MA4230: Problem Sheet 1

AY 2023/24

Q 1 *Matrix manipulation*

Let $A \in \mathbb{R}^{4\times 4}$. Let $B \in \mathbb{R}^{4\times 3}$ be the matrix obtained from A as follows: Double column 1, halve row 3, add row 3 to row 1, interchange columns 1 and 4, subtract row 2 from each of the other rows, replace column 4 by column 3, delete column 1. Find $M_1 \in \mathbb{R}^{4\times 4}$ and $M_2 \in \mathbb{R}^{4\times 3}$ such that $B = M_1 A M_2$.

Q 2 *Estimate for matrix ∞-norm and spectral norm*

Show that for any $A \in \mathbb{R}^{m\times n}$ there holds $\frac{1}{\sqrt{m}}\|A\|_2 \le \|A\|_\infty \le \sqrt{n}\|A\|_2$.
(Hint: First, show that there holds $\frac{1}{\sqrt{d}}\|x\|_2 \le \|x\|_\infty \le \|x\|_2$ for all $x \in \mathbb{R}^d$.)

Q 3 *SVD: computation, matrix properties, and low-rank approximation*

Let $A := \begin{pmatrix} 2 & 11 \\ 10 & -5 \end{pmatrix}$ and $B := \begin{pmatrix} 3 & 2 \\ 2 & -2 \\ 2 & 3 \end{pmatrix}$.

(i) Compute a SVD of A and compute a SVD of B.

(ii) Using (i), find $\mathrm{rk}(B)$, $\mathcal{R}(B)$, $\mathcal{N}(B)$, $\|B\|_2$, and $\|B\|_F$.

(iii) Let $Q \in \mathbb{R}^{2\times 2}$ be a given orthogonal matrix. Find a matrix $C \in \mathbb{R}^{3\times 2}$ with $\mathrm{rk}(C) \le 1$ such that $\|BQ - C\|_F \le \|BQ - M\|_F$ for any $M \in \mathbb{R}^{3\times 2}$ with $\mathrm{rk}(M) \le 1$, and compute $\|BQ - C\|_F$.

(iv) List all SVDs of A. For the SVD $A = U\Sigma V^{\mathrm{T}}$ with $u_{11} \ge 0 \ge u_{12}$, sketch $S := \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$, $V^{\mathrm{T}}S$, $\Sigma V^{\mathrm{T}}S$, $U\Sigma V^{\mathrm{T}}S$, and include the right singular vectors $v_1, v_2 \in S$ and their images $V^{\mathrm{T}}v_i$, $\Sigma V^{\mathrm{T}}v_i$, $U\Sigma V^{\mathrm{T}}v_i$ in the respective diagrams (as in Figure 1 from lecture notes). Compute the area enclosed by the ellipse AS.

Q 4 *Some results surrounding SVD*

Let $A \in \mathbb{R}^{n\times n}$, let $A = U\Sigma V^{\mathrm{T}}$ be a SVD of A, and write $\Sigma = \mathrm{diag}_{n\times n}(\sigma_1, \dots, \sigma_n)$. We define

$$f : \mathbb{R}^{n\times n} \to [0,\infty), \qquad M \mapsto f(M) := \inf_{Z\in\mathbb{R}^{n\times n},\, Z^{\mathrm{T}}Z = I_n} \|M - Z\|_2.$$

Let $k \in \{1, \dots, n\}$ be such that $|\sigma_k - 1| \ge |\sigma_i - 1|$ for all $i \in \{1, \dots, n\}$.

(i) Find an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and a symmetric positive semidefinite matrix $P \in \mathbb{R}^{n\times n}$ (i.e., $P^{\mathrm{T}} = P$ and $\Lambda(P) \subset [0,\infty)$) such that $A = QP$.

(ii) Show that if $A = Q_1P_1$ and $A = Q_2P_2$ for some orthogonal matrices $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ and symmetric positive semidefinite matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, then $P_1^2 = P_2^2$.

(iii) Show that $f(A) = f(\Sigma) = |\sigma_k - 1|$. (Hint: Prove $f(A) \ge f(\Sigma) \ge |\sigma_k - 1| \ge f(A)$.)

(iv) Assuming that $\det(A) \ne 0$, find a SVD of A^{-1}.

Q 5 *An alternative way for computing SVDs of square matrices*

(i) Let $A \in \mathbb{R}^{n\times n}$ and let $A = U\Sigma V^{\mathrm{T}}$ be a SVD of A. Show $AV = U\Sigma$ and $A^{\mathrm{T}}U = V\Sigma$.

(ii) Let $A \in \mathbb{R}^{n\times n}$ and let $A = U\Sigma V^{\mathrm{T}}$ be a SVD of A. Define $H := \left(\begin{array}{c|c} 0_{n\times n} & A^{\mathrm{T}} \\ A & 0_{n\times n} \end{array}\right) \in \mathbb{R}^{2n\times 2n}$ and $X := \left(\begin{array}{c|c} V & V \\ U & -U \end{array}\right) \in \mathbb{R}^{2n\times 2n}$. First, find a diagonal matrix $D \in \mathbb{R}^{2n\times 2n}$ such that $HX = XD$. Next, find an orthogonal matrix $\tilde{X} \in \mathbb{R}^{2n\times 2n}$ such that $H = \tilde{X}D\tilde{X}^{\mathrm{T}}$. What are the eigenvalues and corresponding normalized eigenvectors of H?

(iii) Let $A := \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}$. Compute a SVD of A via an eigenvalue decomposition of H.

Q 6 *MATLAB: geometric interpretation of SVD*

Write a MATLAB program for the following task (you may use the svd command, i.e., [U,Sigma,V] = svd(A)). Input: $A \in \mathbb{R}^{2\times 2}$. Output:

- Figure 1: unit circle $S := \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$ and right singular vectors v_1, v_2,
- Figure 2: ellipse AS and scaled left singular vectors $\sigma_1 u_1$, $\sigma_2 u_2$.

The vectors $v_1, v_2, \sigma_1 u_1, \sigma_2 u_2$ are to be represented as straight lines starting from the origin.

Q 7 *Fill in the gaps from lectures*

(i) Let $m, n \in \mathbb{N}$ and let $\|\cdot\|_{(n)} : \mathbb{R}^n \to [0,\infty)$ be a norm on $\mathbb{R}^n$ and $\|\cdot\|_{(m)} : \mathbb{R}^m \to [0,\infty)$ a norm on $\mathbb{R}^m$. Show that the induced norm $\|\cdot\|_{(m,n)}$ is indeed a norm on $\mathbb{R}^{m\times n}$.

(ii) Let $A \in \mathbb{R}^{m\times n}$. Writing $A^{\mathrm{T}} = (b_1|\cdots|b_m) \in \mathbb{R}^{n\times m}$, prove $\|A\|_\infty = \max_{i\in\{1,\ldots,m\}} \|b_i\|_1$.

Suppose $A \in \mathbb{R}^{3\times 2}$ satisfies $\|A\|_\infty = 1$. What is the largest possible value of $\|A\|_1$?

(iii) Show that the Frobenius norm is submultiplicative. Find a non-submultiplicative norm $\|\cdot\|$ on $\mathbb{R}^{2\times 2}$. (Hint: define $\|M\| := \max_{i,j\in\{1,2\}}|m_{ij}|$.)

Let $n \in \mathbb{N}$ with $n \geq 2$. Prove that there is no norm $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ on $\mathbb{R}^n$ for which $\|A\|_F = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$ holds for all $A \in \mathbb{R}^{n\times n}$. (Hint: consider I_n.)

(iv) Let $A \in \mathbb{R}^{m\times n}$ and define $\alpha := \sup_{x\in S} \|Ax\|_2$, where $S := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Show that the supremum is attained, i.e., that there exists $v \in S$ such that $\|Av\|_2 = \alpha$. (Hint: First, note $\alpha = \|A\|_2 < \infty$ by Q2 and Q7(ii). Next, show that $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|Ax\|_2$ is continuous.)

(v) Let $A \in \mathbb{R}^{m\times n}$. Show that for any invertible matrices $M_m \in \mathbb{R}^{m\times m}$ and $M_n \in \mathbb{R}^{n\times n}$ there holds $\mathrm{rk}(M_m A) = \mathrm{rk}(A)$ (hint: use rank-nullity theorem) and $\mathrm{rk}(AM_n) = \mathrm{rk}(A)$.

(vi) Let $A \in \mathbb{R}^{m\times n}\setminus\{0\}$. Write A as a sum of rank-one matrices without using a SVD.