MA4255 Numerical Methods in Differential Equations

Chapter 1: Introduction/Preliminaries (Part I: ODEs)

What are Ordinary Differential Equations (ODEs)?

An ODE is an equation for a fct $y = y(x)$ containing derivatives of y, e.g.,

- $y'(x) = 3y(x)$,
- $(y(x) + y'(x) + y''''(x))^7 = y''(x) - y''''(x) + x^3$.

Explicit ODE of order n:

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)).$$

Implicit ODE of order n:

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0.$$

In Part I of this course, we focus on **explicit first-order ODEs**, i.e.,

$$y'(x) = f(x, y(x)).$$

Initial-value Problems (IVPs)

Ex.: $y'(x) = 3y(x)$ has infinitely many solns: $y(x) = c\, e^{3x}$, $c \in \mathbb{R}$.

$\Longrightarrow$ To get a unique solution, we need to specify y at some value $x_0 \in \mathbb{R}$.

$$y'(x) = 3y(x), \quad y(x_0) = y_0 \qquad \Longrightarrow \quad y(x) = y_0\, e^{3(x-x_0)}.$$

$y(x_0) = y_0$ is called an **initial condition (i.c.)**. Problems of the form

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0 \tag{1}$$

are called **Initial-value Problems (IVPs)**.

Note: Not all IVPs have a unique solution. E.g.,

$$y'(x) = [y(x)]^{\frac{2}{3}}\,, \qquad y(0) = 0$$

has multiple solutions: $y \equiv 0$ and $y(x) = \frac{1}{27}x^3$. (Here, $f(x,z) := z^{\frac{2}{3}}$)

$\Longrightarrow$ $f(\cdot,\cdot)$ continuous not sufficient for uniqueness. So what is sufficient?

Rk: f continuous $\Longrightarrow$ (1) has at least one solution (Peano Existence Thm)

Picard's Theorem for IVPs $y'(x) = f(x, y(x))$, $y(x_0) = y_0$

Theorem (Picard's Theorem)

Suppose $f(\cdot,\cdot)$ *is continuous in a region* $U \subseteq \mathbb{R}^2$ *containing the rectangle* $\mathsf{R} := [x_0, X_M] \times [y_0 - Y_M, y_0 + Y_M]$, *where* $X_M > x_0, Y_M > 0$, *suppose*

$$\exists L > 0 : \quad |f(x,z) - f(x,\tilde{z})| \le L|z - \tilde{z}| \quad \forall (x,z),(x,\tilde{z}) \in \mathsf{R}, \qquad (2)$$

and with $M := \max_{(x,z)\in\mathsf{R}} |f(x,z)|$, *suppose* $M(X_M - x_0) \le Y_M$. *Then,* $\exists$ *a unique continuously differentiable fct* $y : [x_0, X_M] \to \mathbb{R}$ *s.t.*

$$y'(x) = f(x, y(x)) \quad \text{for} \quad x \in (x_0, X_M), \qquad y(x_0) = y_0.$$

(2) is called **Lipschitz condition** (in 2nd argument of f), L is called **Lipschitz constant**.

Proof: See any introductory ODE course or book. Idea: **Picard iteration:**

$$y_0(x) \equiv y_0, \quad y_n(x) := y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t, \qquad n \in \mathbb{N}.$$

Then, $y_n \to y$ in $C([x_0, X_M])$ and y solves the IVP. □

Theorem (Picard's Theorem)

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$$\exists L > 0 : \quad |f(x,z) - f(x,\tilde{z})| \leq L|z - \tilde{z}| \quad \forall (x,z),(x,\tilde{z}) \in \mathsf{R},$$

and with $M := \max_{(x,z)\in\mathsf{R}} |f(x,z)|$, *suppose* $M(X_M - x_0) \leq Y_M$. *Then,* $\exists$ *a unique continuously differentiable fct* $y : [x_0, X_M] \to \mathbb{R}$ *s.t.*

$$y'(x) = f(x, y(x)) \quad \textit{for} \quad x \in (x_0, X_M), \qquad y(x_0) = y_0.$$

Observation: the graph of y lies in R, i.e., $(x, y(x)) \in \mathsf{R}\ \forall x \in [x_0, X_M]$.

Pf of observation: Suppose this were not true. Then, by continuity of y,

$$\exists x_* \in (x_0, X_M) : \quad |y(x_*) - y_0| = Y_M, \quad |y(x) - y_0| < Y_M \ \forall x \in [x_0, x_*).$$

$$\Rightarrow |y(x_*) - y_0| = \left| \int_{x_0}^{x_*} y'(x)\,\mathrm{d}x \right| \leq \int_{x_0}^{x_*} |y'(x)|\,\mathrm{d}x$$

$$= \int_{x_0}^{x_*} |f(x, y(x))|\,\mathrm{d}x \leq M(x_* - x_0) < Y_M. \ ↯ \quad \square$$

IVPs for systems of ODEs

IVPs for systems of m ODEs:

$$\begin{cases} y_1'(x) = f_1(x, y_1(x), y_2(x), \dots, y_m(x)), & y_1(x_0) = y_{0,1}, \\ y_2'(x) = f_2(x, y_1(x), y_2(x), \dots, y_m(x)), & y_2(x_0) = y_{0,2}, \\ \qquad \vdots & \\ y_m'(x) = f_m(x, y_1(x), y_2(x), \dots, y_m(x)), & y_m(x_0) = y_{0,m}, \end{cases}$$

or equivalently,

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x)), \qquad \mathbf{y}(x_0) = \mathbf{y}_0$$

with given

$$x_0 \in \mathbb{R}, \quad \mathbf{y}_0 = \begin{pmatrix} y_{0,1} \\ \vdots \\ y_{0,m} \end{pmatrix} \in \mathbb{R}^m, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} : [x_0, X_M] \times \mathbb{R}^m \to \mathbb{R}^m,$$

and we are seeking a soln $\mathbf{y} = (y_1, \dots, y_m)^{\mathrm{T}} : [x_0, X_M] \to \mathbb{R}^m$.

Picard's Thm for systems $\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x))$, $\mathbf{y}(x_0) = \mathbf{y}_0$

Introduce the Euclidean norm $\|\cdot\| : \mathbb{R}^m \to [0, \infty)$ on $\mathbb{R}^m$ by

$$\|\mathbf{u}\| := \sqrt{\sum_{i=1}^{m} |u_i|^2}, \quad \text{for } \mathbf{u} = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m.$$

Theorem (Picard's Theorem (version for systems))

Suppose that $\mathbf{f}(\cdot, \cdot)$ *is a continuous in a region* $U \subseteq \mathbb{R}^{1+m}$ *containing*

$$\mathsf{R} = \{(x, \mathbf{z}) \in \mathbb{R} \times \mathbb{R}^m \, : \, x \in [x_0, X_M], \quad \|\mathbf{z} - \mathbf{y}_0\| \leq Y_M\},$$

where $X_M > x_0, Y_M > 0$, *suppose that*

$$\exists L > 0 : \qquad \|\mathbf{f}(x, \mathbf{z}) - \mathbf{f}(x, \tilde{\mathbf{z}})\| \leq L\|\mathbf{z} - \tilde{\mathbf{z}}\| \quad \forall (x, \mathbf{z}), (x, \tilde{\mathbf{z}}) \in \mathsf{R},$$

and with $M := \max_{(x,\mathbf{z}) \in \mathsf{R}} \|\mathbf{f}(x, \mathbf{z})\|$, *suppose* $M(X_M - x_0) \leq Y_M$. *Then,* $\exists$ *a unique continuously differentiable fct* $\mathbf{y} : [x_0, X_M] \to \mathbb{R}^m$ *s.t.*

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x)), \qquad \mathbf{y}(x_0) = \mathbf{y}_0.$$

A sufficient condition guaranteeing the Lipschitz property

Recall Lipschitz condition for IVPs:

$$\exists L > 0 : \qquad |f(x,z) - f(x,\tilde{z})| \le L|z - \tilde{z}| \quad \forall (x,z),(x,\tilde{z}) \in \mathsf{R}. \tag{3}$$

This is automatically satisfied if f is cts on R, differentiable in $\mathrm{int}(\mathsf{R})$, and

$$\exists\, C > 0 : \qquad |f_z(x,z)| := \left|\frac{\partial f}{\partial z}(x,z)\right| \le C \quad \forall (x,z) \in \mathrm{int}(\mathsf{R}). \tag{4}$$

Indeed, suppose that (4) holds. By the Mean-Value Thm, for any $(x,z) \in \mathsf{R}$ we have $f(x,z) - f(x,\tilde{z}) = f_z(x,\xi)(z-\tilde{z})$ for some ξ between z and $\tilde{z}$. So, we obtain (3) with $L := C$.

Recall Lipschitz condition for IVPs for systems:

$$\exists L > 0 : \qquad \|\mathbf{f}(x,\mathbf{z}) - \mathbf{f}(x,\tilde{\mathbf{z}})\| \le L\|\mathbf{z} - \tilde{\mathbf{z}}\| \quad \forall (x,\mathbf{z}),(x,\tilde{\mathbf{z}}) \in \mathsf{R}.$$

This is automatically satisfied with $L = C$ if $\mathbf{f}$ cts on R, diff. in $\mathrm{int}(\mathsf{R})$, and

$$\exists\, C > 0 : \qquad \left\|\frac{\partial \mathbf{f}}{\partial \mathbf{z}}(x,z)\right\| \le C \quad \forall (x,z) \in \mathrm{int}(\mathsf{R}), \tag{5}$$

where $\|\cdot\|$ in (5) is the matrix norm induced by the Euclidean vector norm.

Warning: The converse is not necessarily true!

E.g., consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, z) := |z|$$

and R $:= [x_0, X_M] \times [y_0 - Y_M, y_0 + Y_M]$ with $x_0 := y_0 := 0$ and any chosen $X_M, Y_M > 0$. Then,

- f satisfies the Lipschitz condition with $L = 1$:

$$|f(x, z) - f(x, \tilde{z})| = ||z| - |\tilde{z}|| \leq |z - \tilde{z}| \qquad \forall (x, z), (x, \tilde{z}) \in \mathsf{R},$$

- but f is not differentiable in $\mathrm{int}(\mathsf{R})$ (as $z \mapsto |z|$ is not diff. at $z = 0$).

Stability

Definition (Stability)

Consider an IVP for a system of ODEs $\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x))$, $\mathbf{y}(x_0) = \mathbf{y}_0$.

(i) A solution $\mathbf{y} = \mathbf{v}(x)$ is called **stable** on $[x_0, X_M]$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. for all $\mathbf{z} \in \mathbb{R}^m$ satisfying $\|\mathbf{y}_0 - \mathbf{z}\| < \delta$ the solution $\mathbf{w}$ to

$$\mathbf{w}'(x) = \mathbf{f}(x, \mathbf{w}(x)), \qquad \mathbf{w}(x_0) = \mathbf{z}$$

is defined in $[x_0, X_M]$ and satisfies $\|\mathbf{v}(x) - \mathbf{w}(x)\| < \varepsilon \; \forall x \in [x_0, X_M]$.

(ii) A soln $\mathbf{y} = \mathbf{v}(x)$ which is stable on $[x_0, \infty)$ (i.e. stable on $[x_0, X_M]$ for each X_M and with δ independent of X_M) is called **stable in the sense of Lyapunov**.

(iii) If in addition to (ii) there holds

$$\lim_{x \to \infty} \|\mathbf{v}(x) - \mathbf{w}(x)\| = 0,$$

then the solution $\mathbf{y} = \mathbf{v}(x)$ is called **asymptotically stable**.

Stability: Example

Recall defn: A solution $\mathbf{y} = \mathbf{v}(x)$ is called **stable** on $[x_0, X_M]$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. for all $\mathbf{z} \in \mathbb{R}^m$ satisfying $\|\mathbf{y}_0 - \mathbf{z}\| < \delta$ the solution $\mathbf{w}$ to $\mathbf{w}'(x) = \mathbf{f}(x, \mathbf{w}(x))$, $\mathbf{w}(x_0) = \mathbf{z}$ is defined in $[x_0, X_M]$ and satisfies $\|\mathbf{v}(x) - \mathbf{w}(x)\| < \varepsilon\ \forall x \in [x_0, X_M]$.

For some fixed $\lambda \in \mathbb{R}$, consider the IVP

$$y'(x) = \lambda y(x), \qquad y(0) = 1$$

with unique soln $y = v(x)$ with $v(x) := e^{\lambda x}$.

Question: For what values of λ is the soln $y = v(x)$ stable on $[0, \infty)$ (i.e., stable on $[0, X_M]$ for any $X_M > 0$ and δ independent of X_M)?

For $z \in \mathbb{R}$, the problem

$$w'(x) = \lambda w(x), \qquad w(0) = z$$

has the unique soln $w(x) := ze^{\lambda x}$. Note that $|v(x) - w(x)| = |1 - z|e^{\lambda x}$.

- Case $\lambda \leq 0$: Then, $|v(x) - w(x)| \leq |1 - z|\ \forall x \in [0, \infty)$.
  $\implies y = v(x)$ stable on $[0, \infty)$ when $\lambda \leq 0$.
- Case $\lambda > 0$: Note $\max_{x \in [0, X_M]} |v(x) - w(x)| = |1 - z|e^{\lambda X_M}$.
  $\implies y = v(x)$ is unstable on $[0, \infty)$ when $\lambda > 0$.

  However, $y = v(x)$ is stable on $[0, X_M]$ for fixed X_M.

Main result on stability

Theorem (Stability under assumptions of Picard's Thm)

Under assumptions of Picard's Thm, the unique solution $\mathbf{y} = \mathbf{v}(x)$ *to* $\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x))$, $\mathbf{y}(x_0) = \mathbf{y}_0$ *is stable on* $[x_0, X_M]$.

Proof: Let $\mathbf{w}$ be the soln to $\mathbf{w}'(x) = \mathbf{f}(x, \mathbf{w}(x))$, $\mathbf{w}(x_0) = \mathbf{z}$. Then, we have $\mathbf{v}(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(t, \mathbf{v}(t))\,\mathrm{d}t$ and $\mathbf{w}(x) = \mathbf{z} + \int_{x_0}^{x} \mathbf{f}(t, \mathbf{w}(t))\,\mathrm{d}t$.

$$\begin{aligned}\|\mathbf{v}(x) - \mathbf{w}(x)\| &\le \|\mathbf{y}_0 - \mathbf{z}\| + \left\| \int_{x_0}^{x} (\mathbf{f}(t, \mathbf{v}(t)) - \mathbf{f}(t, \mathbf{w}(t)))\,\mathrm{d}t \right\| \\ &\le \|\mathbf{y}_0 - \mathbf{z}\| + \int_{x_0}^{x} \|\mathbf{f}(t, \mathbf{v}(t)) - \mathbf{f}(t, \mathbf{w}(t))\|\,\mathrm{d}t \\ &\le \|\mathbf{y}_0 - \mathbf{z}\| + L \int_{x_0}^{x} \|\mathbf{v}(t) - \mathbf{w}(t)\|\,\mathrm{d}t.\end{aligned}$$

Gronwall Lemma: $A(x) \le a + L \int_{x_0}^{x} A(t)\,\mathrm{d}t \, \forall x \implies A(x) \le a e^{L(x - x_0)} \, \forall x$.

$$\|\mathbf{v}(x) - \mathbf{w}(x)\| \le e^{L(x - x_0)} \|\mathbf{y}_0 - \mathbf{z}\| \le e^{L(X_M - x_0)} \|\mathbf{y}_0 - \mathbf{z}\|.$$

Given $\varepsilon > 0$, set $\delta = e^{-L(X_M - x_0)} \varepsilon$: $\|\mathbf{v}(x) - \mathbf{w}(x)\| < \varepsilon$ if $\|\mathbf{y}_0 - \mathbf{z}\| < \delta$. □

Proof of Gronwall Lemma: Write $I := [x_0, X_M]$. We need to show that

$$A(x) \leq a + L \int_{x_0}^{x} A(t)\,\mathrm{d}t \quad \forall x \in I \quad \Longrightarrow \quad A(x) \leq a e^{L(x-x_0)} \quad \forall x \in I.$$

Multiplying the inequality $-a - L \int_{x_0}^{x} A(t) + A(x) \leq 0$ by e^{-Lx}:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{a}{L} e^{-Lx} + e^{-Lx} \int_{x_0}^{x} A(t)\,\mathrm{d}t \right] \leq 0 \qquad \forall x \in I.$$

So, the function in the brackets is non-increasing. Thus,

$$\frac{a}{L} e^{-Lx} + e^{-Lx} \int_{x_0}^{x} A(t)\,\mathrm{d}t \leq \frac{a}{L} e^{-Lx_0} \quad \forall x \in I.$$

Multiply by Le^{Lx}:

$$a + L \int_{x_0}^{x} A(t)\,\mathrm{d}t \leq a\, e^{L(x-x_0)} \quad \forall x \in I.$$

It follows that $A(x) \leq a + L \int_{x_0}^{x} A(t)\,\mathrm{d}t \leq a e^{L(x-x_0)} \quad \forall x \in I.$ $\square$

End of “Chapter 1: Introduction/Preliminaries”.