MA4255 Numerical Methods in Differential Equations

Chapter 3: Linear multi-step methods (LMMs)

3.0 Introduction and definition

Introduction

Explicit RK methods are superior to, e.g., explicit Euler in terms of accuracy, but ...

... they are computationally more costly; RK methods require more evaluations of f than would seem necessary. E.g., the 4th-order accurate 4-stage explicit RK method from Ch.2 needs four evaluations of f per step.

For comparison, noting that

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x, y(x))\,\mathrm{d}x,$$

and using Simpson's rule $\int_a^b g(x)\mathrm{d}x \approx \frac{b-a}{6}(g(a) + 4g(\frac{a+b}{2}) + g(b))$:

$$y(x_{n+1}) \approx y(x_{n-1}) + \frac{1}{3}h\left[f(x_{n-1}, y(x_{n-1})) + 4f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right],$$

leads to the **Simpson rule method**

$$y_{n+1} = y_{n-1} + \frac{1}{3}h\left[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right].$$

Note: we need *two* preceding values, y_n and y_{n-1} to calculate y_{n+1}.

Linear multi-step methods (LMMs)

Given a sequence of equally spaced mesh points (x_n) with step size h, we consider the general **linear k-step method**

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k \in \mathbb{R}$ and we assume that $\alpha_k \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$ (i.e., α_0 and β_0 are not both equal to zero).

If $\beta_k = 0$, then y_{n+k} can be computed from the values of y_{n+j} and $f(x_{n+j}, y_{n+j})$ for $j \in \{0, \ldots, k-1\}$, and the method is called **explicit**. If $\beta_k \neq 0$, then the method is called **implicit**.

The linear k-step method is called *linear* because it involves only linear combinations of the $\{y_n\}$ and the $\{f(x_n, y_n)\}$.

Notation: $f_n := f(x_n, y_n)$. The general linear k-step method then reads

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}.$$

Examples of LMMs

- The method which we have derived from Simpson's rule,

$$y_{n+1} = y_{n-1} + \tfrac{1}{3}h\,[f_{n-1} + 4f_n + f_{n+1}],$$

  is an example of an implicit linear 2-step method.

- Explicit Euler: $y_{n+1} = y_n + hf_n$ is an explicit linear 1-step method.
- Implicit Euler: $y_{n+1} = y_n + hf_{n+1}$ is an implicit linear 1-step method.
- Trapezium rule method: $y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n)$ is an implicit linear 1-step method.
- The **four-step Adams–Bashforth method**

$$y_{n+4} = y_{n+3} + \frac{h}{24}\left(55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n\right)$$

  is an explicit linear 4-step method.

- The **four-step Adams–Moulton method**

$$y_{n+4} = y_{n+3} + \frac{h}{720}\left(251f_{n+4} + 646f_{n+3} - 264f_{n+2} + 106f_{n+1} - 19f_n\right)$$

  is an implicit linear 4-step method.

3.1 Construction of linear multi-step methods

Shift operator and forward/backward difference operator

Introduce the shift operator E, the inverse shift operator E^{-1}, the forward difference operator Δ_+, and the backward difference operator Δ_-, which map a sequence of real numbers to another sequence of real numbers, by

$$\begin{aligned}
E &: (u_n)_{n\in\mathbb{N}_0} = (u_0, u_1, u_2, \dots) \mapsto (u_{n+1})_{n\in\mathbb{N}_0} = (u_1, u_2, \dots),\\
E^{-1} &: (u_n)_{n\in\mathbb{N}_0} = (u_0, u_1, u_2, \dots) \mapsto (u_{n-1})_{n\in\mathbb{N}_0} = (0, u_0, u_1, \dots),\\
\Delta_+ &: (u_n)_{n\in\mathbb{N}_0} = (u_0, u_1, u_2, \dots) \mapsto (u_{n+1} - u_n)_{n\in\mathbb{N}_0} = (u_1 - u_0, u_2 - u_1, \dots),\\
\Delta_- &: (u_n)_{n\in\mathbb{N}_0} = (u_0, u_1, u_2, \dots) \mapsto (u_n - u_{n-1})_{n\in\mathbb{N}_0} = (u_0, u_1 - u_0, u_2 - u_1, \dots)
\end{aligned}$$

$(u_{-1} := 0.)$ Example: For $u := (u_n)_{n\in\mathbb{N}_0} := (1, 3, 5, 7, \dots)$, we have

$$Eu = (3, 5, 7, \dots), \quad \Delta_+ u = (3-1, 5-3, 7-5, \dots) = (2, 2, 2, \dots),$$
$$E^{-1}u = (0, 1, 3, \dots), \quad \Delta_- u = (\quad 1, 3-1, 5-3, \dots) = (1, 2, 2, \dots).$$

Note for any $u = (u_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$, we have $E(E^{-1}u) = u$, and

$$\Delta_+ u = Eu - u = E(\Delta_- u), \qquad \Delta_- u = u - E^{-1}u, \qquad E(u - \Delta_- u) = u.$$

Writing $I : u \mapsto u$ for the identity operator, we find $E \circ E^{-1} = I$ and

$$\Delta_+ = E - I = E\Delta_-, \qquad \Delta_- = I - E^{-1}, \qquad E \circ (I - \Delta_-) = I.$$

Notation: For a fct $u : \mathbb{R} \to \mathbb{R}$ whose derivative exists and is integrable on $[x_0, x_n]$ for each $n \in \mathbb{N}_0$, we define $u_n := u(x_n)$ where $x_n = x_0 + nh$ for $n \in \mathbb{N}_0$, and call the resulting sequence

$$(u_n)_{n\in\mathbb{N}_0} = (u_0, u_1, u_2, \dots) = (u(x_0), u(x_1), u(x_2), \dots)$$

again u. (will be clear from context if we mean the fct or the sequence.)

For $s \in \mathbb{N}_0$, note that $E^s u = (u(x_s), u(x_{s+1}), \dots)$. Letting $D := \frac{\mathrm{d}}{\mathrm{d}x}$, we have using Taylor expansion that

$$[E^s u]_n = u(x_{s+n}) = u(x_n + sh) = \sum_{k=0}^{\infty} \frac{(sh)^k}{k!} D^k u(x_n) = [\mathrm{e}^{shD} u]_n.$$

$\implies$ Formally, $E^s = e^{shD}$ and thus, $hD = \ln(E) = -\ln(I - \Delta_-)$ (recall that $\Delta_- = I - E^{-1}$). Using Taylor expansion,

$$hu'(x_n) = \left[\left(\Delta_- + \frac{1}{2}\Delta_-^2 + \frac{1}{3}\Delta_-^3 + \cdots\right) u\right]_n.$$

The BDF methods

Recall: We have obtained

$$\left[\left(\Delta_- + \frac{1}{2}\Delta_-^2 + \frac{1}{3}\Delta_-^3 + \cdots\right) u\right]_n = hu'(x_n).$$

Now let $u(x) = y(x)$ where y is the solution of the IVP. Then,

$$\left[\left(\Delta_- + \frac{1}{2}\Delta_-^2 + \frac{1}{3}\Delta_-^3 + \cdots\right) y\right]_n = hf(x_n, y(x_n)).$$

By truncating the series on the left, we find

$$\begin{aligned}
y(x_n) - y(x_{n-1}) &\approx hf(x_n, y(x_n)), && (n \geq 1) \\
\frac{3}{2}y(x_n) - 2y(x_{n-1}) + \frac{1}{2}y(x_{n-2}) &\approx hf(x_n, y(x_n)), && (n \geq 2) \\
\frac{11}{6}y(x_n) - 3y(x_{n-1}) + \frac{3}{2}y(x_{n-2}) - \frac{1}{3}y(x_{n-3}) &\approx hf(x_n, y(x_n)), && (n \geq 3)
\end{aligned}$$

etc. This leads to the **backward differentiation formulae (BDF)**

$$\begin{aligned}
y_n - y_{n-1} &= hf_n, && (n \geq 1) \\
\frac{3}{2}y_n - 2y_{n-1} + \frac{1}{2}y_{n-2} &= hf_n, && (n \geq 2) \\
\frac{11}{6}y_n - 3y_{n-1} + \frac{3}{2}y_{n-2} - \frac{1}{3}y_{n-3} &= hf_n. && (n \geq 3)
\end{aligned}$$

Constructing further methods via same idea

Similarly, using $E^{-1} = I - \Delta_-$ and $hD = -\ln(I - \Delta_-)$, we find

$$-(I - \Delta_-)\ln(I - \Delta_-) = E^{-1}(hD),$$

and therefore

$$\left[\left(\Delta_- - \frac{1}{2}\Delta_-^2 - \frac{1}{6}\Delta_-^3 + \cdots\right) u\right]_{n+1} = hu'(x_n).$$

Letting $u(x) = y(x)$ where y is the soln of the IVP, and noting $y'(x) = f(x, y(x))$, truncations of the infinite series yield

$$\begin{aligned}
y(x_{n+1}) - y(x_n) &\approx hf(x_n, y(x_n)), \\
\frac{1}{2}y(x_{n+1}) - \frac{1}{2}y(x_{n-1}) &\approx hf(x_n, y(x_n)), \quad (n \geq 1) \\
\frac{1}{3}y(x_{n+1}) + \frac{1}{2}y(x_n) - y(x_{n-1}) + \frac{1}{6}y(x_{n-2}) &\approx hf(x_n, y(x_n)), \quad (n \geq 2)
\end{aligned}$$

etc. Replacing $y(x_n)$ by y_n, $f(x_n, y(x_n))$ by f_n, and $\approx$ by $=$ leads to LMMs. The first is explicit Euler, the 2nd is called explicit midpoint rule.

Adams–Moulton and Adams–Bashforth methods

Further methods can be created using a similar methodology. Without going into detail, one can show that

$$y(x_{n+1}) - y(x_n) \approx h\left[\left(I - \frac{1}{2}\Delta_- - \frac{1}{12}\Delta_-^2 - \frac{1}{24}\Delta_-^3 - \frac{19}{720}\Delta_-^4 - \cdots\right)y'\right]_{n+1} \tag{1}$$

and

$$y(x_{n+1}) - y(x_n) \approx h\left[\left(I + \frac{1}{2}\Delta_- + \frac{5}{12}\Delta_-^2 + \frac{3}{8}\Delta_-^3 + \frac{251}{720}\Delta_-^4 + \cdots\right)y'\right]_{n}. \tag{2}$$

Using $y'(x) = f(x, y(x))$, truncations of (1) yield the family of **Adams–Moulton methods**, while truncations of (2) yield the family of **Adams–Bashforth methods**.

3.2 Zero-stability

Zero-stability

Recall: General linear k-step method:

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = h(\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \cdots + \beta_0 f_n).$$

Observe: need k starting values $y_0, \ldots, y_{k-1}$ to apply this method. We get $y_0 = y(x_0)$ from i.c., but how to get $y_1, \ldots, y_{k-1}$?

$\Longrightarrow$ have to be computed by other means: e.g., by using a RK method.

The starting values contain numerical errors which will affect y_n for $n \geq k$.
Q: Is the method stable w.r.t. small perturbations in starting conditions?

Definition (Zero-stability)

A linear k-step method for the ODE $y'(x) = f(x, y(x))$ is called **zero-stable** if $\exists\ K > 0$ s.t., for any two sequences (y_n) and $(\hat{y}_n)$, which have been generated by the same formulae but with different initial data $y_0, \ldots, y_{k-1}$ and $\hat{y}_0, \ldots, \hat{y}_{k-1}$, respectively, we have

$$|y_n - \hat{y}_n| \leq K \max\{|y_0 - \hat{y}_0|, \ldots, |y_{k-1} - \hat{y}_{k-1}|\}$$

for $n \in \{0, \ldots, N\}$, and as h tends to 0.

Definition (Zero-stability)

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$$|y_n - \hat{y}_n| \leq K \max\{|y_0 - \hat{y}_0|, \dots, |y_{k-1} - \hat{y}_{k-1}|\}$$

for $n \in \{0, \dots, N\}$, and as h tends to 0.

Some comments:

- Why is it called zero-stability?
  $\Longrightarrow$ whether or not a method is zero-stable can be determined from its behavior when applied to the ODE $y'(x) = 0$ (here, $f \equiv 0$).
- This definition seems difficult to check ...
  $\Longrightarrow$ there is an algebraic equivalent of zero-stability, known as the *root condition*, which will simplify this task.

The root condition

Given the linear k-step method $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$, define

- its **first characteristic polynomial**

$$\rho : \mathbb{C} \to \mathbb{C}, \qquad \rho(z) := \sum_{j=0}^{k} \alpha_j z^j,$$

- and its **second characteristic polynomial**

$$\sigma : \mathbb{C} \to \mathbb{C}, \qquad \sigma(z) := \sum_{j=0}^{k} \beta_j z^j.$$

Theorem (Equivalence of zero-stability and root condition)

A LMM is zero-stable for any ODE of the form $y'(x) = f(x, y(x))$ where f satisfies the Lipschitz condition, iff the **root condition** *is satisfied, i.e., all zeros of ρ lie inside $\bar{D}_1(0)$, with any which lie on $\partial D_1(0)$ being simple.*

Notation: For $r \in (0, \infty)$, $a \in \mathbb{C}$, we write $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, $\bar{D}_r(a) := \{z \in \mathbb{C} : |z - a| \leq r\}$, and $\partial D_r(a) := \{z \in \mathbb{C} : |z - a| = r\}$.

Proof that root condition is necessary for zero-stability

Suppose root condition is violated. Goal: show method is not zero-stable.
Apply the linear k-step method to the ODE $y'(x) = 0$ (i.e., $f \equiv 0$):

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = 0.$$

Denote distinct zeros of ρ by $z_1, \ldots, z_S$ with multiplicities $m_1, \ldots, m_S$.
The general soln of this k-th order linear difference equation has the form

$$y_n = \sum_{s=1}^{S} p_s(n) z_s^n,$$

where $p_s(\cdot)$ is a polynomial of degree $m_s - 1$.

If $|z_s| > 1$, then $\exists$ starting values for which the corresponding solns grow like $|z_s|^n$. If $|z_s| = 1$ and $m_s > 1$, then $\exists$ solns growing like $n^{m_s - 1}$.
$\implies \exists$ solns that grow unbounded as $n \to \infty$, i.e. as $h \to 0$ with nh fixed.

Considering starting data $y_0, \ldots, y_{k-1}$ which give rise to such an unbounded solution (y_n), and starting data $\hat{y}_0 = \hat{y}_1 = \cdots = \hat{y}_{k-1} = 0$ for which $\hat{y}_n = 0$ for all n, we see that zero-stability cannot hold. $\square$

Some examples

- Explicit Euler: $y_{n+1} - y_n = hf_n$.

Here, $\rho(z) = z - 1$ which has a simple root at $z = 1$. $\Longrightarrow$ zero-stable.

- Implicit Euler: $y_{n+1} - y_n = hf_{n+1}$.

Again, $\rho(z) = z - 1 \Longrightarrow$ zero-stable.

- Trapezium rule method: $y_{n+1} - y_n = h(\frac{1}{2}f_{n+1} + \frac{1}{2}f_n)$.

Again, $\rho(z) = z - 1 \Longrightarrow$ zero-stable.

- 4-step Adams–Bashforth method:

$y_{n+4} - y_{n+3} = h\left(\frac{55}{24}f_{n+3} - \frac{59}{24}f_{n+2} + \frac{37}{24}f_{n+1} - \frac{9}{24}f_n\right)$.
Here, $\rho(z) = z^4 - z^3 = z^3(z-1)$ which has the root $z_1 = 0$ with multiplicity 3, and the root $z_2 = 1$ with multiplicity 1. $\Longrightarrow$ zero-stable.

- Consider the three-step (sixth-order accurate) LMM

$$11y_{n+3} + 27y_{n+2} - 27y_{n+1} - 11y_n = h\left(3f_{n+3} + 27f_{n+2} + 27f_{n+1} + 3f_n\right).$$

Here, $\rho(z) = 11z^3 + 27z^2 - 27z - 11$ with roots $z_1 = 1$, $z_2 = -\frac{19-4\sqrt{15}}{11}$, $z_3 = -\frac{19+4\sqrt{15}}{11}$. Note $|z_3| = \frac{19+4\sqrt{15}}{11} > 1 \Longrightarrow$ not zero-stable.

3.3 Consistency

Consistency error of a LMM

Consider a LMM $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ with $\alpha_k \neq 0 \neq \alpha_0^2 + \beta_0^2$. Suppose $\sigma(1) = \sum_{j=0}^{k} \beta_j \neq 0$ (we see later that this holds for any convergent LMM). Introduce the **consistency error**

$$T_n := \frac{\sum_{j=0}^{k} [\alpha_j y(x_{n+j}) - h\beta_j y'(x_{n+j})]}{h \sum_{j=0}^{k} \beta_j},$$

where y is a soln to the ODE $y'(x) = f(x, y(x))$.

As for one-step methods, the consistency error can be thought of as the residual obtained by inserting the true soln, and scaling this appropriately.

Definition (Consistent LMM)

The numerical scheme is said to be **consistent** with the ODE if the consistency error is such that $\forall \varepsilon > 0 \; \exists h_\varepsilon > 0$ s.t. $|T_n| < \varepsilon$ for all $h \in (0, h_\varepsilon)$ and for any $(k+1)$ points $(x_n, y(x_n)), \ldots, (x_{n+k}, y(x_{n+k}))$ on any solution curve in R of the IVP.

Order of accuracy of a LMM

Definition (Order of accuracy)

The LMM is said to have **order of accuracy** p (or **order of consistency** p) if $p \in \mathbb{N}$ is the largest natural number s.t. for any sufficiently smooth solution curve in R of the IVP $y'(x) = f(x, y(x))$, $y(x_0) = y_0$, we have

$$|T_n| = \mathcal{O}(h^p),$$

i.e., $\exists h_0, K > 0$ s.t. $|T_n| \leq K h^p$ for all $h \in (0, h_0)$, for any $(k+1)$ points $(x_n, y(x_n)), \ldots, (x_{n+k}, y(x_{n+k}))$ on the solution curve.

Goal: Find conditions on the coefficients α_j, β_j of the LMM from which we can easily see the order of accuracy.

Taylor expansion for the consistency error

Let us expand the consistency error in powers of h:

$$\begin{aligned}
\sigma(1)\, T_n &= \frac{1}{h}\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh) - h\beta_j y'(x_n+jh)\right] \\
&= \frac{1}{h}\sum_{j=0}^{k}\left[\alpha_j \sum_{i=0}^{\infty}\frac{j^i h^i}{i!} y^{(i)}(x_n) - h\beta_j \sum_{i=0}^{\infty}\frac{j^i h^i}{i!} y^{(i+1)}(x_n)\right] \\
&= \sum_{j=0}^{k}\left[\frac{1}{h}\alpha_j y(x_n) + \alpha_j \sum_{i=0}^{\infty}\frac{j^{i+1} h^i}{(i+1)!} y^{(i+1)}(x_n) - \beta_j \sum_{i=0}^{\infty}\frac{j^i h^i}{i!} y^{(i+1)}(x_n)\right] \\
&= \frac{1}{h}\sum_{j=0}^{k}\alpha_j y(x_n) + \sum_{i=0}^{\infty} h^i \left(\sum_{j=0}^{k}\frac{j^{i+1}}{(i+1)!}\alpha_j - \sum_{j=0}^{k}\frac{j^i}{i!}\beta_j\right) y^{(i+1)}(x_n) \\
&= \frac{1}{h} C_0 y(x_n) + \sum_{i=0}^{\infty} h^i C_{i+1} y^{(i+1)}(x_n)
\end{aligned}$$

where $C_0 := \sum_{j=0}^{k}\alpha_j$ and $C_q := \sum_{j=0}^{k}\frac{j^q}{q!}\alpha_j - \sum_{j=0}^{k}\frac{j^{q-1}}{(q-1)!}\beta_j$ for $q \in \mathbb{N}$.

Order conditions

We have obtained that

$$T_n = \frac{1}{h}\frac{C_0}{\sigma(1)}y(x_n) + \sum_{i=0}^{\infty} h^i \frac{C_{i+1}}{\sigma(1)} y^{(i+1)}(x_n),$$

where $C_0 := \sum_{j=0}^{k} \alpha_j$ and $C_q := \sum_{j=0}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=0}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j$ for $q \in \mathbb{N}$.

- The method is consistent iff $C_0 = C_1 = 0$, i.e.,

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1) \neq 0.$$

- The method is of order of accuracy p iff

$$C_0 = C_1 = \cdots = C_p = 0 \quad \text{and} \quad C_{p+1} \neq 0.$$

In this case,

$$T_n = h^p \frac{C_{p+1}}{\sigma(1)} y^{(p+1)}(x_n) + \mathcal{O}(h^{p+1});$$

the number $C_{p+1} \neq 0$ is then called the **error constant** of the method.

Equivalent formulas for the constants C_j

The constants $C_0, C_1, \dots \in \mathbb{R}$ given by

$$C_0 := \sum_{j=0}^{k} \alpha_j, \qquad C_q := \sum_{j=0}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=0}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j \quad \text{for} \quad q \in \mathbb{N}$$

can alternatively be computed as follows:

$$\begin{aligned}
C_0 &= \rho(1),\\
C_1 &= \rho'(1) - \sigma(1),\\
2C_2 &= \rho'(1) - 2\sigma'(1) + \rho''(1),\\
6C_3 &= \rho'(1) - 3\sigma'(1) + 3\rho''(1) - 3\sigma''(1) + \rho'''(1),\\
24C_4 &= \rho'(1) - 4\sigma'(1) + 7\rho''(1) - 12\sigma''(1) + 6\rho'''(1) - 4\sigma'''(1) + \rho^{(4)}(1),\\
120C_5 &= \rho'(1) - 5\sigma'(1) + 15\rho''(1) - 35\sigma''(1) + 25\rho'''(1) - 30\sigma'''(1) + 10\rho^{(4)}(1) - 5\sigma^{(4)}(1) + \rho^{(5)}(1),\\
&\vdots\\
q!C_q &= \sum_{j=1}^{q-1} \Big(S(q,j)\rho^{(j)}(1) - qS(q-1,j)\sigma^{(j)}(1) \Big) + \rho^{(q)}(1), \qquad q \in \mathbb{N}_{\geq 2},
\end{aligned}$$

where $S(q,j) := \frac{1}{j!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}(j-i)^q$ (*Stirling numbers of 2nd kind*).

Example

Task: Construct an implicit linear two-step method of maximum order of accuracy. Determine the order of accuracy and the error constant of the method.

Taking $\alpha_0 = a$ as parameter, the method has the form

$$y_{n+2} + \alpha_1 y_{n+1} + a y_n = h(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n),$$

with $\beta_2 \neq 0$ and $a^2 + \beta_0^2 \neq 0$. Here, $\alpha_2 = 1$, $\alpha_0 = a$. We have

$$\rho(z) = z^2 + \alpha_1 z + a, \qquad \sigma(z) = \beta_2 z^2 + \beta_1 z + \beta_0.$$

Assume $\sigma(1) = \beta_0 + \beta_1 + \beta_2 \neq 0$. We have to determine four unknowns: α_1, β_2, β_1, β_0, so we require four equations; demanding that

$$\begin{aligned}
C_0 &= \rho(1) &&= 1 + a + \alpha_1 &&= 0,\\
C_1 &= \rho'(1) - \sigma(1) &&= 2 + \alpha_1 - \beta_0 - \beta_1 - \beta_2 &&= 0,\\
2\,C_2 &= \rho'(1) - 2\sigma'(1) + \rho''(1) &&= 4 + \alpha_1 - 2\beta_1 - 4\beta_2 &&= 0,\\
6\,C_3 &= \rho'(1) - 3\sigma'(1) + 3\rho''(1) - 3\sigma''(1) + \rho'''(1) &&= 8 + \alpha_1 - 3\beta_1 - 12\beta_2 &&= 0.
\end{aligned}$$

$$\Longrightarrow \alpha_1 = -(1+a),\ \beta_0 = -\tfrac{1}{12}(1+5a),\ \beta_1 = \tfrac{2}{3}(1-a),\ \beta_2 = \tfrac{1}{12}(5+a).$$

We have obtained $\alpha_1 = -1 - a$, $\beta_0 = -\frac{1}{12}(1+5a)$, $\beta_1 = \frac{2}{3}(1-a)$, $\beta_2 = \frac{1}{12}(5+a)$, and the resulting method is

$$y_{n+2} - (1+a)y_{n+1} + ay_n = \frac{h}{12}\left((5+a)f_{n+2} + 8(1-a)f_{n+1} - (1+5a)f_n\right).$$

Note $\sigma(1) = \beta_0 + \beta_1 + \beta_2 = 1 - a \neq 0$ iff $a \neq 1$.

Now compute C_4 and C_5 which gives

$$C_4 = -\frac{1+a}{24}, \qquad C_5 = -\frac{17+13a}{360}.$$

- If $a \notin \{-1, 1\}$, then $C_4 \neq 0$, and the method is third-order accurate and the error constant is $C_4 = -\frac{1}{24}(1+a)$.

- If $a = -1$, then $C_4 = 0$ and $C_5 \neq 0$, and the method is fourth-order accurate and the error constant is $C_5 = -\frac{1}{90}$. The method in this case is the Simpson rule method

$$y_{n+2} - y_n = \frac{h}{3}\left(f_{n+2} + 4f_{n+1} + f_n\right).$$

3.4 Convergence

What is a convergent LMM?

Motivation: Zero-stability and consistency are of great theoretical importance, but what matters most from the practical point of view is that the computed approximations y_n are close to the values of the true solution $y(x_n)$, and that the global error $e_n = y(x_n) - y_n$ decays when the step size h is reduced.

Definition (Convergent LMM)

The LMM $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ is said to be **convergent** if, for all IVPs $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ subject to the hypotheses of Picard's thm, we have

$$\lim_{\substack{h \to 0 \\ nh = x - x_0}} y_n = y(x)$$

for all $x \in [x_0, X_M]$ and for all solutions $\{y_n\}_{n=0}^{N}$ of the difference equation (from the LMM) with **consistent starting conditions**, i.e. with starting conds $y_0 = \eta_0(h)$, $y_1 = \eta_1(h)$, $\ldots$, $y_{k-1} = \eta_{k-1}(h)$, for which $\lim_{h \to 0} \eta_s(h) = y_0$ for $s \in \{0, \ldots, k-1\}$.

The main result on convergence: Dahlquist's theorem

We are going to prove the following result:

Theorem (Necessary conditions for convergence)

A convergent LMM must be consistent and zero-stable.

It can actually be shown that for a consistent LMM, zero-stability is necessary and sufficient for the convergence of the LMM. This is the famous Dahlquist Theorem:

Theorem (Dahlquist)

For a LMM that is consistent with the ODE $y'(x) = f(x, y(x))$ where f is assumed to satisfy a Lipschitz condition, and starting with consistent initial data, zero-stability is necessary and sufficient for convergence. Moreover if the solution y has continuous derivatives of order $(p+1)$ and consistency error $\mathcal{O}(h^p)$, then the global error $e_n = y(x_n) - y_n$ is also $\mathcal{O}(h^p)$, i.e. the method is p-th order convergent.

Proof that Convergence $\Longrightarrow$ Zero-stability

Suppose the LMM $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ is convergent. Apply to IVP $y'(x) = 0$, $y(0) = 0$, on $[0, X_M]$, $X_M > 0$ (note true soln: $y \equiv 0$):

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = 0. \tag{3}$$

Since method is convergent, have $\lim_{\substack{h\to 0\\ nh=x}} y_n = 0 \ \forall x \in [0, X_M]$, for all solns of (3) with $y_s = \eta_s(h)$, $\lim_{h\to 0} \eta_s(h) = 0$, $s \in \{0, \ldots, k-1\}$ $(*)$.

Let $z = r\mathrm{e}^{i\phi}$ with $r \geq 0$, $\phi \in [0, 2\pi)$ be a root of ρ. Then,

$$y_n = hr^n \cos(n\phi)$$

defines a solution to (3) satisfying $(*)$. Observe that if $\phi \notin \{0, \pi\}$, then

$$\frac{y_n^2 - y_{n+1}y_{n-1}}{\sin^2(\phi)} = h^2 r^{2n} \frac{\cos^2(n\phi) - \cos((n+1)\phi)\cos((n-1)\phi)}{\sin^2(\phi)} = h^2 r^{2n}.$$

Since the left-hand side converges to 0 as $h \to 0$, $n \to \infty$, $nh = x$, find $\lim_{n\to\infty} \left(\frac{x}{n}\right)^2 r^{2n} = 0 \ \forall x \in [0, X_M]$. $\Longrightarrow$ $r \in [0, 1]$, i.e., $z \in \bar{D}_1(0)$.

Remains to prove that any root of ρ that lies on $\partial D_1(0)$ is simple.

Assume, instead, that $z = re^{i\phi}$, is a multiple root of ρ, with $|z| = r = 1$ and $\phi \in [0, 2\pi)$. Then,

$$y_n = \sqrt{h}\, n \cos(n\phi)$$

defines a solution to (3). This satisfies $(*)$ as for any $s \in \{0, \ldots, k-1\}$,

$$|\eta_s(h)| = |y_s| \leq \sqrt{h}\, s \leq \sqrt{h}(k-1) \to 0 \quad \text{as} \quad h \to 0.$$

If $\phi \in \{0, \pi\}$, using $nh = x$ find $|y_n| = \sqrt{x}\sqrt{n}$ and hence, $\lim_{n\to\infty, nh=x} |y_n| = \infty$ when $x \neq 0$, contradicting convergence (recall $y \equiv 0$).

If $\phi \notin \{0, \pi\}$, then

$$\frac{z_n^2 - z_{n+1}z_{n-1}}{\sin^2(\phi)} = 1,$$

where $z_n = \frac{1}{n\sqrt{h}} y_n = \frac{\sqrt{h}}{x} y_n$. As z_n converges to 0 as $h \to 0$, $n \to \infty$, $nh = x$, it follows that the left-hand side converges to 0 as $h \to 0$, $n \to \infty$, $nh = x$, a contradiction. □

Proof that Convergence $\Longrightarrow$ Consistency

Suppose the LMM $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ is convergent.

- First show that $C_0 = 0$: Consider the IVP

$$y'(x) = 0, \quad x \in [0, X_M], \qquad y(0) = 1$$

with true soln $y \equiv 1$. Applying the LMM to this IVP gives

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = 0. \tag{4}$$

Take "exact" starting values $y_s = 1$, $s \in \{0, \ldots, k-1\}$. As method is convergent, have $\lim_{\substack{h \to 0 \\ nh = x}} y_n = 1$. Since here, y_n is indep. of h, we find

$$\lim_{n \to \infty} y_n = 1.$$

Taking $n \to \infty$ in (4), we find $C_0 = \rho(1) = \sum_{j=0}^{k} \alpha_j = 0$.

- Now show that $C_1 = 0$: Apply LMM to IVP $y'(x) = 1$, $y(0) = 0$, on $[0, X_M]$, $X_M > 0$ (note true soln $y(x) = x$):

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j, \tag{5}$$

where $X_M = Nh$ and $n \in \{0, \ldots, N-k\}$.

For a convergent method any soln of (5) satisfying $\lim_{h\to 0} \eta_s(h) = 0$ $(*)$, where $y_s = \eta_s(h)$, $s \in \{0, \ldots k-1\}$, must also satisfy $\lim_{\substack{h\to 0\\ nh=x}} y_n = x$.

Since zero-stability is necessary for convergence, we know ρ does not have a multiple root on $\partial D_1(0)$; therefore $\rho'(1) = \sum_{j=1}^{k} j\alpha_j \neq 0$.

Let $\{y_n\}_{n=0}^{N}$ defined by $y_n = Knh$, where $K = \frac{\sigma(1)}{\rho'(1)}$ (note $C_1 = 0 \Leftrightarrow K = 1$). This satisfies $(*)$ for $s \in \{0, \ldots, k-1\}$, and is a soln of (5) as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = hK \sum_{j=0}^{k} \alpha_j (n+j) = KnhC_0 + Kh\rho'(1) = h\sigma(1).$$

$$\Longrightarrow x = \lim_{\substack{h\to 0\\ nh=x}} y_n = \lim_{\substack{h\to 0\\ nh=x}} Knh = Kx \;\forall x \in [0, X_M] \Longrightarrow K = 1. \quad \square$$

3.5 Maximum order of accuracy of a zero-stable linear multi-step method

Highest achievable order of a linear k-step method

Recall: Linear k-step method: $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$.

For consistency, need $C_0 = \rho(1) = 0$, $C_1 = \rho'(1) - \sigma(1) = 0$, $\sigma(1) \neq 0$.

Method has $2k+2$ coefficients: α_j, β_j, $j \in \{0, \ldots, k\}$, of which α_k is set to 1 by normalization.

- $2k+1$ free parameters if method is implicit,
- $2k$ free parameters if the method is explicit ($\beta_k = 0$).

We find that we can achieve

- $C_0 = 0, C_1 = 0, \ldots, C_{2k} = 0$ ($2k+1$ eqns) if method is implicit,
- $C_0 = 0, C_1 = 0, \ldots, C_{2k-1} = 0$ ($2k$ eqns) if method is explicit,

and we cannot impose more constraints.

$\Longrightarrow$ Maximum order: $p = 2k$ if implicit, and $p = 2k-1$ if explicit.

Highest achievable order of a zero-stable LMM

Bad news: For $k \geq 3$, k-step LMMs of maximum order ($2k$ if implicit, $2k-1$ if explicit) are not zero-stable $\implies$ should not be used in practice.

Theorem (Upper bound on order of accuracy of zero-stable LMMs)

There is no zero-stable linear k-step method whose order of accuracy exceeds $k+1$ if k is odd or $k+2$ if k is even.

Definition (Optimal method)

A zero-stable linear k-step method of order of accuracy $k+2$ is called an **optimal method**.

Rk: For an optimal LMM, all roots of ρ lie on $\partial D_1(0)$.

Ex.: Task: Find a zero-stable LMM which is of max. order and optimal.

Note k must be even (as otherwise, order $\leq k+1$ and thus, not optimal).
$\implies$ Want zero-stable method with k even, order $p = 2k = k+2$.
$\implies$ Want fourth-order accurate zero-stable 2-step method.
$\implies$ Only such method is the Simpson rule method.

3.6 Absolute stability of linear multi-step methods

Motivation

Up to now: discussed stability and accuracy properties of LMMs in limit $h \to 0$, $n \to \infty$, nh fixed.

However, it is of practical significance to investigate the performance of methods in the case of $h > 0$ fixed and $n \to \infty$.

Specifically, we would like to ensure that when applied to an IVP whose soln decays to 0 as $x \to \infty$, the LMM has a similar behaviour, for $h > 0$ fixed and $x_n = x_0 + nh \to \infty$. Model problem:

$$y'(x) = \lambda y(x), \qquad y(0) = y_0,$$

where $\lambda < 0$, $y_0 \neq 0$. True soln is $y(x) = y_0 \, e^{\lambda x}$ and hence,

$$\lim_{x \to \infty} y(x) = 0.$$

Apply LMM to model problem

Now consider the linear k-step method $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ and apply it to the model problem

$$y'(x) = \lambda y(x), \qquad y(0) = y_0,$$

where $\lambda < 0$, $y_0 \neq 0$. Noting that $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, z) = \lambda z$, this yields

$$0 = \sum_{j=0}^{k} \left(\alpha_j y_{n+j} - h\beta_j f(x_{n+j}, y_{n+j})\right) = \sum_{j=0}^{k} \left(\alpha_j - h\lambda\beta_j\right) y_{n+j}.$$

Since the general soln y_n to this homogeneous difference equation can be expressed as a linear combination of powers of roots of the associated characteristic polynomial

$$\pi(z; \bar{h}) := \sum_{j=0}^{k} \left(\alpha_j - \bar{h}\beta_j\right) z^j = \rho(z) - \bar{h}\,\sigma(z), \qquad z \in \mathbb{C}, \qquad (\bar{h} := \lambda h),$$

it follows that y_n will converge to zero for $h > 0$ fixed and $n \to \infty$ iff all roots of $\pi(z; \bar{h})$ have modulus less than 1, i.e., iff all roots lie in $D_1(0)$.

Absolute stability of LMMs

Definition (Absolute stability of LMMs)

The LMM $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ is called **absolutely stable** for a given $\bar{h}$ iff for that $\bar{h}$ all the roots $r_s = r_s(\bar{h})$ of the stability polynomial

$$\mathbb{C} \ni z \mapsto \pi(z; \bar{h}) := \rho(z) - \bar{h}\, \sigma(z)$$

satisfy $|r_s| < 1$, $s \in \{1, \ldots, k\}$. Otherwise, the method is called **absolutely unstable**.

An interval $(\alpha, \beta) \subset \mathbb{R}$ is called the **interval of absolute stability** if it is the largest open interval with the property that the method is absolutely stable for all $\bar{h} \in (\alpha, \beta)$. If the method is absolutely unstable for all $\bar{h}$, it is said to have **no interval of absolute stability**.

Rk: It can be shown that an optimal k-step method, i.e., a zero-stable linear k-step method of order $k+2$, has no interval of absolute stability.

Convergent LMMs are absolutely unstable for $\bar{h} > 0$ small

Since for $\lambda > 0$ the solution $y(x) = y_0 e^{\lambda x}$ of the model problem has exponential growth, we expect that a consistent and zero-stable (and, therefore, convergent) LMM has a similar behaviour for $h > 0$ sufficiently small, and will therefore be absolutely unstable for small $\bar{h} > 0$.

Theorem

Every consistent zero-stable LMM is absolutely unstable for $\bar{h} > 0$ small.

Proof: Consistency $\Longrightarrow \exists p \in \mathbb{N}$: $C_0 = C_1 = \cdots = C_p = 0 \neq C_{p+1}$.
PS 2 $\Longrightarrow \pi(\mathrm{e}^{\bar{h}}; \bar{h}) = \mathcal{O}(\bar{h}^{p+1})$. Note $\pi(z; \bar{h}) = (\alpha_k - \bar{h}\beta_k) \prod_{s=1}^{k} (z - r_s)$, where $r_s = r_s(\bar{h})$, $s \in \{1, \ldots, k\}$, denote the roots of $z \mapsto \pi(z; \bar{h})$. Thus,

$$(\alpha_k - \bar{h}\beta_k)(e^{\bar{h}} - r_1(\bar{h})) \cdots (e^{\bar{h}} - r_k(\bar{h})) = \pi(\mathrm{e}^{\bar{h}}; \bar{h}) = \mathcal{O}(\bar{h}^{p+1}). \qquad (6)$$

As $\bar{h} \to 0$, $\alpha_k - \bar{h}\beta_k \to \alpha_k \neq 0$ and $r_s(\bar{h}) \to \zeta_s$, $s \in \{1, \ldots, k\}$, where ζ_s, $s \in \{1, \ldots, k\}$, are the roots of ρ. By consistency, 1 is a root of ρ; by zero-stability, 1 is simple root of ρ. WLOG $\zeta_1 = 1$. As $\zeta_s \neq 1$ for $s \neq 1$, only factor converging to 0 in (6) is $e^{\bar{h}} - r_1(\bar{h})$. $\Longrightarrow e^{\bar{h}} - r_1(\bar{h}) = \mathcal{O}(\bar{h}^{p+1})$
$\Longrightarrow r_1(\bar{h}) = e^{\bar{h}} + \mathcal{O}(\bar{h}^{p+1}) > 1 + \frac{1}{2}\bar{h}$ for $\bar{h} > 0$ sufficiently small. □

Locating the interval of absolute stability: Schur criterion

Consider the polynomial

$$\phi : \mathbb{C} \to \mathbb{C}, \quad \phi(z) = c_k z^k + c_{k-1} z^{k-1} + \cdots + c_1 z + c_0,$$

with $c_0, c_1, \ldots, c_k \in \mathbb{C}$ and $c_k \neq 0$, $c_0 \neq 0$. The polynomial ϕ is called a **Schur polynomial** if all of its roots lie in $D_1(0)$.

Define the polynomial

$$\hat{\phi} : \mathbb{C} \to \mathbb{C}, \quad \hat{\phi}(z) = \bar{c}_0 z^k + \bar{c}_1 z^{k-1} + \cdots + \bar{c}_{k-1} z + \bar{c}_k,$$

where $\bar{c}_j$ denotes the complex conjugate of c_j, and define the polynomial

$$\phi_1 : \mathbb{C} \to \mathbb{C}, \quad \phi_1(z) = \frac{\hat{\phi}(0)\phi(z) - \phi(0)\hat{\phi}(z)}{z}.$$

Theorem (Schur's criterion)

The polynomial ϕ is a Schur polynomial iff

$$\left|\hat{\phi}(0)\right| > |\phi(0)| \qquad \text{and} \qquad \phi_1 \text{ is a Schur polynomial.}$$

Example: Interval of absolute stability via Schur criterion

Task: Find interval of abs. stab. of the LMM $y_{n+2} - y_n = \frac{h}{2}(f_{n+1} + 3f_n)$.

We have $\rho(z) = z^2 - 1$ and $\sigma(z) = \frac{1}{2}(z+3)$. Therefore,

$$\pi(z;\bar{h}) = \rho(z) - \bar{h}\sigma(z) = z^2 - \frac{1}{2}\bar{h}z - \left(1 + \frac{3}{2}\bar{h}\right).$$

Suppose $1 + \frac{3}{2}\bar{h} \neq 0$, i.e., $\bar{h} \neq -\frac{2}{3}$ s.t. we can apply Schur crit. We have

$$\hat{\pi}(z;\bar{h}) = -\left(1 + \frac{3}{2}\bar{h}\right) z^2 - \frac{1}{2}\bar{h}z + 1.$$

Note $|\hat{\pi}(0;\bar{h})| > |\pi(0;\bar{h})|$ iff $1 > |1 + \frac{3}{2}\bar{h}|$ iff $\bar{h} \in (-\frac{4}{3}, 0)$. For such $\bar{h}$,

$$\pi_1(z;\bar{h}) = \frac{\hat{\pi}(0;\bar{h})\pi(z;\bar{h}) - \pi(0;\bar{h})\hat{\pi}(z;\bar{h})}{z} = -\frac{1}{2}\bar{h}\left(2 + \frac{3}{2}\bar{h}\right)(3z+1)$$

has unique root $-\frac{1}{3} \in D_1(0)$. $\implies z \mapsto \pi_1(z;\bar{h})$ is Schur polynomial
By Schur crit., $z \mapsto \pi(z;\bar{h})$, $\bar{h} \neq -\frac{2}{3}$, is Schur polynomial iff $\bar{h} \in (-\frac{4}{3}, 0)$.
Finally, for $\bar{h} = -\frac{2}{3}$, $\pi(z; -\frac{2}{3}) = z(z + \frac{1}{3})$ is Schur polynomial.
$\implies$ interval of absolute stability is $(-\frac{4}{3}, 0)$.

Locating interval of abs. stab.: Routh–Hurwitz criterion

Consider the bijections $m_1 : D_1(0) \to \mathbb{C}^-$ and $m_2 = m_1^{-1} : \mathbb{C}^- \to D_1(0)$,

$$m_1(z) := \frac{z-1}{z+1}, \qquad m_2(z) := \frac{1+z}{1-z},$$

where $\mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Re}\,(z) < 0\}$. Consider the polynomial

$$(1-z)^k \left[\pi\left(\frac{1+z}{1-z}; \bar{h}\right)\right] = a_0 z^k + a_1 z^{k-1} + \cdots + a_k. \tag{7}$$

The roots of $z \mapsto \pi(z;\bar{h})$ lie inside $D_1(0)$ iff $a_0 \neq 0$ and the roots of (7) lie in $\mathbb{C}^-$. (Note $a_0 = (-1)^k \pi(-1;\bar{h})$ and thus, $a_0 = 0$ iff $\pi(-1;\bar{h}) = 0$.)

Theorem (Routh–Hurwitz criterion)

The roots of a polynomial $P : \mathbb{C} \to \mathbb{C}$, $P(z) := a_0 z^k + a_1 z^{k-1} + \cdots + a_k$ with $a_0, \ldots, a_k \in \mathbb{R}$ and $a_0 > 0$ lie in $\mathbb{C}^-$ iff all leading principal minors of

$$H := \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & a_0 & a_2 & \cdots & a_{2k-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_k \end{bmatrix} \in \mathbb{R}^{k\times k}$$

are positive, where we set $a_j := 0$ if $j > k$.

Theorem (Routh–Hurwitz criterion)

The roots of a polynomial $P : \mathbb{C} \to \mathbb{C}$, $P(z) := a_0 z^k + a_1 z^{k-1} + \cdots + a_k$ *with* $a_0, \ldots, a_k \in \mathbb{R}$ *and* $a_0 > 0$ *lie in* $\mathbb{C}^-$ *iff all leading principal minors of*

$$H := \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & a_0 & a_2 & \cdots & a_{2k-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_k \end{bmatrix} \in \mathbb{R}^{k \times k}$$

are positive, where we set $a_j := 0$ *if* $j > k$.

The necessary and sufficient conditions for $k \in \{1, 2, 3, 4\}$ for ensuring that all roots of $P : \mathbb{C} \to \mathbb{C}$, $p(z) := a_0 z^k + a_1 z^{k-1} + \cdots + a_k$ with $a_0, \ldots, a_k \in \mathbb{R}$ and $a_0 > 0$ lie in $\mathbb{C}^-$ are the following:

$k = 1$ $a_1 > 0$.

$k = 2$ $a_1 > 0$, $a_2 > 0$.

$k = 3$ $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 a_0 > 0$.

$k = 4$ $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, $a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 > 0$.

Example: Interval of absolute stability via RH criterion

Task: Find interval of abs. stab. of the LMM $y_{n+2} - y_n = \frac{h}{2}(f_{n+1} + 3f_n)$.

We have $\rho(z) = z^2 - 1$ and $\sigma(z) = \frac{1}{2}(z+3)$. Therefore,

$$\pi(z;\bar{h}) = \rho(z) - \bar{h}\sigma(z) = z^2 - \frac{1}{2}\bar{h}z - \left(1 + \frac{3}{2}\bar{h}\right).$$

We compute

$$P(z) := (1-z)^2 \left[\pi\left(\frac{1+z}{1-z};\bar{h}\right)\right] = -\bar{h}z^2 + (4+3\bar{h})z - 2\bar{h} =: a_0 z^2 + a_1 z + a_2.$$

All roots of $z \mapsto \pi(z;\bar{h})$ lie inside $D_1(0)$ iff $a_0 = -\bar{h} \neq 0$ and all roots of P lie in $\mathbb{C}^-$. So, for $\bar{h} = 0$ we are unstable. For $\bar{h} \neq 0$, we use RH crit.:

- Case $\bar{h} < 0$: all roots of P lie in $\mathbb{C}^-$ iff (RH) $4 + 3\bar{h} > 0$ and $-2\bar{h} > 0$, i.e., iff $\bar{h} \in (-\frac{4}{3}, 0)$.
- Case $\bar{h} > 0$: all roots of P lie in $\mathbb{C}^-$ iff all roots of $-P$ lie in $\mathbb{C}^-$ iff (RH) $-(4+3\bar{h}) > 0$ and $2\bar{h} > 0$; impossible.

$\Longrightarrow$ interval of absolute stability is $(-\frac{4}{3}, 0)$.

k-step Adams–Bashforth methods

p: order of accuracy, C_{p+1}: error const., I_{as} interval of absolute stability.

k=1 $p = 1$, $C_{p+1} = \frac{1}{2}$, $I_{as} = (-2, 0)$,

$$y_{n+1} - y_n = hf_n;$$

k=2 $p = 2$, $C_{p+1} = \frac{5}{12}$, $I_{as} = (-1, 0)$,

$$y_{n+2} - y_{n+1} = \frac{h}{2}(3f_{n+1} - f_n);$$

k=3 $p = 3$, $C_{p+1} = \frac{3}{8}$, $I_{as} = (-\frac{6}{11}, 0)$,

$$y_{n+3} - y_{n+2} = \frac{h}{12}(23f_{n+2} - 16f_{n+1} + 5f_n);$$

k=4 $p = 4$, $C_{p+1} = \frac{251}{720}$, $I_{as} = (-\frac{3}{10}, 0)$,

$$y_{n+4} - y_{n+3} = \frac{h}{24}(55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n).$$

k-step Adams–Moulton methods

p: order of accuracy, C_{p+1}: error const., I_{as} interval of absolute stability.

k=1 $p = 2$, $C_{p+1} = -\frac{1}{12}$, $I_{as} = (-\infty, 0)$,

$$y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n);$$

k=2 $p = 3$, $C_{p+1} = -\frac{1}{24}$, $I_{as} = (-6, 0)$,

$$y_{n+2} - y_{n+1} = \frac{h}{12}(5f_{n+2} + 8f_{n+1} - f_n);$$

k=3 $p = 4$, $C_{p+1} = -\frac{19}{720}$, $I_{as} = (-3, 0)$,

$$y_{n+3} - y_{n+2} = \frac{h}{24}(9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n);$$

k=4 $p = 5$, $C_{p+1} = -\frac{27}{1440}$, $I_{as} = (-\frac{90}{49}, 0)$,

$$y_{n+4} - y_{n+3} = \frac{h}{720}(251f_{n+4} + 646f_{n+3} - 264f_{n+2} + 106f_{n+1} - 19f_n).$$

End of “Chapter 3: Linear multi-step methods”.