MA4255 Numerical Methods in Differential Equations

Chapter 7: FD approximation of elliptic problems

7.1 FD approximation of an elliptic BVP in 2D: Existence and uniqueness, stability, consistency, and convergence

The BVP and the mesh

Let $\Omega := (0,1)^2$. We consider the BVP

$$-\Delta u + cu := -(\partial_{xx}^2 u + \partial_{yy}^2 u) + cu = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $f, c \in C(\overline{\Omega})$ and $c(x,y) \geq 0 \ \forall (x,y) \in \overline{\Omega}$. This problem has a unique weak soln $u \in H_0^1(\Omega)$. We make the assumption that $u \in C^4(\overline{\Omega})$.

First, define the mesh: Let $N \in \mathbb{N}_{\geq 2}$ and set $h := \frac{1}{N}$.
The mesh-points are $(x_i, y_j) := (ih, jh)$ for $i, j \in \{0, \ldots, N\}$.

Define the set of interior mesh-points

$$\Omega_h := \{(x_i, y_j) \,|\, i, j \in \{1, \ldots, N-1\}\},$$

the set of boundary mesh-points

$$\Gamma_h := \{(x_i, y_j) \,|\, i \in \{0, N\} \text{ or } j \in \{0, N\}\},$$

and the mesh, i.e., the set of all-mesh points,

$$\overline{\Omega}_h := \Omega_h \cup \Gamma_h = \{(x_i, y_j) \,|\, i, j \in \{0, \ldots, N\}\}.$$

The five-point FD scheme

We use the second divided difference operator to approximate $\partial_{xx}^2 u$ and $\partial_{yy}^2 u$ in the mesh points. This yields the FD scheme

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) + c(x_i, y_j) U_{i,j} &= f(x_i, y_j) \qquad \text{for } (x_i, y_j) \in \Omega_h, \\ U &= 0 \qquad \text{on } \Gamma_h, \end{aligned}$$

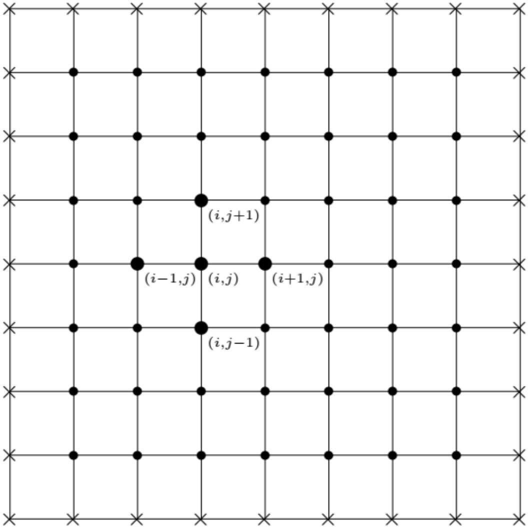
or equivalently,

$$-\left[\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \right] + c(x_i, y_j) U_{i,j} = f(x_i, y_j)$$

for $i, j \in \{1, \ldots, N-1\}$, and

$$\begin{aligned} U_{0,j} = U_{N,j} = 0 \quad & \forall i \in \{0, \ldots, N\}, \\ U_{i,0} = U_{i,N} = 0 \quad & \forall j \in \{0, \ldots, N\}. \end{aligned}$$

For each i and j with $i, j \in \{1, \ldots, N-1\}$, the FD scheme involves five values of the approximate solution U: $U_{i,j}$, $U_{i-1,j}$, $U_{i+1,j}$, $U_{i,j-1}$, $U_{i,j+1}$, and is therefore called the **five-point FD scheme**.



The mesh $\Omega_h(\cdot)$, the boundary mesh $\Gamma_h(\times)$, and a typical five-point difference stencil.

The FD scheme as linear system $AU = F$

Writing $c_{i,j} := c(x_i, y_j)$ and $F_{i,j} := f(x_i, y_j)$, the five-point FD scheme

$$-\left[\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}\right] + c_{i,j}U_{i,j} = F_{i,j}, \quad i,j \in \{1,\dots,N-1\}$$

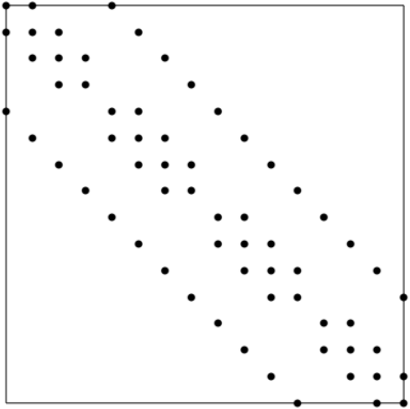
with $U_{0,j} = U_{N,j} = U_{i,0} = U_{i,N} = 0 \ \forall i,j \in \{0,\dots,N\}$, can be written as

$$AU = \underbrace{\begin{bmatrix} T_1 & -D & & & \\ -D & T_2 & -D & & \\ & \ddots & \ddots & \ddots & \\ & & -D & T_{N-2} & -D \\ & & & -D & T_{N-1} \end{bmatrix}}_{=:A \in \mathbb{R}^{(N-1)^2 \times (N-1)^2}} \underbrace{\begin{bmatrix} U_{1,:} \\ U_{2,:} \\ \vdots \\ U_{N-2,:} \\ U_{N-1,:} \end{bmatrix}}_{=:U \in \mathbb{R}^{(N-1)^2}} = \underbrace{\begin{bmatrix} F_{1,:} \\ F_{2,:} \\ \vdots \\ F_{N-2,:} \\ F_{N-1,:} \end{bmatrix}}_{=:F \in \mathbb{R}^{(N-1)^2}} = F,$$

where $U_{k,:} := \begin{bmatrix} U_{k,1} \\ \vdots \\ U_{k,N-1} \end{bmatrix} \in \mathbb{R}^{N-1}$, $F_{k,:} := \begin{bmatrix} F_{k,1} \\ \vdots \\ F_{k,N-1} \end{bmatrix} \in \mathbb{R}^{N-1}$, $D := \frac{1}{h^2} I_{N-1}$, and

$$T_k := \begin{bmatrix} \frac{4}{h^2} + c_{k,1} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{4}{h^2} + c_{k,2} & -\frac{1}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h^2} & \frac{4}{h^2} + c_{k,N-2} & -\frac{1}{h^2} \\ & & & -\frac{1}{h^2} & \frac{4}{h^2} + c_{k,N-1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}.$$

$A =$



The sparsity structure of the matrix $A \in \mathbb{R}^{(N-1)^2 \times (N-1)^2}$ (illustration for $N = 5$)

Rk: If $c > 0$ in $\overline{\Omega}$, then A is strictly diagonally dominant (as then $a_{ii} > \frac{4}{h^2} \geq \sum_{j \neq i} |a_{ij}|$ for all i). Therefore, in this case A is invertible and the FD scheme has the unique soln $U = A^{-1} F$.

Next: Show invertibility of A under the weaker assumption $c \geq 0$ in $\overline{\Omega}$.

1. (ExUn) Proof of invertibility of A: the idea

Observe: A invertible iff the only soln to $AV = 0$ is $V = 0 \in \mathbb{R}^{(N-1)^2}$.

The argument which we develop is based on mimicking, at the discrete level, the following procedure based on integration-by-parts: (recall that $u = 0$ on $\partial\Omega$, and $c(x,y) \geq 0$ for all $(x,y) \in \overline{\Omega}$)

$$\begin{aligned} &\int_\Omega (-\Delta u(x,y) + c(x,y)u(x,y))u(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_\Omega \nabla u(x,y) \cdot \nabla u(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega c(x,y)|u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad \geq \int_\Omega \left(|\partial_x u(x,y)|^2 + |\partial_y u(x,y)|^2\right)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

$\Longrightarrow$ If $-\Delta u + cu = 0$, then $\partial_x u = 0$, $\partial_y u = 0$, giving $u = 0$ (by b.c.).

For two functions V and W defined on Ω_h, we define the inner product

$$(V, W)_h := \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} h^2 V_{i,j} W_{i,j},$$

resembling the L^2-inner product $(v, w)_{L^2(\Omega)} := \int_\Omega v(x,y)\,w(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

1. (ExUn) Proof of invertibility of A: the key tool

Our key technical tool is the following summation-by-parts identity, which is the discrete counterpart of the integration-by-parts identity $(-\Delta u, u)_{L^2(\Omega)} = (\partial_x u, \partial_x u)_{L^2(\Omega)} + (\partial_y u, \partial_y u)_{L^2(\Omega)} = \|\partial_x u\|^2_{L^2(\Omega)} + \|\partial_y u\|^2_{L^2(\Omega)}$ satisfied by the fct u, obeying b.c. $u = 0$ on $\partial\Omega$.

Lemma (summation-by-parts (2D version))

Suppose that V is a function defined on $\overline{\Omega}_h$ and that $V = 0$ on Γ_h. Then, there holds

$$(-D_x^+ D_x^- V, V)_h + (-D_y^+ D_y^- V, V)_h$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{i,j}|^2.$$

(Recall $(V, W)_h := \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} h^2 V_{i,j} W_{i,j}$.)

This follows immediately from the summation-by-parts identity from Ch.6.

1. (ExUn) Proof of invertibility of A

Let $V \in \mathbb{R}^{(N-1)^2}$ be s.t. $AV = 0$. We prove that $V = 0$.

We set $V|_{\Gamma_h} := 0$. Then, by summation-by-parts, and using that $c(x, y) \geq 0 \; \forall (x, y) \in \overline{\Omega}$, we have

$$
\begin{aligned}
0 &= (-D_x^+ D_x^- V - D_y^+ D_y^- V + cV, V)_h \\
&= (-D_x^+ D_x^- V, V)_h + (-D_y^+ D_y^- V, V)_h + (cV, V)_h \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} h^2 c(x_i, y_j) |V_{i,j}|^2 \\
&\geq \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{i,j}|^2.
\end{aligned}
$$

$\implies D_x^- V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{h} = 0 \; \forall i \in \{1, \ldots, N\}, j \in \{1, \ldots, N-1\}$ and $D_y^- V_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{h} = 0 \; \forall i \in \{1, \ldots, N-1\}, j \in \{1, \ldots, N\}$.
$\implies V = 0$ (as $V = 0$ on Γ_h) and hence, A is invertible. $\square$

Thus, the FD scheme has a unique solution: $U = A^{-1} F$.

2. (Stab) Stability of the FD scheme

Goal: Prove a discrete version of the stability bound

$$\|u\|_{H^1(\Omega)} \le \frac{1}{c_0}\|f\|_{L^2(\Omega)}.$$

Recall pf:
$c_0\|u\|^2_{H^1(\Omega)} \le a(u,u) = (f,u)_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}.$

Define the **discrete L^2-norm** $\|\cdot\|_h$ and the **discrete H^1-norm** $\|\cdot\|_{1,h}$ by

$$\|V\|_h := \sqrt{(V,V)_h} = \sqrt{\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} h^2|V_{i,j}|^2},$$

$$\|V\|_{1,h} := \sqrt{\|V\|_h^2 + \|D_x^- V]|_x^2 + \|D_y^- V]|_y^2},$$

where $\|V]|_x := \sqrt{(V,V]_x}$ with $(V,W]_x := \sum_{i=1}^{N}\sum_{j=1}^{N-1} h^2 V_{i,j}W_{i,j}$, and $\|V]|_y := \sqrt{(V,V]_y}$ with $(V,W]_y := \sum_{i=1}^{N-1}\sum_{j=1}^{N} h^2 V_{i,j}W_{i,j}$.
Using this notation, we have shown on the previous slide that

$$(f,U)_h = (-D_x^+D_x^-U - D_y^+D_y^-U + cU, U)_h \ge \|D_x^-U]|_x^2 + \|D_y^-U]|_y^2.$$

2. (Stab) Proof of stability of the FD scheme

Lemma (Discrete Poincaré–Friedrichs inequality (2D version))

Let V be a fct defined on $\overline{\Omega}_h$, and such that $V = 0$ on Γ_h. Then, $\exists$ a constant $c_\star > 0$, independent of V and h, s.t., for all such V,

$$\|V\|_h^2 \leq c_* \left(\|D_x^- V]|_x^2 + \|D_y^- V]|_y^2 \right).$$

Rk: The constant $c_\star$ can be take to be $c_\star = \frac{1}{4}$.

$\Longrightarrow \|U\|_h^2 \leq \frac{1}{4} \left(\|D_x^- U]|_x^2 + \|D_y^- U]|_y^2 \right)$. Using $(f, U)_h \geq \|D_x^- U]|_x^2 + \|D_y^- U]|_y^2$, we find

$$\begin{aligned} \frac{4}{5}\|U\|_{1,h}^2 &= \frac{4}{5}\|U\|_h^2 + \frac{4}{5}\left(\|D_x^- U]|_x^2 + \|D_y^- U]|_y^2 \right) \\ &\leq \|D_x^- U]|_x^2 + \|D_y^- U]|_y^2 \leq (f, U)_h. \end{aligned}$$

Noting that $(f, U)_h \leq \|f\|_h \|U\|_h \leq \|f\|_h \|U\|_{1,h}$, we proved that the FD scheme is stable with stability bound

$$\|U\|_{1,h} \leq \frac{5}{4}\|f\|_h.$$

3. (Conv) Convergence of the FD scheme

Define the **global error** e by $e_{i,j} := u(x_i, y_j) - U_{i,j}$ for $i, j \in \{0, \ldots, N\}$. Note $e = 0$ on Γ_h. For $i, j \in \{1, \ldots, N-1\}$, we have

$$\begin{aligned}
&- D_x^+ D_x^- e_{i,j} - D_y^+ D_y^- e_{i,j} + c(x_i, y_j) e_{i,j} \\
&\quad = -D_x^+ D_x^- u(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j) + c(x_i, y_j) u(x_i, y_j) - f(x_i, y_j) \\
&\quad = -D_x^+ D_x^- u(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j) + \Delta u(x_i, y_j) \\
&\quad = \left[\partial_{xx}^2 u(x_i, y_j) - D_x^+ D_x^- u(x_i, y_j)\right] + \left[\partial_{yy}^2 u(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j)\right],
\end{aligned}$$

where we have used that $f = -\Delta u + cu$. Thus,

$$-D_x^+ D_x^- e_{i,j} - D_y^+ D_y^- e_{i,j} + c(x_i, y_j) e_{i,j} = \varphi_{i,j}, \quad i, j \in \{1, \ldots, N-1\}$$

and $e = 0$ on Γ_h, where

$$\begin{aligned}
\varphi_{i,j} &:= -D_x^+ D_x^- u(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j) + c(x_i, y_j) u(x_i, y_j) - f(x_i, y_j) \\
&= \left[\partial_{xx}^2 u(x_i, y_j) - D_x^+ D_x^- u(x_i, y_j)\right] + \left[\partial_{yy}^2 u(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j)\right]
\end{aligned}$$

is the **consistency error** (or **truncation error**). By the stability bound,

$$\|u - U\|_{1,h} = \|e\|_{1,h} \leq \frac{5}{4} \|\varphi\|_h.$$

$\Longrightarrow$ It remains to estimate the term $\|\varphi\|_h$.

3. (Conv) Convergence of FD scheme: Pf of error bound

Taylor expansion yields

$$\begin{aligned}\varphi_{i,j} &= \partial^2_{xx}u(x_i,y_j) - \frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{h^2} + \partial^2_{yy}u(x_i,y_j) - \frac{u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})}{h^2} \\ &= -\frac{h^2}{12}\left(\partial^4_{xxxx}u(\xi_i,y_j) + \partial^4_{yyyy}u(x_i,\eta_j)\right)\end{aligned}$$

for some $\xi_i \in (x_{i-1}, x_{i+1})$, $\eta_j \in (y_{j-1}, y_{j+1})$. Thus,

$$|\varphi_{i,j}| \le \frac{h^2}{12}\left(\left\|\partial^4_{xxxx}u\right\|_{C(\overline{\Omega})} + \left\|\partial^4_{yyyy}u\right\|_{C(\overline{\Omega})}\right),$$

and hence,

$$\begin{aligned}\|\varphi\|_h = \sqrt{\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} h^2|\varphi_{i,j}|^2} &\le \frac{h^2}{12}\left(\left\|\partial^4_{xxxx}u\right\|_{C(\overline{\Omega})} + \left\|\partial^4_{yyyy}u\right\|_{C(\overline{\Omega})}\right)\sqrt{\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} h^2} \\ &\le \frac{h^2}{12}\left(\left\|\partial^4_{xxxx}u\right\|_{C(\overline{\Omega})} + \left\|\partial^4_{yyyy}u\right\|_{C(\overline{\Omega})}\right).\end{aligned}$$

(Note $(N-1)h \le Nh = 1$.) Combining with $\|u-U\|_{1,h} \le \frac{5}{4}\|\varphi\|_h$, we have proved the following convergence theorem/error bound:

Theorem (Convergence of the soln U of the FD scheme to the true soln u)

Let $f, c \in C(\overline{\Omega})$ with $c(x,y) \geq 0 \ \forall (x,y) \in \overline{\Omega}$, and suppose that the unique weak soln $u \in H_0^1(\Omega)$ to the BVP satisfies $u \in C^4(\overline{\Omega})$. Then,

$$\|u - U\|_{1,h} \leq \frac{5h^2}{48} \left(\left\| \partial^4_{xxxx} u \right\|_{C(\overline{\Omega})} + \left\| \partial^4_{yyyy} u \right\|_{C(\overline{\Omega})} \right) = \mathcal{O}(h^2).$$

7.2 Nonaxiparallel domains and nonuniform meshes

FD approximation of more general elliptic PDE

We have carried out an error analysis of FD schemes for the PDE $-\Delta u + cu = f$ on a square domain Ω. The error analysis of FD schemes for more general elliptic equations would proceed similarly. Consider, e.g.,

$$-\left[\partial_x(a_1\partial_x u) + \partial_y(a_2\partial_y u)\right] + b_1\partial_x u + b_2\partial_y u + cu = f$$

on $\Omega := (0,1)^2$, which we can approximate in the mesh point (x_i, y_j) by

$$\begin{aligned} f(x_i,y_j) = &-\frac{a_1(x_{i+1/2},y_j)\frac{U_{i+1,j}-U_{i,j}}{h} - a_1(x_{i-1/2},y_j)\frac{U_{i,j}-U_{i-1,j}}{h}}{h} \\ &-\frac{a_2(x_i,y_{j+1/2})\frac{U_{i,j+1}-U_{i,j}}{h} - a_2(x_i,y_{j-1/2})\frac{U_{i,j}-U_{i,j-1}}{h}}{h} \\ &+b_1(x_i,y_j)\frac{U_{i+1,j}-U_{i-1,j}}{2h} + b_2(x_i,y_j)\frac{U_{i,j+1}-U_{i,j-1}}{2h} \\ &+c(x_i,y_j)U_{i,j}. \end{aligned}$$

This is still a five point difference scheme that is second-order consistent.

How to deal with nonaxiparallel domains?

When Ω has a curved boundary, a nonuniform mesh has to be used near $\partial\Omega$ to avoid a loss of accuracy. Let us introduce the following notation: let $h_{i+1} := x_{i+1} - x_i$, $h_i := x_i - x_{i-1}$, and let $\hbar_i := \frac{h_{i+1}+h_i}{2}$. We define

$$D_x^+ U_i := \frac{U_{i+1} - U_i}{\hbar_i}, \quad D_x^- U_i := \frac{U_i - U_{i-1}}{h_i}, \quad D_x^+ D_x^- U_i := \frac{1}{\hbar_i}\left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}\right).$$

Similarly, let $k_{j+1} := y_{j+1} - y_j$, $k_j := y_j - y_{j-1}$, $\hbar := \frac{k_{j+1}+k_j}{2}$, and

$$D_y^+ U_j := \frac{U_{j+1} - U_j}{\hbar_j}, \quad D_y^- U_j := \frac{U_j - U_{j-1}}{k_j}, \quad D_y^+ D_y^- U_j := \frac{1}{\hbar_j}\left(\frac{U_{j+1} - U_j}{k_{j+1}} - \frac{U_j - U_{j-1}}{k_j}\right).$$

Note that, whereas on a uniform mesh $D_x^- U_{i+1} = D_x^+ U_i$ and $D_y^- U_{j+1} = D_y^+ U_j$, on nonuniform meshes this is no longer the case. For the same reason, on a nonuniform mesh $D_x^+ D_x^- U_i \neq D_x^- D_x^+ U_i$ and $D_y^+ D_y^- U_j \neq D_y^- D_y^+ U_j$. On a general nonuniform mesh

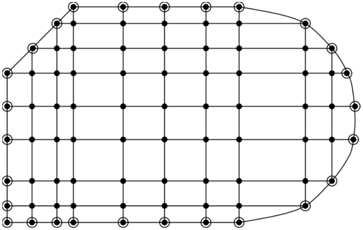
$$\overline{\Omega}_h := \{(x_i, y_j) \in \overline{\Omega} : x_{i+1} - x_i = h_{i+1},\ y_{j+1} - y_j = k_{j+1}\},$$

the Laplace operator Δ can be approximated by $D_x^+ D_x^- + D_y^+ D_y^-$.

Consider, e.g., the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω and the nonuniform mesh $\overline{\Omega}_h$ are depicted below:



$\bullet\ \Omega_h$; $\odot\ \Gamma_h$, $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$.

The FD approximation of this BVP is

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) = f(x_i, y_j) \qquad & \text{in } \Omega_h, \\ U_{i,j} = 0 \qquad & \text{on } \Gamma_h, \end{aligned}$$

or equivalently,

$$\begin{aligned} -\frac{1}{\hbar_i}\left(\frac{U_{i+1,j} - U_{i,j}}{h_{i+1}} - \frac{U_{i,j} - U_{i-1,j}}{h_i}\right) - \frac{1}{\bar{k}_j}\left(\frac{U_{i,j+1} - U_{i,j}}{k_{j+1}} - \frac{U_{i,j} - U_{i,j-1}}{k_j}\right) = f(x_i, y_j) \quad & \text{in } \Omega_h, \\ U_{i,j} = 0 \quad & \text{on } \Gamma_h. \end{aligned}$$

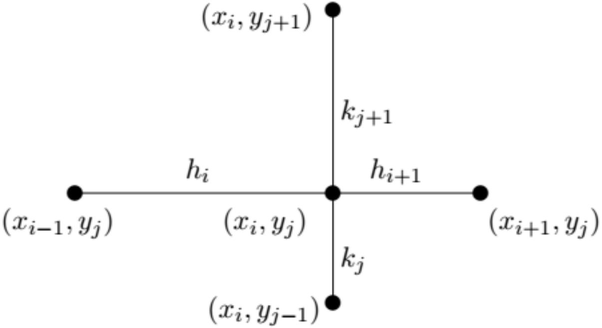


Figure: Five-point stencil on a nonuniform mesh.

7.3 The discrete maximum principle

The BVP and the FD scheme

For given $f \in C(\Omega)$, $g \in C(\partial\Omega)$, we consider the BVP

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega,$$

on a general nonaxiparallel domain $\Omega \subset \mathbb{R}^2$, and the FD scheme

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) &= f(x_i, y_j) \qquad && \text{in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) && \text{on } \Gamma_h, \end{aligned}$$

or equivalently,

$$\begin{aligned} -\frac{1}{\hbar_i}\left(\frac{U_{i+1,j} - U_{i,j}}{h_{i+1}} - \frac{U_{i,j} - U_{i-1,j}}{h_i}\right) - \frac{1}{\bar{k}_j}\left(\frac{U_{i,j+1} - U_{i,j}}{k_{j+1}} - \frac{U_{i,j} - U_{i,j-1}}{k_j}\right) &= f(x_i, y_j) \quad && \text{in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) && \text{on } \Gamma_h, \end{aligned}$$

where we consider a general nonuniform mesh

$$\overline{\Omega}_h = \{(x_i, y_j) \in \overline{\Omega} : x_{i+1} - x_i = h_{i+1},\ y_{j+1} - y_j = k_{j+1}\}.$$

Goal: show U satisfies a discrete counterpart of the max./min. principle.

Discrete maximum principle: Case $f < 0$ in Ω_h

Assume $f(x_i, y_j) < 0 \ \forall (x_i, y_j) \in \Omega_h$. Suppose that the maximum value of U is attained at an interior mesh point $(x_{i_0}, y_{j_0}) \in \Omega_h$. Rewriting the FD scheme, we see that for any $(x_i, y_j) \in \Omega_h$ we have

$$\left(\frac{1}{\hbar_i}\left(\frac{1}{h_{i+1}}+\frac{1}{h_i}\right)+\frac{1}{\hbar_j}\left(\frac{1}{k_{j+1}}+\frac{1}{k_j}\right)\right)U_{i,j} = \frac{U_{i+1,j}}{\hbar_i\, h_{i+1}}+\frac{U_{i-1,j}}{\hbar_i\, h_i}+\frac{U_{i,j+1}}{\hbar_j\, k_{j+1}}+\frac{U_{i,j-1}}{\hbar_j\, k_j}+f(x_i,y_j).$$

Therefore, as $U_{i_0\pm1,j_0} \le U_{i_0,j_0}$, $U_{i_0,j_0\pm1} \le U_{i_0,j_0}$, and $f(x_{i_0}, y_{j_0}) < 0$,

$$\left(\frac{1}{\hbar_{i_0}}\left(\frac{1}{h_{i_0+1}}+\frac{1}{h_{i_0}}\right)+\frac{1}{\hbar_{j_0}}\left(\frac{1}{k_{j_0+1}}+\frac{1}{k_{j_0}}\right)\right)U_{i_0,j_0} < \frac{U_{i_0,j_0}}{\hbar_{i_0}\, h_{i_0+1}}+\frac{U_{i_0,j_0}}{\hbar_{i_0}\, h_{i_0}}+\frac{U_{i_0,j_0}}{\hbar_{j_0}\, k_{j_0+1}}+\frac{U_{i_0,j_0}}{\hbar_{j_0}\, k_{j_0}},$$

a contradiction (LHS = RHS).

$\Longrightarrow$ If $f < 0$ in Ω_h, then the maximum value of U is attained on the boundary Γ_h of Ω_h, or equivalently,

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} U_{i,j},$$

which is called the **discrete maximum principle**.

Discrete maximum principle: Case $f \le 0$ in Ω_h

Now assume only $f(x_i, y_j) \le 0 \; \forall (x_i, y_j) \in \Omega_h$. We claim that the discrete maximum principle still holds in this case.

For $\varepsilon > 0$, define $V_{i,j} := U_{i,j} + \frac{\varepsilon}{4}(x_i^2 + y_j^2)$ for $(x_i, y_j) \in \overline{\Omega}_h$. Then,

$$-(D_x^+ D_x^- V_{i,j} + D_y^+ D_y^- V_{i,j}) = -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) - \varepsilon = f(x_i, y_j) - \varepsilon < 0$$

$\forall (x_i, y_j) \in \Omega_h$. Hence, $\max_{(x_i,y_j)\in\overline{\Omega}_h} V_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} V_{i,j}$. Then,

$$\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} \left[V_{i,j} - \frac{\varepsilon}{4}(x_i^2 + y_j^2) \right]$$

$$\ge \max_{(x,y)\in\Gamma_h} V_{i,j} - \frac{\varepsilon}{4} \max_{(x_i,y_j)\in\Gamma_h} (x_i^2 + y_j^2) = \max_{(x_i,y_j)\in\overline{\Omega}_h} V_{i,j} - \frac{\varepsilon}{4} \max_{(x_i,y_j)\in\Gamma_h} (x_i^2 + y_j^2)$$

$$\ge \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} - \frac{\varepsilon}{4} \max_{(x_i,y_j)\in\Gamma_h} (x_i^2 + y_j^2).$$

$\varepsilon \searrow 0$: $\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} \ge \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j}$. (Note also $\le$ trivially.)

$\Longrightarrow$ If $f \le 0$ in Ω_h, then the **discrete maximum principle** holds:

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} U_{i,j}.$$

Discrete minimum principle when $f \geq 0$ in Ω_h

Assuming $f(x_i, y_j) \geq 0 \; \forall (x_i, y_j) \in \Omega_h$, we can apply the discrete maximum principle to $-U$, which yields the **discrete minimum principle**:

$$\min_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \min_{(x_i,y_j)\in\Gamma_h} U_{i,j}.$$

7.4 Stability in the discrete maximum norm

Same set-up as before: The BVP and the FD scheme

For given $f \in C(\Omega)$, $g \in C(\partial\Omega)$, we consider the BVP

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega,$$

on a general nonaxiparallel domain $\Omega \subset \mathbb{R}^2$, and the FD scheme

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) &= f(x_i, y_j) \qquad &\text{in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) \qquad &\text{on } \Gamma_h, \end{aligned}$$

or equivalently,

$$\begin{aligned} -\frac{1}{\hbar_i}\left(\frac{U_{i+1,j} - U_{i,j}}{h_{i+1}} - \frac{U_{i,j} - U_{i-1,j}}{h_i}\right) - \frac{1}{\bar{k}_j}\left(\frac{U_{i,j+1} - U_{i,j}}{k_{j+1}} - \frac{U_{i,j} - U_{i,j-1}}{k_j}\right) &= f(x_i, y_j) \quad &\text{in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) \quad &\text{on } \Gamma_h, \end{aligned}$$

where we consider a general nonuniform mesh

$$\overline{\Omega}_h = \{(x_i, y_j) \in \overline{\Omega} \,:\, x_{i+1} - x_i = h_{i+1},\; y_{j+1} - y_j = k_{j+1}\}.$$

We have seen that U satisfies the discrete maximum principle (DMP):

$$f \leq 0 \quad \text{in } \Omega_h \qquad \Longrightarrow \qquad \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} U_{i,j}.$$

Existence & uniqueness of solns to FD scheme via DMP

Claim: The FD scheme has a unique solution U.

Proof: We denote the total number of mesh-points in Ω_h by M_h. Then,

- the FD scheme can be written as a linear system $AU = F$ where $U \in \mathbb{R}^{M_h}$ (containing the values $U_{i,j}$ such that $(x_i, y_j) \in \Omega_h$), $F \in \mathbb{R}^{M_h}$, and $A \in \mathbb{R}^{M_h \times M_h}$.
- the FD scheme has a unique soln iff A is invertible, i.e., iff the only solution to $AV = 0$ is $V = 0 \in \mathbb{R}^{M_h}$.

Equivalently, the FD scheme has a unique soln iff

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) &= 0 \qquad \text{in } \Omega_h, \\ U_{i,j} &= 0 \qquad \text{on } \Gamma_h \end{aligned} \tag{1}$$

only has the solution $U = 0$ (i.e., $U_{i,j} = 0 \ \forall (x_i, y_j) \in \overline{\Omega}_h$).

By discrete max. principle & discrete min. principle, for any soln U of (1), we have $\max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = 0$ and $\min_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = 0$, i.e., $U = 0$. □

Stability of FD scheme w.r.t. perturbation in bdry data

Consider mesh functions $U^{(1)}$ and $U^{(2)}$ satisfying

$$-(D_x^+ D_x^- U_{i,j}^{(1)} + D_y^+ D_y^- U_{i,j}^{(1)}) = f(x_i, y_j) \text{ in } \Omega_h, \quad U_{i,j}^{(1)} = g^{(1)}(x_i, y_j) \text{ on } \Gamma_h,$$

$$-(D_x^+ D_x^- U_{i,j}^{(2)} + D_y^+ D_y^- U_{i,j}^{(2)}) = f(x_i, y_j) \text{ in } \Omega_h, \quad U_{i,j}^{(2)} = g^{(2)}(x_i, y_j) \text{ on } \Gamma_h$$

for given $f, g^{(1)}, g^{(2)}$. Let $U := U^{(1)} - U^{(2)}$ and $g := g^{(1)} - g^{(2)}$. Then,

$$-(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) = 0 \text{ in } \Omega_h, \quad U_{i,j} = g(x_i, y_j) \text{ on } \Gamma_h.$$

By the discrete maximum principle, we have that

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} g(x_i, y_j) \le \max_{(x_i,y_j)\in\Gamma_h} |g(x_i, y_j)| =: M.$$

By the discrete minimum principle, we have that

$$\min_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \min_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \min_{(x_i,y_j)\in\Gamma_h} g(x_i, y_j) \ge \min_{(x_i,y_j)\in\Gamma_h} (-|g(x_i, y_j)|) = -M.$$

Together: $|U_{i,j}| \le M \ \forall (x_i, y_j) \in \overline{\Omega}_h$ and thus $\max_{(x_i,y_j)\in\overline{\Omega}_h} |U_{i,j}| \le M$.

Therefore, we have proved the following stability result:

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} |U_{i,j}^{(1)} - U_{i,j}^{(2)}| \le \max_{(x_i,y_j)\in\Gamma_h} |g^{(1)}(x_i, y_j) - g^{(2)}(x_i, y_j)|.$$

End of “Chapter 7: FD approximation of elliptic problems”.