MA4230: Problem Sheet 2

AY 2023/24

Q 1 *Hadamard's inequality*

Let $a_1, \ldots, a_n \in \mathbb{R}^n \backslash \{0\}$. Set $S := \{a_1, \ldots, a_n\} \subset \mathbb{R}^n \backslash \{0\}$ and $A := (a_1 | \cdots | a_n) \in \mathbb{R}^{n \times n}$.

(i) Show that $|\det(A)| \leq \prod_{i=1}^n \|a_i\|_2$. (Hint: Take a QR factorization $A = QR = Q(r_1 | \cdots | r_n)$.)

(ii) Show that $|\det(A)| = \prod_{i=1}^n \|a_i\|_2$ if, and only if, the set S is orthogonal.

(iii) Show that $|\det(A)| = n^{\frac{n}{2}}$ if A is a Hadamard matrix of order n, that is, $AA^{\mathrm{T}} = nI_n$ and all entries of A belong to $\{-1, 1\}$.

Q 2 *Projectors*

Let $P \in \mathbb{R}^{n \times n} \backslash \{0\}$ be a projector. We denote its singular values by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, and denote its rank by $r := \mathrm{rk}(P)$.

(i) First, show that $\sigma_1 \geq 1$. (Hint: show $\|P\|_2 \leq \|P\|_2^2$.) Now, show that $\sigma_i \geq 1$ for all $i \in \{1, \ldots, r\}$. (Hint: take a SVD $P = U\Sigma V^{\mathrm{T}}$ of P, prove $\Sigma = \Sigma W \Sigma$ for some orthogonal matrix W, and deduce that $\sigma_i \leq \sigma_i^2 \ \forall i \in \{1, \ldots, n\}$.)

(ii) Show that $\Lambda(P) \subseteq \{0, 1\}$. Find an eigenvalue decomposition $P = XDX^{-1}$ of P. (Hint: use Remark 3.8(i) and Theorem 2.3(ii) to find eigenvectors.) Prove $\mathrm{tr}(P) = r$.

(iii) Show that $\det(P) = 0$ if $P \neq I_n$.

(iv) Show that $I_n - 2P$ is an orthogonal matrix if P is an orthogonal projector.

(v) Suppose $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ for some $A \in \mathbb{R}^{n \times m}$ with $m \leq n$ and $\mathrm{rk}(A) = m$. Prove $\mathrm{rk}(P) = m$ (hint: $\mathrm{rk}(P) = \mathrm{tr}(P)$; see (ii)). What are the singular values of P? (Hint: use Theorem 3.6.)

Q 3 *Computing QR factorizations and orthogonal projectors*

(i) Compute a reduced QR factorization and a (full) QR factorization for each of the following matrices:

$$A_1 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 4 & 0 & 2 & 1 \\ 2 & -6 & -2 & 0 \\ 4 & 2 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Use Householder for A_2, and Gram–Schmidt for A_3. (No constraints for A_1.)

(ii) Find all QR factorizations of A_2. Using the QR factorization $A_2 = QR$ with R having positive diagonal entries, find $x \in \mathbb{R}^3$ such that $A_2 x = (2, -3, 4)^{\mathrm{T}} =: b$ by reducing to an upper-triangular system $Rx = \tilde{b}$ which you should solve by backward substitution.

(iii) Compute the orthogonal projector onto $\mathcal{R}(A_1^{\mathrm{T}})$, the orthogonal projector onto $\mathcal{N}(A_2)$, and the orthogonal projector onto $\mathcal{R}(A_3)$.

Q 4 *QR factorization of a wide matrix*

A QR factorization of a "wide" matrix $A \in \mathbb{R}^{m\times n}$ with $m < n$ is defined as a factorization $A = QR$ where $Q \in \mathbb{R}^{m\times m}$ is orthogonal and $R \in \mathbb{R}^{m\times n}$ is upper-triangular, i.e., $R = (\hat{R}\,|\,\tilde{R})$ for some upper-triangular $\hat{R} \in \mathbb{R}^{m\times m}$ and some matrix $\tilde{R} \in \mathbb{R}^{m\times(n-m)}$. State a procedure for computing a QR factorization of a given matrix $A \in \mathbb{R}^{m\times n}$ with $m < n$. (Hint: One step should be to compute a QR factorization of some square matrix.)

Optional: Compute a QR factorization of $A := A_3^{\mathrm{T}} \in \mathbb{R}^{4\times 5}$ with A_3 from Q3.

Q 5 *LQ, RQ, QL factorizations*

Having the QR factorization at hand, we can find similar factorizations. Let $A \in \mathbb{R}^{n\times n}$.

(i) LQ factorization: Show that there exist $L \in \mathbb{R}^{n\times n}$ lower-triangular and $Q \in \mathbb{R}^{n\times n}$ orthogonal such that $A = LQ$. (Hint: take a QR factorization of a certain matrix.)

(ii) RQ factorization: Show that there exist $R \in \mathbb{R}^{n\times n}$ upper-triangular and $Q \in \mathbb{R}^{n\times n}$ orthogonal such that $A = RQ$. (Hint: QR factorize $(PA)^{\mathrm{T}}$, where $P := (e_n|\cdots|e_2|e_1) \in \mathbb{R}^{n\times n}$.)

(iii) QL factorization: Show that there exist $L \in \mathbb{R}^{n\times n}$ lower-triangular and $Q \in \mathbb{R}^{n\times n}$ orthogonal such that $A = QL$. (Hint: take a RQ factorization of a certain matrix.)

(iv) Compute $Q, M \in \mathbb{R}^{3\times 3}$ such that $Q^{\mathrm{T}}Q = I_3$, $m_{12} = m_{22} = m_{13} = 0$, and $QM = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 2 & -2 \\ -3 & 4 & -1 \end{pmatrix}$.

Q 6 *Householder reflectors*

Let $v \in \mathbb{R}^n$ with $\|v\|_2 = 1$, and let $F := I_n - 2vv^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. Show that $F^{\mathrm{T}} = F = F^{-1}$, and compute the eigenvalues, the determinant, and the singular values of F.

Q 7 *MATLAB: Gram–Schmidt*

Write a MATLAB function [Qhat,Rhat]=mgs(A) performing Algorithm 3.3. Input: $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, $\mathrm{rk}(A) = n$. Output: Qhat$\in \mathbb{R}^{m\times n}$, Rhat$\in \mathbb{R}^{n\times n}$ such that $A =$ Qhat Rhat is a reduced QR factorization of A.

Q 8 *Properties of triangular matrices, uniqueness result for LU factorization*

Let $U, \tilde{U} \in \mathbb{R}^{n\times n}$ be upper-triangular and $L, \tilde{L} \in \mathbb{R}^{n\times n}$ lower-triangular matrices.

(i) Show that $U\tilde{U}$ is upper-triangular and $L\tilde{L}$ is lower-triangular.

(ii) If U and L are invertible, prove U^{-1} is upper-triangular and L^{-1} is lower-triangular.

(iii) Show that if $A \in \{U, L\}$ is an orthogonal matrix, then A is diagonal.

(iv) Suppose that $L, \tilde{L}$ are unit lower-triangular, $A := LU = \tilde{L}\tilde{U}$ and that A is invertible. Show that $L = \tilde{L}$ and $U = \tilde{U}$.

Q 9 *LU factorization of a tridiagonal matrix*

Let $A = (a_{ij}) \in \mathbb{R}^{5\times 5}$ be an invertible matrix with $\det(A_{1:i,1:i}) \neq 0$ for all $i \in \{1, \ldots, 5\}$. We then know (Theorem 4.2 and Q8(iv)) that A can be uniquely factorized as $A = LU$ with L unit lower-triangular and U upper-triangular. Suppose A is a tridiagonal matrix, that is, $a_{ij} = 0$ whenever $|i - j| > 1$. Show that L and U are bidiagonal (i.e., except for the main diagonal and one adjacent diagonal, there are no non-zero entries).