MA4255: Problem Sheet 4

AY 2022/23

Q 1 *Discrete Parseval's identity*

Prove Lemma 9 of the lecture notes. (Hint: Note $|\hat{U}(k)|^2 = \hat{U}(k)\overline{\hat{U}(k)}$.)

Q 2 *FD schemes for the heat equation*

Let us consider the IVP for the heat equation:

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx}^2 u(x,t) \quad &&\text{for } (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) &= u_0(x) &&\text{for } x \in \mathbb{R}. \end{aligned} \tag{1}$$

For $M \in \mathbb{N}$ fixed, we consider a mesh with spacing $\Delta x > 0$ in the x-direction and spacing $\Delta t := \frac{T}{M}$ in the t-direction. Let $x_j := j\Delta x$ for $j \in \mathbb{Z}$, and $t_m := m\Delta t$ for $m \in \{0, 1, \ldots, M\}$.

(i) State the explicit Euler FD scheme for this problem.

(ii) We seek another FD scheme for this problem based on approximating the time derivative $\partial_t u$ in a mesh point (x_j, t_m) by $\partial_t u(x_j, t_m) \approx \frac{u(x_j,t_{m+1}) - u(x_j,t_m)}{\Delta t}$, and the second spatial derivative $\partial_{xx}^2 u$ in a mesh point (x_j, t_m) by a divided difference of the form

$$\partial_{xx}^2 u(x_j, t_m) \approx \frac{c_1 u(x_{j+2}, t_m) + c_2 u(x_j, t_m) + c_3 u(x_{j-2}, t_m)}{(\Delta x)^2}.$$

Determine appropriate values $c_1, c_2, c_3 \in \mathbb{R}$ and state the resulting FD scheme.

(iii) State a FD scheme for this problem obtained by combining the FD schemes from (i) and (ii), in the sense that you should add $\frac{1}{2}$ times the FD scheme from (i) to $\frac{1}{2}$ times the FD scheme from (ii).

(iv) Show that the FD scheme from (iii) is conditionally practically stable, with the stability condition being $\frac{\Delta t}{(\Delta x)^2} \leq 1$.

(Hint: the amplification factor can be written as $\lambda(k) = P(S(k))$, where P is a quadratic polynomial and $S(k) := \sin^2\left(\frac{k\Delta x}{2}\right)$.)

Q 3 *Practical stability of the θ-scheme*

Consider the θ-scheme for the approximation of the IVP (1). Prove the following results:

(i) If $\theta \in [0, \frac{1}{2})$, then the θ-scheme is conditionally practically stable, with the stability condition being $\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2(1-2\theta)}$.

(ii) If $\theta \in [\frac{1}{2}, 1]$, then the θ-scheme is unconditionally practically stable.

Q 4 *Crandall's FD scheme*

For the numerical solution of (1), consider the Crandall FD scheme

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{1}{2}\left(1 - \frac{\zeta}{\mu}\right) \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} + \frac{1}{2}\left(1 + \frac{\zeta}{\mu}\right) \frac{U_{j+1}^{m} - 2U_j^{m} + U_{j-1}^{m}}{(\Delta x)^2}$$

for $j \in \mathbb{Z}$ and $m \in \{0, 1, \ldots, M-1\}$, and $U_j^0 = u_0(x_j)$ for $j \in \mathbb{Z}$. Here, $\zeta \in \mathbb{R}$ is a fixed real number, and $\mu := \frac{\Delta t}{(\Delta x)^2}$. Show that the consistency error T_j^m satisfies $|T_j^m| = \mathcal{O}((\Delta x)^2)$ if $\zeta \neq \frac{1}{6}$, and $|T_j^m| = \mathcal{O}((\Delta x)^4)$ if $\zeta = \frac{1}{6}$, provided that μ is a fixed real number as $\Delta t, \Delta x \searrow 0$.

Q 5 *FD approximation of the PDE $\partial_t u - \partial_{xx}^2 u = \partial_x u$*

We consider the IVP

$$\begin{aligned} \partial_t u(x,t) - \partial_{xx}^2 u(x,t) &= \partial_x u(x,t) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) &= u_0(x) \qquad \text{for } x \in \mathbb{R}. \end{aligned}$$

For $M \in \mathbb{N}_{\geq 2}$ fixed, we consider a mesh with spacing $0 < \Delta x \leq 1$ in x-direction and spacing $\Delta t := \frac{T}{M}$ in t-direction. Let $x_j := j\Delta x$ for $j \in \mathbb{Z}$, and $t_m := m\Delta t$ for $m \in \{0, 1, \ldots, M\}$.

(i) We consider the FD scheme

$$\begin{aligned} \frac{U_j^{m+1} - U_j^m}{\Delta t} - \frac{U_{j+1}^{m} - 2U_j^{m} + U_{j-1}^{m}}{(\Delta x)^2} &= \frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} \quad \text{for } j \in \mathbb{Z},\ m \in \{0, 1, \ldots, M-1\}, \\ U_j^0 &= u_0(x_j) \qquad \text{for } j \in \mathbb{Z}. \end{aligned}$$

1) Show that the FD scheme is conditionally practically stable, with the stability condition being $\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$.

2) Show that the consistency error T_j^m satisfies $|T_j^m| = \mathcal{O}((\Delta x)^2 + \Delta t)$.

(ii) We consider the FD scheme

$$\begin{aligned} \frac{U_j^{m+1} - U_j^m}{\Delta t} - \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2} &= \frac{U_{j+1}^{m+1} - U_{j-1}^{m+1}}{2\Delta x} \quad \text{for } j \in \mathbb{Z},\ m \in \{0, 1, \ldots, M-1\}, \\ U_j^0 &= u_0(x_j) \qquad \text{for } j \in \mathbb{Z}. \end{aligned}$$

Show that the FD scheme is unconditionally practically stable.