{\theta}_i$ is the time derivative of θ_i; and, depending on actual instrumentation, $f_{ij}(t)$ ($j = 1, 2, \ldots, M_i$ with M_i being the number of feedbacks used to control the ith link) are variables related to the bending of the ith link, or any combination of bending variables of the ith link. The condition on choosing $f_{ij}(t)$ is that they must be zero when the ith link undergoes no deformation. Some examples of bending variables of link i described in frame $x_iO_iy_i$ are $y_i(x_i, t)$ (deflection at x_i), $y'_i(x_i, t)$ (rotation at x_i), $y''_i(x_i, t)$ (strain at x_i), and $y'''_i(x_i, t)$ (shearing force at x_i). Obviously x_i should not exceed the length of the ith link.

Before further derivation, we shall make the following assumption: the support rollers in Fig. 1 are frictionless, and any other damping and frictions are neglected. Thus, by recalling that the robot is operated without gravitational influence, we can conclude that the total change in system energy must be equal to the work done by the N motor torques, i.e.

$$E_k - E_{k0} + E_p - E_{p0} = \sum_{i=1}^{N} \int_0^t \tau_i \dot{\theta}_i(t)\,dt, \tag{2}$$

where E_{k0} and E_{p0} are kinetic energy and potential energy of the system at the initial moment; τ_i is the torque generated by motor i. Differentiating with respect to time on both sides of Eqn (2) yields

$$\dot{E}_k + \dot{E}_p = \sum_{i=1}^{N} \tau_i \dot{\theta}_i(t). \tag{3}$$

Now, we are ready to present the following theorem on the stability of the multi-link flexible robot system.

Theorem 1. The closed-loop multi-link flexible robot system is stable if the N control torques are given by

$$\tau_i = -k_{pi}[\theta_i(t) - \theta_{fi}] - k_{vi}\dot{\theta}_i - \sum_{j=1}^{M_i} k_{ij} f_{ij}(t) \int_0^t \dot{\theta}_i(s) f_{ij}(s)\,ds \tag{4}$$

$$i = 1, 2, \ldots, N,$$

where $k_{vi} > 0$.

Proof. Recalling Eqn (3), the time derivative of energy function V in Eqn (1) is given by

$$\dot{V} = \sum_{i=1}^{N} \tau_i \dot{\theta}_i(t) + \sum_{i=1}^{N} k_{pi}\dot{\theta}_i[\theta_i(t) - \theta_{fi}] + \sum_{i=1}^{N}\sum_{j=1}^{M_i} k_{ij}\dot{\theta}_i(t) f_{ij}(t) \int_0^t \dot{\theta}_i(s) f_{ij}(s)\,ds. \tag{5}$$

Substituting Eqn (4) into Eqn (5) yields

$$\dot{V} = -\sum_{i=1}^{N} k_{vi}\dot{\theta}_i^2, \tag{6}$$

which is negative semi-definite. [QED]

Remarks

(1) The joint PD controller for the ith link is a special case of τ_i in Eqn (4) by setting k_{ij}s to be zeros. Although the joint PD controller will not destabilize the system, the system performance, as we will show in the next section, is not satisfactory

because the flexible modes of the beam are seriously excited and not effectively suppressed. The introduction of the k_{ij} items into V allows us to explicitly consider the bending of the flexible beam, and subsequently have a direct control effect on elastic vibrations.

(2) The integral type items in Eqn (1) avoid measurements of high order signals. This is very desirable for easy engineering implementation.

(3) The controller, Eqn (4), is independent of system parameters and thus possesses stability robustness to system parameter uncertainties. As a matter of fact, the closed-loop system is stable as long as k_{pi}, $k_{vi} > 0$ and $k_{ij} \geq 0$.

(4) The controllers in Eqn (4) are very easy to implement in practice. The joint position θ_i and the joint velocity $\dot{\theta}_i$ can be obtained by a rotary encoder and tachometer attached to the rotor of motor i, and $f_{ij}(t)$ can be determined from the available sensor facilities. Unlike the controllers suggested in the literature [3–6, 9–14], measurements/observers for state feedback are not necessary.

(5) The stability proof is independent of the system dynamics and thus the drawbacks/problems associated with model-based controllers mentioned in section 1 are avoided.

(6) When the bending functions $f_{ij}(t)$ are chosen in the local reference frame $x_iO_iy_i$, for example those mentioned previously, the controllers presented in Eqn (4) are of decentralized type, which has the advantage of requiring few computer resources, and allows ease of implementation and tolerance to failure, since the N controllers in Eqn (4) can be implemented in parallel.

(7) In **Theorem 1**, we only claim closed-loop stability of the system. To prove asymptotic stability of the system is not easy due to the infinite dimensionality of the system. Asymptotic tracking control of an Euler–Bernoulli beam has been achieved in [17]; however, it was assumed that there was no hub inertia, i.e. $I_h = 0$, which is not realizable. In the following, instead of giving rigorous proof, we shall show that practically the flexible robot can only possibly stop at $\theta_i = \theta_{fi}$ ($i = 1, 2, \ldots, N$) without vibrating. Assuming that the N links stop at the position $\theta_i \equiv \alpha_i$ (hence $\dot{\theta}_i \equiv 0$ for $i = 1, 2, \ldots, N$) with $\alpha_i \neq \theta_{fi}$, thus there is no energy input to the system since $\dot{\theta}_i \equiv 0$. Due to the existence of internal structural damping in a flexible link in practice (structural damping is neglected in the proof of **Theorem 1**), the links must tend to stop vibrating and finally be static at the undeformed position. Consequently, the first term in τ_i is a nonzero constant, the middle term is zero, and the last term approaches zero [note that $f_{ij}(t)$ are zero when the link i is not deformed]. Therefore, τ_i approaches a nonzero constant and thus $\theta_i \equiv \alpha_i$ cannot hold. The only possibility is that the flexible link is at the final position $\theta_i \equiv \theta_{fi}$ without vibrating, which implies that tip regulation can be practically achieved.

4. SIMULATION TESTS ON A TWO-LINK FLEXIBLE ROBOT

In this section, some numerical simulations are carried out on a two-link flexible robot. The plant is simulated by a FEM model in which each link is divided into four elements with the same length. A fourth-order Runge–Kutta program with adaptive step-size is used to numerically solve the differential equations. The sampling interval is set to be 0.01 s.

The system parameters are given in Table 1, in which M_{t1} actually denotes the 2nd motor, and M_{t2} is the payload attached to the end tip of Link 2.

The initial position and the final position of the robot are described in Fig. 3, i.e. the initial joint positions of both links are all zeros, and the set point values of the two links are $\theta_{f1} = 20°$ and $\theta_{f2} = 10°$.

4.1. Joint PD control

Firstly, we shall give the simulation results of the following joint PD controller:

$$\tau_{PDi} = -k_{pi}(\theta_i - \theta_{fi}) - k_{vi}\dot{\theta}_i, \qquad i = 1, 2.$$

From **Theorem 1**, any k_{pi}, $k_{vi} > 0$ will not destabilize the closed-loop system. However, different selection of k_{pi} and k_{vi} will lead to very different performance. Here, they are determined to make the equivalent rigid motion critically damped as follows. If the flexible links are assumed to be rigid, using joint PD control leads to the following rigid motion error equations:

$$I_{ei}\ddot{e}_i(t) + k_{vi}\dot{e}_i(t) + k_{pi}e_i = 0, \qquad i = 1, 2,$$

where I_{ei} represents the equivalent inertia of the ith joint, $e_i = \theta_i - \theta_{fi}$, and $\ddot{e}_i = \ddot{\theta}_i$ and $\dot{e}_i = \dot{\theta}_i$ since θ_{fi} is constant. It should be noted that because of the rotational

Table 1. System parameters

	Link 1	Link 2
Length	$L_1 = 1.0$ m	$L_2 = 0.8$ m
Flexural rigidity	$EI_1 = 5.0$ Nm^2	$EI_2 = 3.0$ Nm^2
Linear density	$\rho_1 = 0.1$ kg/m	$\rho_2 = 0.1$ kg/m
Hub initial	$I_{h1} = 3.0$ kgm^2	$I_{h2} = 1.5$ kgm^2
Payload	$M_{t1} = 0.1$ kg	$M_{t2} = 0.05$ kg

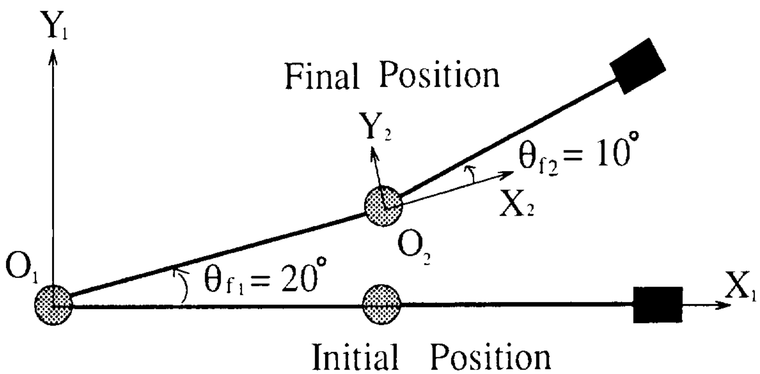

Fig. 3. Two-link flexible robot.

movement of the 2nd link in its own local reference frame $X_iO_iY_i$, I_{e1} is not constant even when the two links are both assumed to be rigid. For simplicity, in our simulations, I_{e1} is determined by further assuming that motor 2 is locked at $\theta_2 = 0$, i.e. the two links are aligned. Subsequently, we have $I_{e1} = 3.46\,\text{kgm}^2$ and $I_{e2} = 1.55\,\text{kgm}^2$. For the 2nd-order systems above, if critical damping is assumed ($\xi = 1$), the PD feedback gains can be determined by

$$k_{pi} = I_{ei}\omega_{ni}^2$$

$$k_{vi} = 2I_{ei}\omega_{ni},$$

where ω_{ni} ($i = 1, 2$) are the corresponding natural frequencies. We let $\omega_{n1} = 2.5$ and $\omega_{n2} = 3.0$, and the two joint PD controllers are given by

$$\tau_{PD1} = -21.6(\theta_1 - \theta_{f1}) - 17.3\dot{\theta}_1 \quad (7)$$

$$\tau_{PD2} = -14.0(\theta_2 - \theta_{f2}) - 9.3\dot{\theta}_2. \quad (8)$$

In what follows we shall construct the energy-based robust controllers based on the above joint PD controllers, whose performance will be plotted by dashed lines in figures for comparison.

4.2. *Energy-Based Robust Control (EBRC)*

From Eqn (4), one can see that in addition to joint PD control effort, a summation has been introduced to explicitly control the elastic vibrations of the flexible links. As stated in section 3, the introduction of the summation into the controller allows great freedom in feedback design according to actual instrumentation.

Case 1. Tip deflection feedback. We firstly consider the case of using the tip deflection of each link as the bending variable $f_{ij}(t)$ in Eqn (4). In practice, the tip deflections can be detected by a vision system, which has been used to control a single-link flexible robot in [9].

With tip deflection feedbacks, the controllers used in our simulations are given by

$$\tau_1 = \tau_{PD1} - 32000y_1(L_1, t)\int_0^t \dot{\theta}_1(s)y_1(L_1, s)\,ds \quad (9)$$

$$\tau_2 = \tau_{PD2} - 32000y_2(L_2, t)\int_0^t \dot{\theta}_2(s)y_2(L_2, s)\,ds, \quad (10)$$

where τ_{PD1} and τ_{PD2} are given in Eqns (7) and (8). Since only tip deflection of each link is used, from Eqn (4), we have $M_1 = M_2 = 1$, $k_{11} = k_{21} = 32{,}000$, $f_{11}(t) = y_1(L_1, t)$ and $f_{21}(t) = y_2(L_2, t)$, which are the tip deflections of link 1 and link 2, respectively. It should be noted that the values of k_{11} and k_{21} given above may not be optimal. The simulations here are only used to show the effectiveness of the EBRC approach. According to **Theorem 1**, τ_1 and τ_2 will guarantee stability of the closed-loop system.

The tip deflections and joint motions of τ_1 [Eqn (9)] and τ_2 [Eqn (10)] are given in Figs 4 and 5. Compared with the results of joint PD control (dashed lines), one can see that the EBRC can suppress the elastic deflections more effectively without

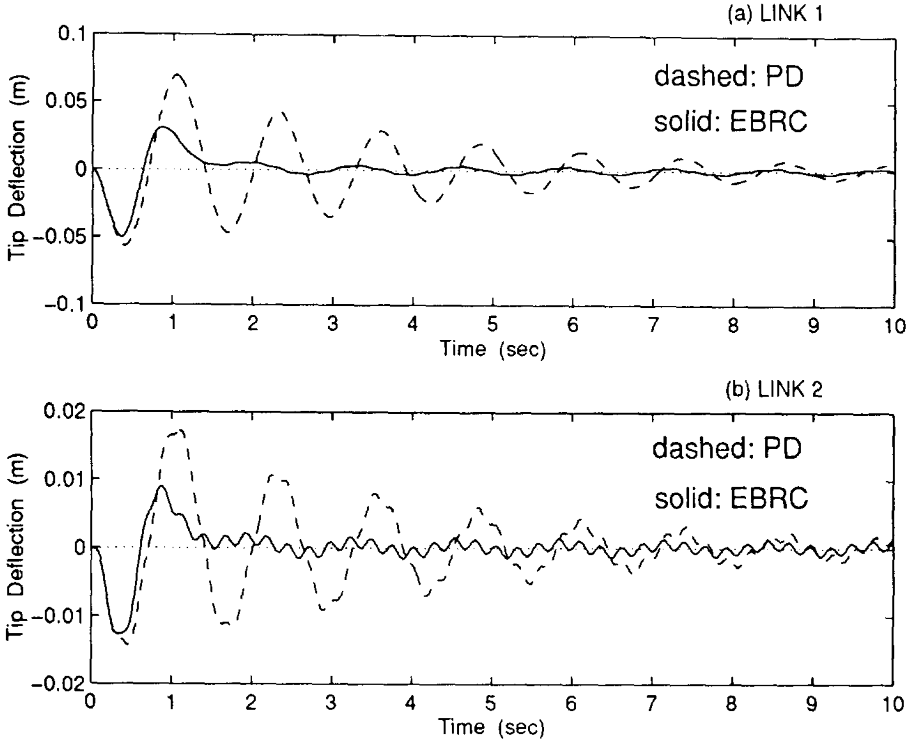

Fig. 4. Tip deflection in Case 1. (a) $y_1(L_1, t)$; (b) $y_2(L_2, t)$.

slowing down the joint motions. Furthermore, the joint motions of EBRC exhibit less vibrations and overshoots and are smoother than that of the joint PD control.

What we care about most in tip position control are the tip trajectories, which are required to converge fast with as small vibrations/overshoots as possible to improve positioning accuracy. Under the assumption of small deflection, the tip position of the two links can be approximated by

$$p_1 = L_1\theta_1 + y_1(L_1, t)$$
$$p_2 = L_2\theta_2 + y_2(L_2, t),$$

in which the angular displacements θ_1 and θ_2 should be represented in radians instead of degrees. The tip positions p_1 and p_2 are plotted in Fig. 6. It is seen that the results of joint PD control exhibit serious vibrations and large overshoots, which are very undesirable for accurate tip position control of flexible robots, while the tip trajectories of EBRC are quite good since they are smooth, fast converging and there are very little vibration and overshoot.

For completeness, Fig. 7 shows the control efforts of Case 1. The joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$, which are used in both PD and EBRC controllers, are plotted in Fig. 8.

Case 2. Base strain feedback. Considering that strain gauges have been very widely used in control of flexible robots, we assume that they are available and the base

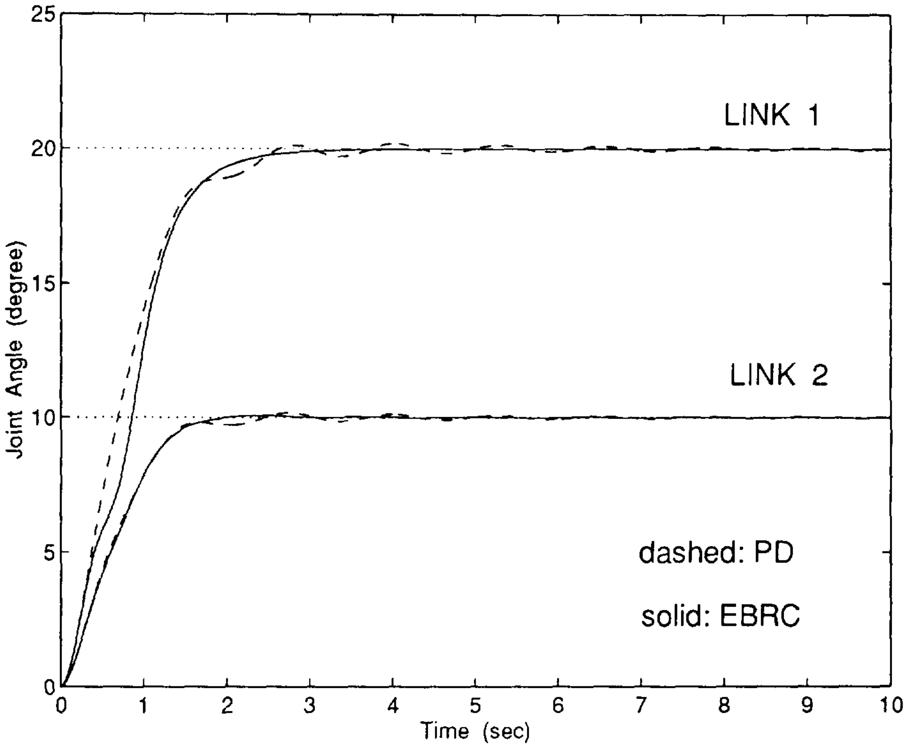

Fig. 5. Joint angle in Case 1.

strain of each link is used for feedback in the following simulations. From Eqn (4), we have the following controllers:

$$\tau_1 = \tau_{PD1} - 1800y_1''(0, t)\int_0^t \dot{\theta}_1(s)y_1''(0, s)\,ds \tag{11}$$

$$\tau_2 = \tau_{PD2} - 1800y_2''(0, t)\int_0^t \dot{\theta}_2(s)y_2''(0, s)\,ds, \tag{12}$$

where τ_{PD1} and τ_{PD2} are also given in Eqns (7) and (8). Deducing from Eqn (4), we arrive at $M_1 = M_2 = 1$, $k_{11} = k_{21} = 1800$, $f_{11}(t) = y_1''(0, t)$ and $f_{21}(t) = y_2''(0, t)$, which are the base strains of link 1 and link 2, respectively. Similar to Case 1, the values of k_{11} and k_{21} may not be optimal and are only used to show the effectiveness of the EBRC approach. The stability of the closed-loop system is guaranteed by **Theorem 1**.

The tip deflections, joint motions and tip trajectories in this case are shown in Figs 9–11. The results of EBRC, compared with that of joint PD control, exhibit similar improvements to those in Case 1, i.e. the tip deflections are suppressed effectively, the joint motions are very smooth, and the tip trajectories converge fast with only very little vibration and overshoot. The base strain feedback signals $y_1''(0, t)$ and $y_2''(0, t)$ are shown in Fig. 12, in which the base strains of joint PD control are also plotted with dashed lines for comparison. The simulation results in

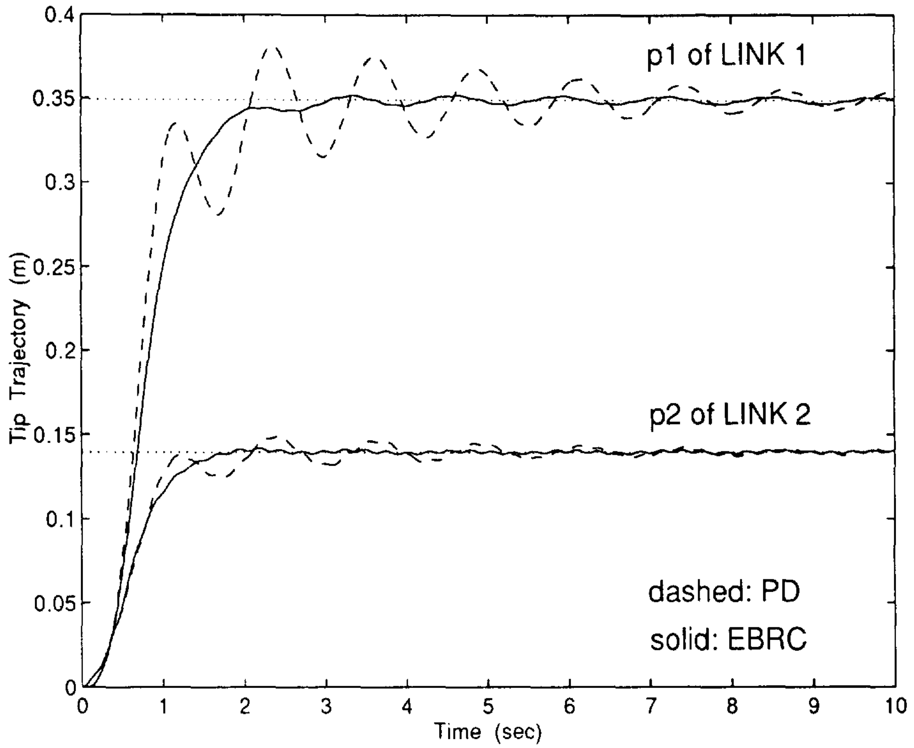

Fig. 6. Tip trajectory in Case 1.

the two cases are quite alike. This can also be seen from the two kinds of feedback signals, i.e. the tip deflections in Case 1 (Fig. 4) and the base strains in Case 2 (Fig. 12), which are very similar to each other in shape except for different magnitudes. In practice, the selection of feedbacks can be determined by available sensor facilities. Finally, the corresponding control efforts and joint velocities are given in Figs 13 and 14 for completeness.

In the above, through simulations, we have shown the effectiveness of the controller in Eqn (4). It should be pointed out that the approaches presented in this paper are not complete, and other kinds of feedbacks and/or other kinds of combinations of feedbacks can also be considered depending on the available sensor facilities, since the controller in Eqn (4) actually allows great freedom of feedback design.

5. CONCLUSION

In this paper, we have presented a non-model-based controller design approach for a multi-link flexible robot. In this method, no information of the system dynamics is needed; instead, it only makes use of the very basic energy relationship of the system.

Using this method, a general form of robust stable controller is constructed for a

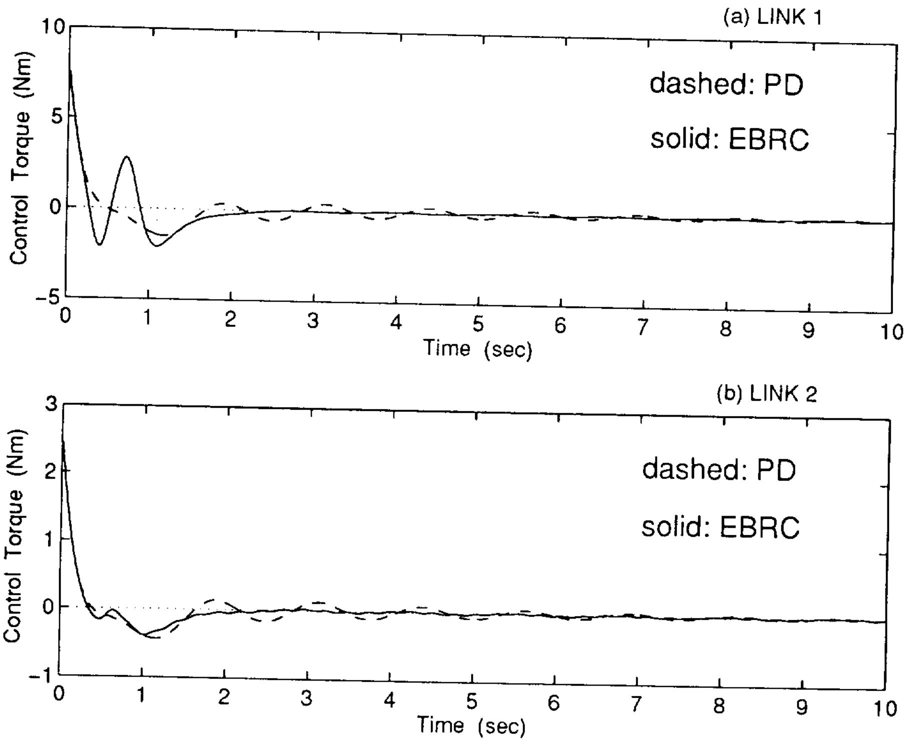

Fig. 7. Control effort in Case 1.

multi-link flexible robot. The controller is independent of system parameters and subsequently possesses stability robustness to parameter variations. Furthermore, the controller is very simple and very flexible in its form and can be easily implemented according to actual instrumentation. Because the controller design is independent of system dynamics, the drawbacks/problems associated with truncated-/un-truncated-model-based controller design methods are essentially avoided.

Numerical simulations showed that two very simple special cases of the controller, in which only tip deflection or base strain feedback is used to represent the bending of each flexible link, can give reasonably good performance of a two-link flexible robot in the sense that the elastic vibrations are effectively suppressed, and the tip positions converge fast along smooth trajectories with negligible overshoots.

Finally, it should be noted that we do not currently have theoretical guidelines on choosing $f_{ij}(t)$ to obtain the most effect control effort and optimize the system's performance. The effect of the terms of bending functions $f_{ij}(t)$ on the system dynamics will be investigated in detail in future research.

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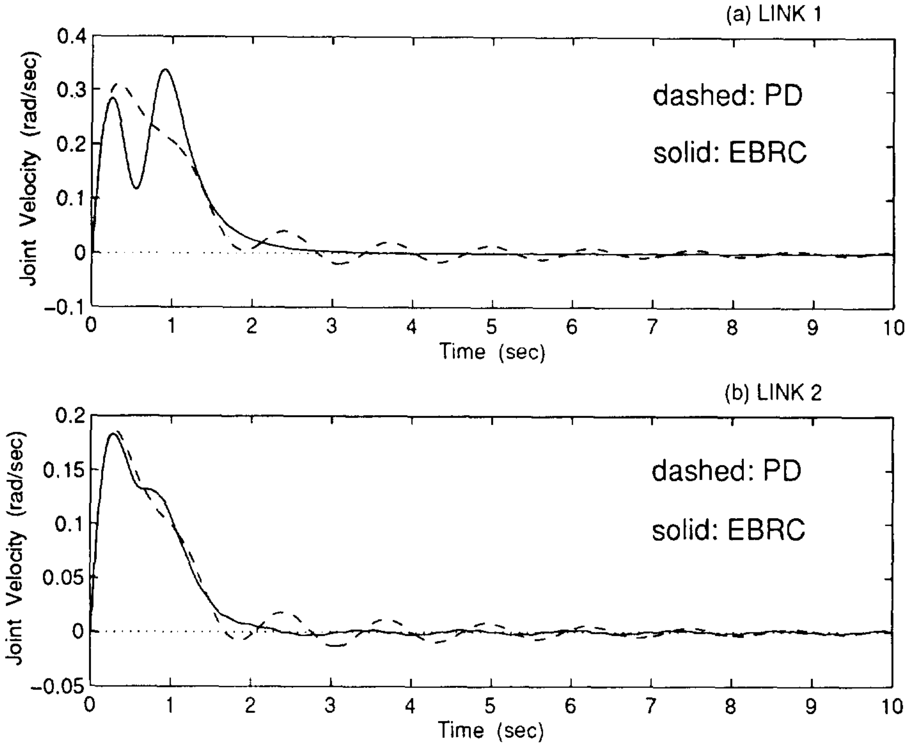

Fig. 8. Joint velocity in Case 1. (a) $\dot{\theta}_1$; (b) $\dot{\theta}_2$.

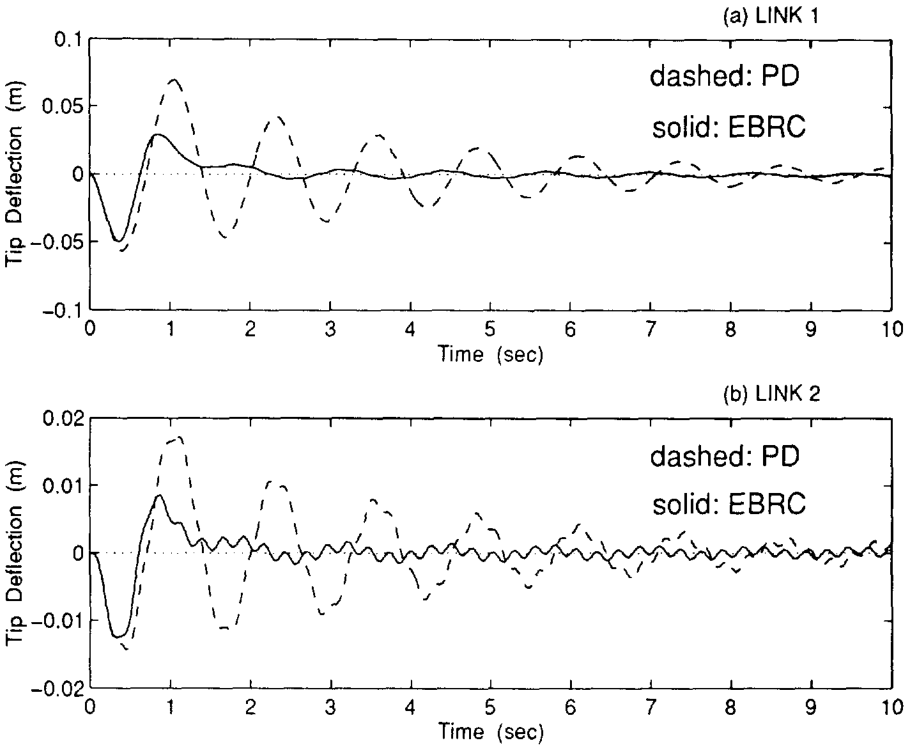

Fig. 9. Tip deflection in Case 2. (a) $y_1(L_1, t)$; (b) $y_2(L_2, t)$.

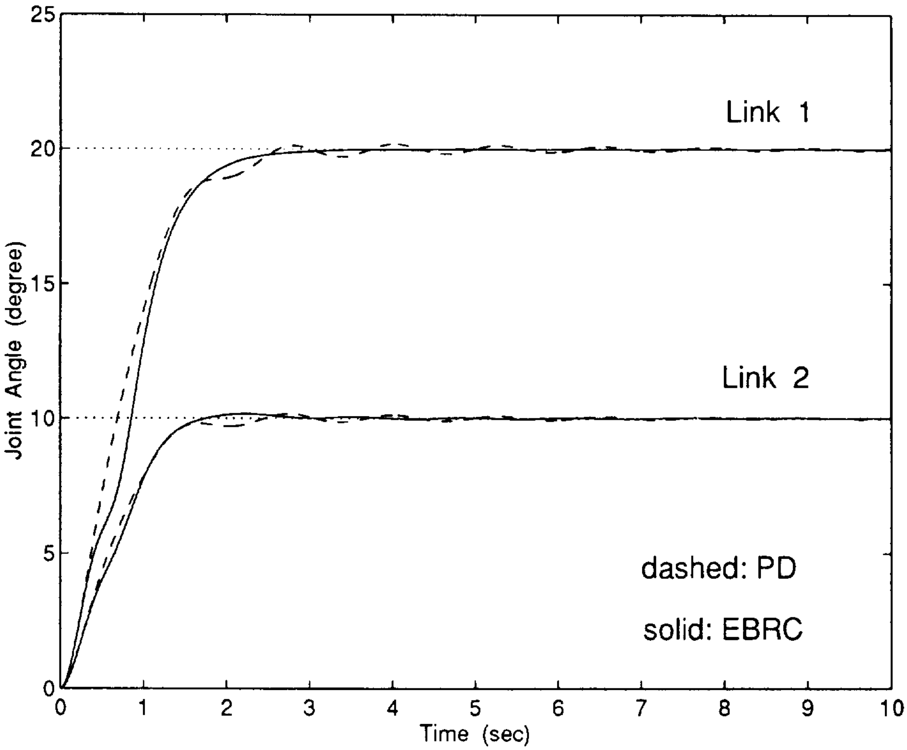

Fig. 10. Joint angle in Case 2.

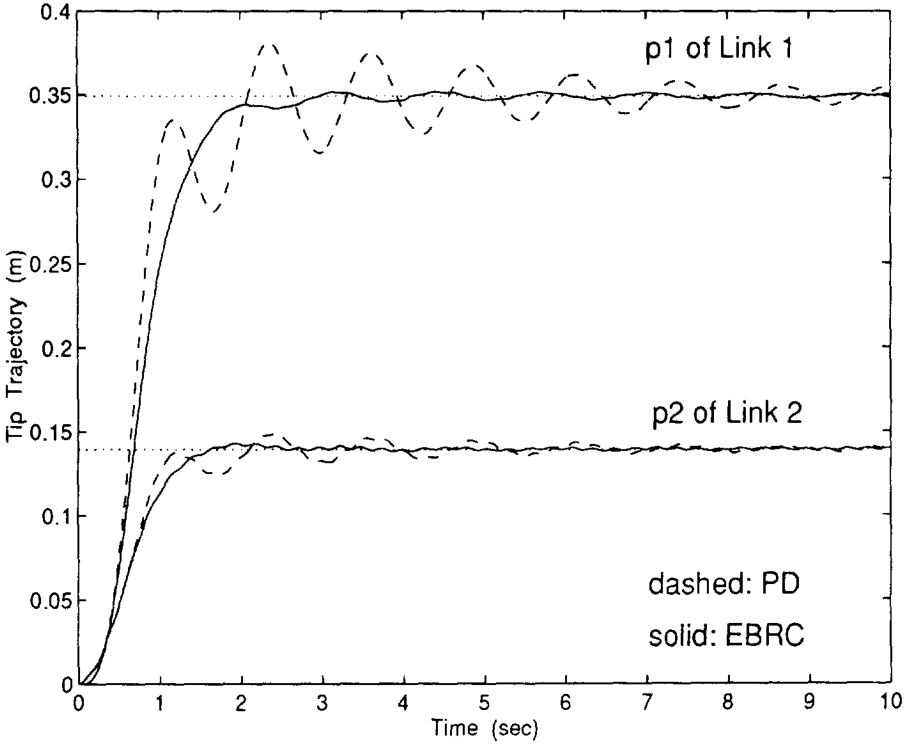

Fig. 11. Tip trajectory in Case 2.

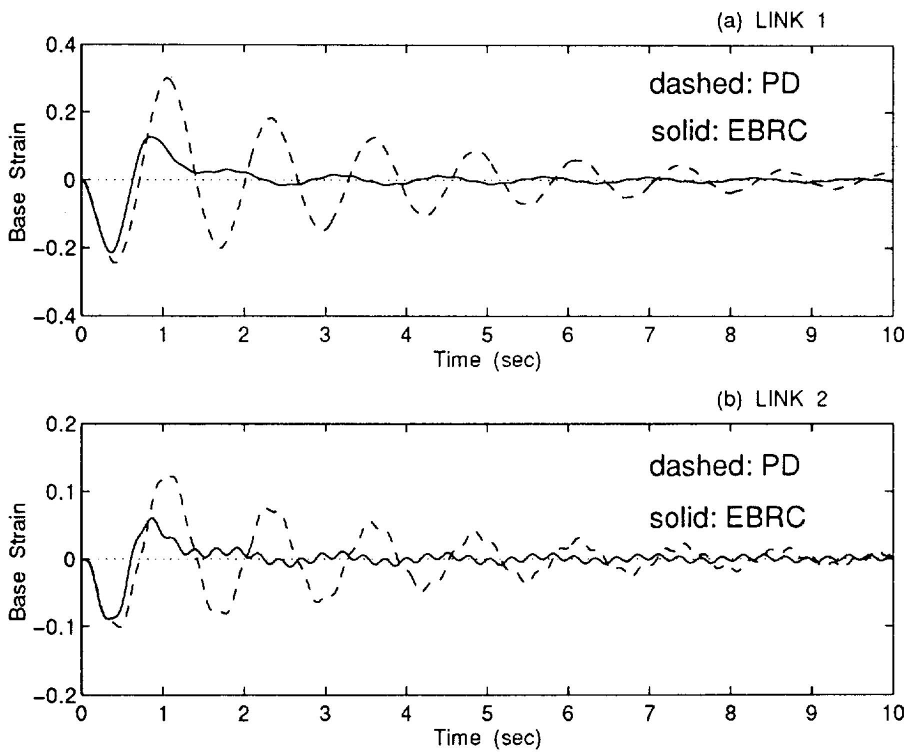

Fig. 12. Base strain feedback in Case 2. (a) $y_1''(0, t)$; (b) $y_2''(0, t)$.

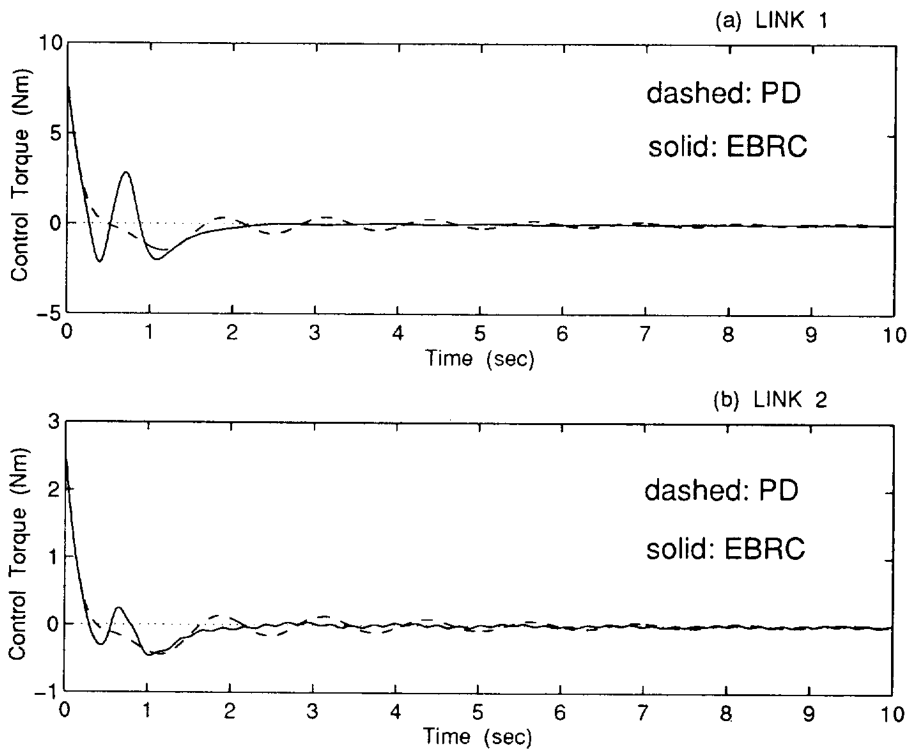

Fig. 13. Control effort in Case 2.

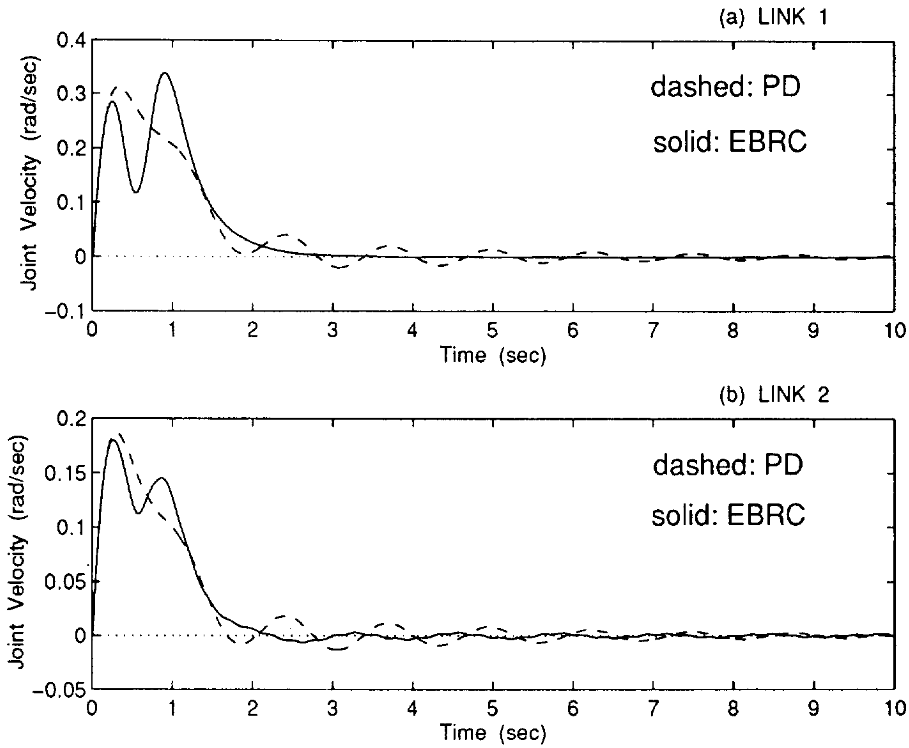

Fig. 14. Joint velocity in Case 2. (a) $\dot{\theta}_1$; (b) $\dot{\theta}_2$.

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