MA4230: Problem Sheet 3

AY 2023/24

Q 1 *Schur complement: A block-version of Gaussian elimination*

Let $d, n \in \mathbb{N}$ with $d > n$. Let $M := \left(\begin{array}{c|c} A & B \\ C & D \end{array}\right) \in \mathbb{R}^{d\times d}$ be a block-matrix with blocks $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times(d-n)}$, $C \in \mathbb{R}^{(d-n)\times n}$, $D \in \mathbb{R}^{(d-n)\times(d-n)}$, and suppose that $\det(M_{1:i,1:i}) \neq 0$ for all $i \in \{1, \dots, d\}$. Let $S := D - CA^{-1}B \in \mathbb{R}^{(d-n)\times(d-n)}$ denote the so-called Schur complement of A in M (check S is well-defined).

(i) Find $L_1 = \left(\begin{array}{c|c} I_n & 0_{n\times(d-n)} \\ \hline X & I_{(d-n)} \end{array}\right) \in \mathbb{R}^{d\times d}$ such that $L_1 M = \left(\begin{array}{c|c} A & B \\ \hline 0_{(d-n)\times n} & S \end{array}\right)$. (Find X.)

(ii) Using the previous part, find a LU factorization of $M := \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -2 & 0 & 3 \\ 5 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Q 2 *Surrounding LU/PA=LU factorization*

Let $f : \mathbb{R}^{n\times n} \to [0, \infty)$ be given by $f(M) := \max_{i,j\in\{1,\dots,n\}} |m_{ij}|$ for $M = (m_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$. Let

$$\mathcal{L}_n := \{M \in \mathbb{R}^{n\times n} \mid f(M) = 1 \text{ and } M \text{ is unit lower-triangular}\},$$
$$\mathcal{U}_n := \{M \in \mathbb{R}^{n\times n} \mid f(M) > 0 \text{ and } M \text{ is upper-triangular}\},$$

and $\mathcal{X}_n := \mathcal{L}_n \times \mathcal{U}_n$.

(i) Show that

$$\sup_{(L,U)\in\mathcal{X}_n} \frac{\|U\|_\infty}{\|LU\|_\infty} = 2^{n-1}.$$

(ii) Suppose $PA = LU$ is a PA=LU factorization obtained via Gaussian elimination with partial pivoting applied to $A \in \mathbb{R}^{n\times n}$. Show that $f(U) \le 2^{n-1} f(A)$ and show that the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ with $a_{ii} = 1 = a_{in}$ for all $1 \le i \le n$, $a_{ij} = -1$ for all $1 \le j < i \le n$, and $a_{ij} = 0$ otherwise satisfies the inequality with equality.

Q 3 *MATLAB*

Write a MATLAB function [P,L,U]=Gauss(A) performing Gaussian elimination with partial pivoting (Algorithm 4.2). Input: $A \in \mathbb{R}^{n\times n}$. Output: $P, L, U \in \mathbb{R}^{n\times n}$ such that $PA = LU$ is a PA=LU factorization of A.

Q 4 *Results surrounding positive definite matrices and Cholesky factorization*

(i) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that $A \succ 0$ if, and only if, $\Lambda(A) \subseteq (0,\infty)$.

(ii) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and let $X \in \mathbb{R}^{n\times r}$ with $r \le n$ and $\operatorname{rk}(X) = r$. Show that $X^{\mathrm{T}}AX \in \mathbb{R}^{r\times r}$ is symmetric positive definite.

(iii) Let $A \in \mathbb{R}^{n\times n}$ be invertible and let $M := A^{\mathrm{T}}A$. Suppose that $A = Q\tilde{R}$ is a QR factorization of A, that $M = R^{\mathrm{T}}R$ is a Cholesky factorization of M, and that the diagonal entries of $\tilde{R}$ and R are positive. Show that $\tilde{R} = R$.

(iv) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix with Cholesky factorization $A = R^{\mathrm{T}}R$. Show that $\|A\|_2 = \|R\|_2^2$.

Q 5 *Computation of LU, PA=LU, AQ=LU, PAQ=LU, and Cholesky factorizations*

Consider the matrix

$$A := \begin{pmatrix} 2 & -1 & 3 & 1 \\ -1 & 1 & -2 & 1 \\ 2 & 8 & -6 & 0 \\ 1 & 0 & -1 & -1 \end{pmatrix}.$$

(i) Does A have a LU factorization? Compute a LU factorization of the submatrix $A_{1:3,1:3}$.

(ii) Apply Gaussian elimination with partial pivoting to A to obtain a PA=LU factorization.

(iii) Compute a permutation matrix $Q \in \mathbb{R}^{4\times 4}$, a lower-triangular matrix L, and an upper-triangular matrix $U \in \mathbb{R}^{4\times 4}$ such that $AQ = LU$.

(iv) Apply Gaussian elimination with full pivoting to A to obtain a PAQ=LU factorization.

(v) Compute the Cholesky factorization of $A^{\mathrm{T}}A$.

Q 6 *Moore–Penrose inverse*

For $A \in \mathbb{R}^{m\times n}$, we call $A^\dagger \in \mathbb{R}^{n\times m}$ the Moore–Penrose inverse of A if, and only if, all of the following holds: $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^{\mathrm{T}} = AA^\dagger$, $(A^\dagger A)^{\mathrm{T}} = A^\dagger A$.

(i) 1) First, show that any square diagonal matrix $D = \operatorname{diag}_{d\times d}(\alpha_1,\dots,\alpha_d) \in \mathbb{R}^{d\times d}$ has a Moore–Penrose inverse.

(Hint: Set $D^\dagger := \operatorname{diag}_{d\times d}(\alpha_1^\dagger,\dots,\alpha_d^\dagger)$ where $\alpha_i^\dagger := \alpha_i^{-1}$ if $\alpha_i \ne 0$ and $\alpha_i^\dagger = 0$ if $\alpha_i = 0$.)

2) Next, show that any diagonal matrix $M = \operatorname{diag}_{m\times n}(\alpha_1,\dots,\alpha_p) \in \mathbb{R}^{m\times n}$ (here, $p := \min(m,n)$) has a Moore–Penrose inverse.

3) Now, show that any matrix $A \in \mathbb{R}^{m\times n}$ has a unique Moore–Penrose inverse $A^\dagger$, and that $A^\dagger$ is given by $A^\dagger := V\Sigma^\dagger U^{\mathrm{T}}$ when $A = U\Sigma V^{\mathrm{T}}$ is a SVD of A.

(Hint: for uniqueness, show that if $A_1^\dagger, A_2^\dagger \in \mathbb{R}^{n\times m}$ are Moore–Penrose inverses of A, then $AA_1^\dagger = AA_2^\dagger$ and $A_1^\dagger A = A_2^\dagger A$.)

(ii) Let $A \in \mathbb{R}^{m\times n}$ and $\alpha \in \mathbb{R}\backslash\{0\}$. Show that $(A^\dagger)^\dagger = A$, $(A^{\mathrm{T}})^\dagger = (A^\dagger)^{\mathrm{T}}$, $(\alpha A)^\dagger = \alpha^{-1}A^\dagger$, and the relations $A^\dagger = (A^{\mathrm{T}}A)^\dagger A^{\mathrm{T}} = A^{\mathrm{T}}(AA^{\mathrm{T}})^\dagger$.

(Hint: For the last claim, you may use that $\forall M \in \mathbb{R}^{n\times n} : A^{\mathrm{T}}AM = 0 \Rightarrow AM = 0$, and that $\forall M \in \mathbb{R}^{m\times m} : MAA^{\mathrm{T}} = 0 \Rightarrow MA = 0$. Optional: show these facts.)

Further, show that if $m = n$ and A is invertible, then $A^\dagger = A^{-1}$.

(iii) For $A \in \mathbb{R}^{m\times n}$, $m \ge n$, and $\operatorname{rk}(A) = n$, show that $A^\dagger = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ and $A^\dagger A = I_n$.

For $A \in \mathbb{R}^{m\times n}$, $m \le n$, and $\operatorname{rk}(A) = m$, show that $A^\dagger = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ and $AA^\dagger = I_m$.

(iv) Let $A \in \mathbb{R}^{m\times n}$, $m > n$, and write $A = \begin{pmatrix} B \\ C \end{pmatrix} \in \mathbb{R}^{m\times n}$ with $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{(m-n)\times n}$. Suppose that B is invertible. Show that $\|A^\dagger\|_2 \le \|B^{-1}\|_2$.

Q 7 *Least squares problems*

(i) Let $A \in \mathbb{R}^{m\times n}$. Consider the matrices $P := AA^\dagger \in \mathbb{R}^{m\times m}$ and $\tilde{P} := A^\dagger A \in \mathbb{R}^{n\times n}$. Show that P and $\tilde{P}$ are orthogonal projectors, and that

$$1)\ \mathcal{R}(P) = \mathcal{R}(A), \quad 2)\ \mathcal{N}(P) = \mathcal{N}(A^{\mathrm{T}}), \quad 3)\ \mathcal{R}(\tilde{P}) = \mathcal{R}(A^{\mathrm{T}}), \quad 4)\ \mathcal{N}(\tilde{P}) = \mathcal{N}(A).$$

(Hint: Show $A^\dagger A = A^{\mathrm{T}}(A^{\mathrm{T}})^\dagger$.) This proves the following important result:

Theorem (orthogonal projector onto range of matrix): For any $A \in \mathbb{R}^{m\times n}$, the orthogonal projector $P_{\mathcal{R}(A)} \in \mathbb{R}^{m\times m}$ onto $\mathcal{R}(A)$ is given by $P_{\mathcal{R}(A)} = AA^\dagger$, and this projector $P_{\mathcal{R}(A)}$ is the projector onto $\mathcal{R}(A)$ along $\mathcal{N}(A^{\mathrm{T}})$.

(ii) Let $A := \begin{pmatrix} 1 & -2 \\ 2 & 2 \\ -1 & 1 \\ 2 & 1 \end{pmatrix}$ and $b := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Compute $A^\dagger$ and the orthogonal projector onto $\mathcal{N}(A^{\mathrm{T}})$, and find all solutions $x \in \mathbb{R}^2$ to the least squares problem $\|Ax - b\|_2 = \inf_{v\in\mathbb{R}^2} \|Av - b\|_2$.

(iii) Let $A := \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 2 & -2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{10\times 2}$ and $b := (b_i) \in \mathbb{R}^{10}$ with $b_i := 1$ for all $i \in \{1, \ldots, 10\}$. Show that the least squares problem $\|Ax - b\|_2 = \inf_{v\in\mathbb{R}^2} \|Av - b\|_2$ has a unique solution $x \in \mathbb{R}^2$, and compute x.