MA4230: Problem Sheet 4

AY 2023/24

Q 1 *Bauer–Fike theorem*

Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix with an eigenvalue decomposition $A = XDX^{-1}$, where $X \in \mathbb{C}^{n\times n}$ is invertible and $D = \operatorname{diag}_{n\times n}(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^{n\times n}$ is diagonal with $\{\lambda_1, \dots, \lambda_n\} = \Lambda(A)$. Let $\Delta A \in \mathbb{C}^{n\times n}$. Show that for any $\mu \in \Lambda(A + \Delta A)$ there holds

$$\min_{i\in\{1,\dots,n\}} |\mu - \lambda_i| \leq \kappa_p(X)\, \|\Delta A\|_p \qquad \text{(here, } \kappa_p(X) := \|X\|_p \|X^{-1}\|_p\text{)}$$

for any $p \geq 1$. [Instructions: 1) Show the claim for $\mu \in \Lambda(A + \Delta A) \cap \Lambda(A)$. 2) Now consider $\mu \in \Lambda(A+\Delta A)\backslash\Lambda(A)$ and define $M := -(D - \mu I_n)^{-1}X^{-1}(\Delta A)X$. Show that $\|M\|_p \geq 1$ by proving that $I_n - M$ is singular. 3) Conclude the proof.]

Q 2 *Perturbed linear systems*

(i) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $|||\cdot|||$ be the matrix norm on $\mathbb{R}^{n\times n}$ induced by $\|\cdot\|$. Let $A \in \mathbb{R}^{n\times n}$ be invertible, let $\Delta A \in \mathbb{R}^{n\times n}$ with $|||\Delta A||| < |||A^{-1}|||^{-1}$, let $b \in \mathbb{R}^n\backslash\{0\}$, and let $\Delta b \in \mathbb{R}^n$. Let $x := A^{-1}b$ and $\hat{x} := (A + \Delta A)^{-1}(b + \Delta b)$. Show that there holds

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \frac{|||\Delta A|||}{|||A|||}\kappa(A)} \left(\frac{|||\Delta A|||}{|||A|||} + \frac{\|\Delta b\|}{\|b\|} \right) \qquad \text{(here, } \kappa(A) := |||A|||\, |||A^{-1}|||\text{)}.$$

(ii) For fixed $A \in \mathbb{R}^{n\times n}\backslash\{0\}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, consider the problem of computing $\frac{1}{2}x^{\mathrm{T}}AA^{\dagger}x + b^{\mathrm{T}}x + c$ from $x \in \mathbb{R}^n$. Write this problem as a mathematical problem $f : X \to Y$ with normed vector spaces X, Y, which you should equip with Euclidean norms. Compute the absolute condition number $\hat{\kappa} = \hat{\kappa}(x)$ and show that $\hat{\kappa}(x) \leq \|x\|_2 + \|b\|_2$ for any $x \in \mathbb{R}^n$.

(iii) Let

$$A := \begin{pmatrix} 6 & 4 & 3 \\ 1 & -3 & -2 \\ 8 & 5 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}.$$

- Show that for any $\Delta A \in \mathbb{R}^{3\times 3}$ with $\frac{\|\Delta A\|_1}{\|A\|_1} \leq \frac{1}{150}$ and $\Delta b \in \mathbb{R}^3$ with $\frac{\|\Delta b\|_1}{\|b\|_1} \leq \frac{1}{110}$, the linear system $(A + \Delta A)\hat{x} = b + \Delta b$ has a unique solution $\hat{x} = \hat{x}(\Delta A, \Delta b) \in \mathbb{R}^3$ and that $\frac{\|\hat{x}-x\|_1}{\|x\|_1} < \frac{1}{4}$, where $x \in \mathbb{R}^3$ denotes the unique solution to $Ax = b$.
- Find the value of

$$s := \sup_{\substack{\Delta A \in \mathbb{R}^{3\times 3},\, \Delta b \in \mathbb{R}^3 \\ \frac{\|\Delta A\|_1}{\|A\|_1} \leq \frac{1}{150},\, \frac{\|\Delta b\|_1}{\|b\|_1} \leq \frac{1}{110}}} \|\hat{x}(\Delta A, \Delta b) - x\|_1$$

and prove your claim.

Q 3 *Application of Gerschgorin's theorem*

Let $A \in \mathbb{R}^{5\times 5}$ be defined as

$$A := \begin{pmatrix} -10 & \frac{1}{2} & \frac{1}{2} & 1 & 2 \\ -\frac{1}{2} & 5 & 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & 10 & 5 & 0 \\ 1 & 1 & 5 & 20 & 5 \\ 2 & \frac{1}{2} & 0 & 5 & 40 \end{pmatrix}$$

and write $\Lambda(A) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} \subseteq \mathbb{C}$ with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq |\lambda_5|$. Using Gerschgorin's theorem, show that $\frac{|\lambda_1|}{|\lambda_5|} \leq 19$.

Q 4 *MATLAB: QR algorithm and simultaneous iteration*

Implement the QR algorithm and simultaneous iteration in MATLAB. In Example 6.3, add a few more iterations produced by the QR algorithm, perform simultaneous iteration and verify the results from Theorem 6.13.

Q 5 *Power iteration, inverse iteration*

Let $A \in \mathbb{R}^{3\times 3}$ be defined as

$$A := \begin{pmatrix} 5 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and write $\Lambda(A) = \{\lambda_1, \lambda_2, \lambda_3\} \subseteq \mathbb{R}$ with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$. Consider the condition number $\kappa_2(A) := \|A\|_2 \|A^{-1}\|_2$ of A.

(i) Show that $\kappa_2(A) = \frac{|\lambda_1|}{|\lambda_3|}$.

(ii) Find an approximation to $\kappa_2(A)$ by performing power iteration and inverse iteration to approximate λ_1 and λ_3. Perform three steps each with $v^{(0)} := (1, 0, 0)^{\mathrm{T}} \in \mathbb{R}^3$.

Q 6 *Hessenberg decomposition, QR algorithm, Rayleigh quotient shift*

Let $A \in \mathbb{R}^{4\times 4}$ be defined as

$$A := \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 4 & -1 & 1 \\ 2 & -1 & 2 & 1 \\ 2 & 1 & 1 & 4 \end{pmatrix}.$$

(i) Compute a Hessenberg decomposition $A = QHQ^{\mathrm{T}}$ of A.

(ii) Perform three steps of the QR algorithm applied to H. What are the approximations to the eigenvalues of A?

(iii) Perform three steps of the QR algorithm with Rayleigh quotient shift applied to H. What are the approximations to the eigenvalues of A?

For parts (ii) and (iii), you may perform the respective algorithms in floating point arithmetic using MATLAB.

Q 7 *Special matrices*

For $n \in \mathbb{N}$, we define $C(n) := \{M \in \mathbb{R}^{n\times n} : M \text{ is } \textit{special}\}$ and $\mathcal{S}^n := \{M \in \mathbb{R}^{n\times n} : M^{\mathrm{T}} = M\}$, where we call a matrix $M \in \mathbb{R}^{n\times n}$ *special* iff

$$\exists\,\delta \in (0,1]: \quad \|M\|_F \le \frac{\operatorname{tr}(M)}{\sqrt{n-1+\delta}}.$$

Further, we define the map $\gamma : \mathbb{R}^{n\times n}\backslash\{0\} \to \mathbb{R},\ M \mapsto \gamma(M) := \|M\|_F^{-2}\operatorname{tr}(M)$. Let $A \in \mathcal{S}^n$ and suppose that there exist $\alpha, \beta > 0$ such that $\alpha\|\xi\|_2^2 \le \xi^{\mathrm{T}} A\xi \le \beta\|\xi\|_2^2$ for any $\xi \in \mathbb{R}^n$.

(i) Show that $\frac{\alpha}{\beta^2} \le \gamma(A) \le \frac{\beta}{\alpha^2}$.

(ii) If $n = 2$, show that $A \in C(2)$ and $\alpha\|B\|_F^2 \le \beta(\|B\|_F^2 - |\gamma(A)\langle A, B\rangle_F - \operatorname{tr}(B)|^2)$ for any $B \in \mathcal{S}^2$.