MA4255 Numerical Methods in Differential Equations

Chapter 5: Function Spaces

5.1 Spaces of continuous functions

Multi-index notation

An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is called a **multi-index**.

- $|\alpha| := \alpha_1 + \cdots + \alpha_n$ is called the **length** of the multi-index α.
- We define

$$D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$$=: \partial^{|\alpha|}_{\underbrace{x_1 \ldots x_1}_{\alpha_1 \text{ times}} \ldots\ldots \underbrace{x_n \ldots x_n}_{\alpha_n \text{ times}}}.$$

Example. Let $u : \mathbb{R}^3 \to \mathbb{R}$, $u(x) := u(x_1, x_2, x_3) := x_1^3 x_2^3 x_3^3$. Then,

- For $\alpha := (1, 2, 3)$ have $D^\alpha u(x) = \partial^6_{x_1 x_2 x_2 x_3 x_3 x_3} u(x) = 108 x_1^2 x_2$.
- For $\alpha := (0, 1, 0)$ have $D^\alpha u(x) = \partial_{x_2} u(x) = 3 x_1^3 x_2^2 x_3^3$.
- For $\alpha := (2, 0, 0)$ have $D^\alpha u(x) = \partial^2_{x_1 x_1} u(x) = 6 x_1 x_2^3 x_3^3$.
- We have $\sum_{\substack{\alpha \in \mathbb{N}_0^3, \\ |\alpha| = 3}} D^\alpha u = \partial^3_{x_1 x_1 x_1} u + \partial^3_{x_1 x_1 x_2} u + \partial^3_{x_1 x_1 x_3} u + \partial^3_{x_1 x_2 x_2} u + \partial^3_{x_1 x_3 x_3} u + \partial^3_{x_2 x_2 x_2} u + \partial^3_{x_1 x_2 x_3} u + \partial^3_{x_2 x_2 x_3} u + \partial^3_{x_2 x_3 x_3} u + \partial^3_{x_3 x_3 x_3} u.$

The spaces $C^k(\Omega)$ and $C^k(\overline{\Omega})$

- Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}_0$. We define ("cts":="continuous")

$$C^k(\Omega) := \{u : \Omega \to \mathbb{R} |\, D^\alpha u \text{ is cts on } \Omega \text{ for any } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k\}.$$

- Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $k \in \mathbb{N}_0$. We define $C^k(\overline{\Omega})$ to be the set of all fcts u in $C^k(\Omega)$ for which $D^\alpha u$ can be extended from Ω to a cts fct on $\overline{\Omega}$ (the closure of Ω) for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.
- We write $C(\Omega) := C^0(\Omega)$ and $C(\overline{\Omega}) := C^0(\overline{\Omega})$.
- We write $C^\infty(\Omega) := \bigcap_{k=0}^\infty C^k(\Omega)$ and $C^\infty(\overline{\Omega}) := \bigcap_{k=0}^\infty C^k(\overline{\Omega})$.
- The space $C^k(\overline{\Omega})$ is equipped with the norm

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)| = \sum_{|\alpha| \leq k} \max_{x \in \overline{\Omega}} |D^\alpha u(x)| \quad \text{for } u \in C^k(\overline{\Omega}).$$

  - $\|u\|_{C(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)|$,
  - $\|u\|_{C^1(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)| + \sum_{j=1}^n \sup_{x \in \Omega} |\partial_{x_j} u(x)|$,
  - $\|u\|_{C^2(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)| + \sum_{j=1}^n \sup_{x \in \Omega} |\partial_{x_j} u(x)| + \sum_{i,j=1}^n \sup_{x \in \Omega} |\partial^2_{x_i x_j} u(x)|$.

Examples

- Let $\Omega := (0,1)$. Define $u : \Omega \to \mathbb{R}$, $u(x) := \frac{1}{x}$. Then,

$$u \in C^k(\Omega) \setminus C^k(\overline{\Omega}) \qquad \forall k \in \mathbb{N}_0.$$

- Let $\Omega := (-1,1)$. Define $u : \Omega \to \mathbb{R}$,

$$u(x) := \begin{cases} 0 & \text{, if } x \in (-1,0) \\ x^2 & \text{, if } x \in [0,1). \end{cases}$$

Then, $u \in C^1(\overline{\Omega})$ (in particular also $u \in C(\overline{\Omega})$, $u \in C^1(\Omega)$, $u \in C(\Omega)$), but $u \notin C^k(\Omega)\ \forall k \geq 2$ (in particular also $u \notin C^k(\overline{\Omega})\ \forall k \geq 2$). We have

$$\|u\|_{C(\bar{\Omega})} = \sup_{x\in\Omega} |u(x)| = 1,$$
$$\|u\|_{C^1(\overline{\Omega})} = \sup_{x\in\Omega} |u(x)| + \sup_{x\in\Omega} |u'(x)| = 1 + 2 = 3.$$

The spaces $C_c^k(\Omega)$

The **support** of a function $u \in C(\Omega)$, denoted supp(u), is defined as the closure in Ω of the set $\{x \in \Omega : u(x) \neq 0\}$, i.e.,

$$\text{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

Rk: supp(u) is smallest closed subset of Ω s.t. $u(x) = 0\ \forall x \in \Omega \backslash \text{supp}(u)$.

For $\Omega \subseteq \mathbb{R}^n$ open, and $k \in \mathbb{N}_0$, we define

$$\begin{aligned} C_c^k(\Omega) &:= \left\{ u \in C^k(\Omega) \,\middle|\, \text{supp}(u) \subset \Omega \text{ and supp}(u) \text{ is compact} \right\} \\ &= \left\{ u \in C^k(\Omega) \,\middle|\, \text{supp}(u) \subset \Omega \text{ and supp}(u) \text{ is bounded} \right\}, \end{aligned}$$

and write $C_c(\Omega) := C_c^0(\Omega)$ and $C_c^\infty(\Omega) := \bigcap_{k=0}^\infty C_c^k(\Omega)$.

Ex: Consider the fct $u : \mathbb{R}^n \to \mathbb{R}$ given by

$$u(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \qquad \left[\text{here, } |x| := \sqrt{x_1^2 + \cdots + x_n^2}. \right]$$

Then, supp$(u) = \{x \in \mathbb{R}^n : |x| \leq 1\}$. There holds $u \in C_c^\infty(\mathbb{R}^n)$.

5.2 Spaces of integrable functions

The spaces $L^p(\Omega)$

- Let $\Omega \subseteq \mathbb{R}^n$ be open, $p \in [1, \infty)$. We define $L^p(\Omega)$ to be the set of all (measurable) fcts $u : \Omega \to \mathbb{R}$ for which

$$\int_\Omega |u(x)|^p \, \mathrm{d}x < \infty.$$

Functions which are **equal almost everywhere (a.e.)** on Ω (i.e., equal, except on a set of measure zero) are identified with each other.

- The space $L^p(\Omega)$ is equipped with the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \qquad \text{for } u \in L^p(\Omega).$$

- Case $p = 2$: then, $\|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega |u(x)|^2 \, \mathrm{d}x}$. The space $L^2(\Omega)$ is equipped with the inner product

$$(u, v)_{L^2(\Omega)} := \int_\Omega u(x) v(x) \, \mathrm{d}x \qquad \text{for } u, v \in L^2(\Omega).$$

Then, $\|u\|_{L^2(\Omega)} = \sqrt{(u, u)_{L^2(\Omega)}}$.

Cauchy–Schwarz and triangle inequalities

Lemma (Cauchy–Schwarz inequality)

Let $u, v \in L^2(\Omega)$. *Then,* $\left|(u,v)_{L^2(\Omega)}\right| \leq \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$.

Proof: We have that

$$0 \leq \|tv+u\|^2_{L^2(\Omega)} = \|v\|^2_{L^2(\Omega)}t^2 + 2(u,v)_{L^2(\Omega)}t + \|u\|^2_{L^2(\Omega)} \quad \forall t \in \mathbb{R}.$$

$$\implies |2(u,v)_{L^2(\Omega)}|^2 - 4\|u\|^2_{L^2(\Omega)}\|v\|^2_{L^2(\Omega)} \leq 0. \qquad \square$$

Corollary (Triangle inequality)

Let $u, v \in L^2(\Omega)$. *Then,* $\|u+v\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}$.

Proof: Using the Cauchy–Schwarz inequality, we have that

$$\|u+v\|^2_{L^2(\Omega)} = \|u\|^2_{L^2(\Omega)} + 2(u,v)_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}$$
$$\leq \|u\|^2_{L^2(\Omega)} + 2\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} = (\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)})^2. \qquad \square$$

The space $L^2(\Omega)$ is a Hilbert space

The space $L^2(\Omega)$ equipped with the inner product

$$(u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\,\mathrm{d}x \qquad \text{for } u,v \in L^2(\Omega),$$

and the associated norm $\|u\|_{L^2(\Omega)} = \sqrt{(u,u)_{L^2(\Omega)}}$ is a Hilbert space.

Recall: a vector space X equipped with an inner product $(\cdot,\cdot)_X$ and associated norm $\|u\|_X := \sqrt{(u,u)_X}$ is called a **Hilbert space** iff for any Cauchy sequence $(u_m)_{m\in\mathbb{N}} \subset X$ ($\lim_{n,\,m\to\infty} \|u_n - u_m\|_X = 0$), there exists $u \in X$ s.t. $\lim_{m\to\infty} \|u_m - u\|_X = 0$.

5.3 Sobolev spaces

Motivation: An integration-by-parts formula

Observe: Let $u \in C^k(\Omega)$. Then, for any $v \in C_c^\infty(\Omega)$ we have that

$$\int_\Omega D^\alpha u(x)\, v(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha v(x)\,\mathrm{d}x \qquad \forall\, \alpha \,:\, |\alpha| \le k.$$

This follows from the fact that by the divergence theorem,

$$\begin{aligned}
\int_\Omega (\partial_{x_i} u(x))\, v(x)\,\mathrm{d}x &= -\int_\Omega u(x)\,\partial_{x_i} v(x)\,\mathrm{d}x + \int_\Omega \partial_{x_i}(u(x)v(x))\,\mathrm{d}x \\
&= -\int_\Omega u(x)\,\partial_{x_i} v(x)\,\mathrm{d}x + \int_\Omega \mathrm{div}(u(x)v(x)e_i)\,\mathrm{d}x \\
&= -\int_\Omega u(x)\,\partial_{x_i} v(x)\,\mathrm{d}x + \int_{\partial\Omega} u(x)v(x)e_i \cdot \nu\,\mathrm{d}s(x) \\
&= -\int_\Omega u(x)\,\partial_{x_i} v(x)\,\mathrm{d}x + \int_{\partial\Omega} u(x)v(x)\nu_i\,\mathrm{d}s(x),
\end{aligned}$$

where ν is the unit outward normal vector to the boundary $\partial\Omega$ of Ω. Note that the boundary integral vanishes for $v \in C_c^\infty(\Omega)$, i.e.,

$$\int_\Omega (\partial_{x_i} u(x))\, v(x)\,\mathrm{d}x = -\int_\Omega u(x)\,\partial_{x_i} v(x)\,\mathrm{d}x \quad \forall v \in C_c^\infty(\Omega).$$

Weak derivatives

Suppose that $u \in L^1_{\mathrm{loc}}(\Omega)$, that is, $u \in L^1(\omega)$ for each bounded open set ω with $\overline{\omega} \subset \Omega$. Suppose also that there exists a fct $w_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ s.t.

$$\int_\Omega w_\alpha(x)\, v(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha v(x)\, \mathrm{d}x \qquad \forall\, v \in C_c^\infty(\Omega).$$

Then we call w_α a **weak derivative** of u of order $|\alpha|$ and write

$$w_\alpha = D^\alpha u.$$

If u is a smooth function then its weak derivatives coincide with those in the classical (pointwise) sense. To simplify the notation, we use the letter D to denote both a classical and a weak derivative.

Example

Let $\Omega := \mathbb{R}$, and consider the function

$$u : \Omega \to \mathbb{R}, \quad u(x) := \max\{1 - |x|, 0\}.$$

Note that u is not differentiable in the classical sense. However, we will show that it has a weak derivative. Indeed, for any $v \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} u(x)\, v'(x)\, \mathrm{d}x &= \int_{-1}^{0} (1+x)\, v'(x)\, \mathrm{d}x + \int_{0}^{1} (1-x)\, v'(x)\, \mathrm{d}x \\ &= -\int_{-1}^{0} v(x)\, \mathrm{d}x + [(1+x)v(x)]_{x=-1}^{x=0} + \int_{0}^{1} v(x)\, \mathrm{d}x + [(1-x)v(x)]_{x=0}^{x=1} \\ &= -\left(\int_{-1}^{0} v(x)\, \mathrm{d}x - \int_{0}^{1} v(x)\, \mathrm{d}x\right) = -\int_{-\infty}^{\infty} w(x)\, v(x)\, \mathrm{d}x,\end{aligned}$$

where

$$w(x) = \begin{cases} 0, & x < -1, \\ 1, & x \in (-1, 0), \\ -1, & x \in (0, 1), \\ 0, & x > 1. \end{cases}$$

The fct w is the first weak derivative of u, and we write $u' = w$.

The Sobolev spaces $H^k(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}_0$. We define the **Sobolev space**

$$H^k(\Omega) := \{ u \in L^2(\Omega) \,\big|\, D^\alpha u \in L^2(\Omega) \;\; \forall \alpha : |\alpha| \leq k \},$$

where derivatives are understood in the weak sense. It is equipped with the norm $\| \cdot \|_{H^k(\Omega)}$ and inner product $(\cdot, \cdot)_{H^k(\Omega)}$ given by

$$\|u\|_{H^k(\Omega)} := \sqrt{\sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L^2(\Omega)}}, \quad (u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

for $u, v \in H^k(\Omega)$. The space $H^k(\Omega)$ is a Hilbert space. Note that

$$\|u\|_{H^k(\Omega)} = \sqrt{\sum_{j=0}^{k} |u|^2_{H^j(\Omega)}}, \quad \text{where} \quad |u|_{H^j(\Omega)} := \sqrt{\sum_{|\alpha| = j} \|D^\alpha u\|^2_{L^2(\Omega)}}$$

for $u \in H^k(\Omega)$. The map $u \mapsto |u|_{H^j(\Omega)}$ is called the $H^j(\Omega)$-seminorm.

Most frequently used Sobolev spaces: $H^1(\Omega)$ and $H^2(\Omega)$

Of particular importance are the Sobolev spaces

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) \,\middle|\, \partial_{x_j} u \in L^2(\Omega) \ \ \forall j \in \{1, \dots, n\} \right\},$$

$$H^2(\Omega) := \left\{ u \in L^2(\Omega) \,\middle|\, \partial_{x_j} u \in L^2(\Omega),\ \partial^2_{x_i x_j} u \in L^2(\Omega) \ \ \forall i, j \in \{1, \dots, n\} \right\}.$$

The $H^1(\Omega)$-norm $\|\cdot\|_{H^1(\Omega)}$ and the $H^1(\Omega)$-seminorm $|\cdot|_{H^1(\Omega)}$ are given by

$$\|u\|_{H^1(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L^2(\Omega)}}, \quad |u|_{H^1(\Omega)} := \sqrt{\sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L^2(\Omega)}}.$$

The $H^2(\Omega)$-norm $\|\cdot\|_{H^2(\Omega)}$ and the $H^2(\Omega)$-seminorm $|\cdot|_{H^2(\Omega)}$ are given by

$$\|u\|_{H^2(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L^2(\Omega)} + \sum_{i,j=1}^{n} \|\partial^2_{x_i x_j} u\|^2_{L^2(\Omega)}},$$

$$|u|_{H^2(\Omega)} := \sqrt{\sum_{i,j=1}^{n} \|\partial^2_{x_i x_j} u\|^2_{L^2(\Omega)}}.$$

The Sobolev space $H_0^1(\Omega)$ & Poincaré–Friedrichs inequality

Finally, we define a special Sobolev space:

$$H_0^1(\Omega) := \{ u \in H^1(\Omega) \,|\, u = 0 \text{ on } \partial\Omega \}.$$

We will use this space when considering a PDE that is coupled with the boundary condition $u = 0$ on $\partial\Omega$. The space $H_0^1(\Omega)$ is a Hilbert space, with the same norm and inner product as $H^1(\Omega)$.

Lemma (Poincaré–Friedrichs inequality)

Let $\Omega \subset \mathbb{R}^n$ be open and bounded (and assume $\partial\Omega$ is sufficiently smooth). Then, there exists a constant $c_\star > 0$, depending only on Ω, s.t.

$$\|u\|_{L^2(\Omega)}^2 \leq c_\star \sum_{i=1}^{n} \|\partial_{x_i} u\|_{L^2(\Omega)}^2 \qquad \forall u \in H_0^1(\Omega).$$

End of “Chapter 5: Function Spaces”.