MA4255: Problem Sheet 3

AY 2022/23

Q 1 *Sobolev spaces*

(i) Set $\Omega := (-2, 2)$ and consider the function $u : \overline{\Omega} \to \mathbb{R}$, $u(x) := |1 - x^2|$.

- Find all $k \in \mathbb{N}_0$ for which $u \in C^k(\Omega)$. Find all $k \in \mathbb{N}_0$ for which $u \in C^k(\overline{\Omega})$.
- Find the first weak derivative of u. Show that $u \in H^1(\Omega)$ and compute $\|u\|_{H^1(\Omega)}$.

(ii) Set $\Omega := (0, 1)$. Let $\alpha \in (0, \frac{1}{2}]$ be fixed. Define the function $u : \Omega \to \mathbb{R}$, $u(x) := x^\alpha$. Show that $u \in C^\infty(\Omega)$, but $u \notin H^1(\Omega)$.

Q 2 *Existence and uniqueness of weak solutions via Lax–Milgram*

(i) Let $\Omega := (0, 1)$ and let $f \in L^2(\Omega)$. Let $p : \overline{\Omega} \to \mathbb{R}$, $p(x) := x + 1$. Consider the problem

$$-(pu')' + 5u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Show that there exists a unique weak solution $u \in H_0^1(\Omega)$ to this problem.

(ii) Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and let $f \in L^2(\Omega)$. Consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Show that there exists a unique weak solution $u \in H_0^1(\Omega)$ to this problem.

Q 3 *Construction of divided difference operators*

(i) Find a second-order accurate one-sided divided difference operator of the form

$$D_x^{-,2} u(x_i) = \frac{c_1 u(x_i) + c_2 u(x_i - h) + c_3 u(x_i - 2h)}{h}$$

to approximate $u'(x_i)$, i.e., $D_x^{-,2} u(x_i) = u'(x_i) + \mathcal{O}(h^2)$ for any sufficiently smooth function $u : \mathbb{R} \to \mathbb{R}$.

(ii) Find a fourth-order accurate central divided difference operator of the form

$$D_x^{0,4} u(x_i) = \frac{c_1 u(x_i - 2h) + c_2 u(x_i - h) + c_3 u(x_i) + c_4 u(x_i + h) + c_5 u(x_i + 2h)}{h}$$

to approximate $u'(x_i)$, i.e., $D_x^{0,4} u(x_i) = u'(x_i) + \mathcal{O}(h^4)$ for any sufficiently smooth function $u : \mathbb{R} \to \mathbb{R}$.

Q 4 *Analyzing a FD scheme for a 1D BVP*

Let $\Omega := (0,1) \subset \mathbb{R}$. We consider the problem

$$-u'' + pu' + qu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $f \in C(\overline{\Omega})$, and $p, q : \overline{\Omega} \to \mathbb{R}$ are given by $p(x) := \cos(x)$ and $q(x) := \exp(x)$ for $x \in \overline{\Omega}$. For a fixed $N \in \mathbb{N}_{\geq 2}$, we define $x_i := ih$, $i \in \{0, 1, \ldots, N\}$, where $h := \frac{1}{N}$. We consider the FD scheme

$$-D_x^+ D_x^- U_i + p(x_i) D_x^0 U_i + q(x_i) U_i = f(x_i) \quad \text{for } i \in \{1, \ldots, N-1\}, \qquad U_0 = U_N = 0, \tag{2}$$

where $D_x^+ D_x^-$ denotes the symmetric second divided difference operator and D_x^0 the central first divided difference operator.

(i) Show that (1) has a unique weak solution, i.e., there exists a unique $u \in H_0^1(\Omega)$ such that

$$\int_\Omega u'v' \,\mathrm{d}x + \int_\Omega pu'v \,\mathrm{d}x + \int_\Omega quv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \qquad \forall v \in H_0^1(\Omega).$$

(Hint: Use that $v'v = \frac{1}{2}(v^2)'$ $\forall v \in H_0^1(\Omega)$ to show that $\int_\Omega pv'v \,\mathrm{d}x \geq 0$ $\forall v \in H_0^1(\Omega)$.)

(ii) Find $A \in \mathbb{R}^{(N-1)\times(N-1)}$ and $F \in \mathbb{R}^{N-1}$ such that (2) can be written as $AU = F$, $U_0 = U_N = 0$, where $U = (U_1, \ldots, U_{N-1})^{\mathrm{T}}$.

(iii) Show that (2) has a unique solution. (Hint: Show that A is strictly diagonally dominant.)

(iv) For the consistency error $\varphi_i := -D_x^+ D_x^- u(x_i) + p(x_i) D_x^0 u(x_i) + q(x_i) u(x_i) - f(x_i)$ show that $|\varphi_i| = \mathcal{O}(h^2)$, assuming that u is sufficiently smooth.

MATLAB (optional): For f chosen such that the true solution to (1) is $u(x) = x^2(1-x)^2$, implement (2) and compute the error $\max_{i\in\{0,1,\ldots,N\}} |U_i - u(x_i)|$ for various values of N. What order of convergence do you observe?

Q 5 *9-point difference operator for 2D Laplacian*

We consider the 9-point difference operator

$$\begin{aligned}\Delta_h^{(9)} u(x_i, y_j) := &- \frac{u(x_i - 2h, y_j) - 16u(x_i - h, y_j) + 30u(x_i, y_j) - 16u(x_i + h, y_j) + u(x_i + 2h, y_j)}{12h^2} \\ &- \frac{u(x_i, y_j - 2h) - 16u(x_i, y_j - h) + 30u(x_i, y_j) - 16u(x_i, y_j + h) + u(x_i, y_j + 2h)}{12h^2}\end{aligned}$$

for the approximation of $\Delta u(x_i, y_j)$. Show that

$$\Delta_h^{(9)} u(x_i, y_j) = \Delta u(x_i, y_j) - \frac{1}{90} h^4 \left[u_{xxxxxx} + u_{yyyyyy}\right](x_i, y_j) + \mathcal{O}(h^6)$$

for any sufficiently smooth function $u : \mathbb{R}^2 \to \mathbb{R}$.

Q 6 *Application of the maximum principle for elliptic PDEs*

Let $\Omega \subseteq (0,\pi)^2$ be an open region with smooth boundary. Set $f : \overline{\Omega} \to \mathbb{R}$, $f(x,y) := \sin(x) + \sin(y)$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is such that $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. Prove $0 \leq u \leq 2$ in $\overline{\Omega}$. (Hint: Consider $w := u - f$.)