MA4255: Problem Sheet 2

AY 2022/23

Q 1 *Adams–Bashforth and Adams–Moulton methods*

(i) For $k \in \{1, 2\}$, derive the k-step Adams–Moulton method from (40) of the lecture notes (Hint: truncate after the $(k+1)$-th term).

(ii) For $k \in \{1, 2, 3, 4\}$, derive the k-step Adams–Bashforth method from (41) of the lecture notes (Hint: truncate the series after the k-th term).

(iii) For the two-step Adams–Moulton method, compute the order of accuracy, the error constant, and the interval of absolute stability.

(iv) For the four-step Adams–Bashforth method, compute the order of accuracy, the error constant, and the interval of absolute stability.

MATLAB: Test the performance of the various Adams–Bashforth methods at the IVP from PS1 Q4 (use a one-step method to obtain the starting values).

Q 2 *Properties of the θ-method*

For which values of $\theta \in [0, 1]$ is the θ-method (i) zero-stable? (ii) A-stable? (iii) $A(\alpha)$-stable for some $\alpha \in (0, \frac{\pi}{2})$?

Q 3 *Surrounding absolute stability*

(i) Find the interval of absolute stability and the region of absolute stability of the trapezium rule method and the Simpson rule method.

(ii) Show that the BDF method $3y_{n+2} - 4y_{n+1} + y_n = 2hf_{n+2}$ is A-stable.

Q 4 *LMMs: Order of accuracy, zero-stability, convergence*

(i) Construct a linear 1-step method of maximum order of accuracy. Is the method zero-stable?

(ii) Find the order of accuracy of the linear 3-step method

$$11y_{n+3} + 27y_{n+2} - 27y_{n+1} - 11y_n = 3h(f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n).$$

Is the method convergent?

(iii) For $a, b \in \mathbb{R}$ fixed, consider a LMM with $\rho(z) = (z-1)(az+1-a)$ and $\sigma(z) = (z-1)^2 b + (z-1)a + \frac{1}{2}(z+1)$. Find the order or accuracy of the method. For which values of a, b is the method zero-stable?

(iv) Show that a linear explicit three-step method cannot be both fourth-order accurate and convergent.

Q 5 *Order of accuracy of LMMs*

(i) For $i \in \mathbb{N}_0$ define $g_i : \mathbb{R} \to \mathbb{R}$ by $g_i(x) := x^i$. Show that a LMM with $\sigma(1) \neq 0$ has order of accuracy $p \in \mathbb{N}$ iff

$$L_h g_i \equiv 0 \quad \forall i \in \{0, 1, \ldots, p\}, \quad \text{and} \quad L_h g_{p+1} \not\equiv 0,$$

where L_h is the operator which maps a continuously differentiable function $u : \mathbb{R} \to \mathbb{R}$ to the function $L_h u : \mathbb{R} \to \mathbb{R}$ defined by

$$[L_h u](x) := \sum_{j=0}^{k} \left(\alpha_j u(x + jh) - h\beta_j u'(x + jh)\right).$$

(Hint: First, show that $[L_h g_i](x) = \sum_{q=0}^{\infty} C_q h^q g_i^{(q)}(x)$ for any $i \in \mathbb{N}_0$. Then, carefully prove both directions of the "iff" statement.)

(ii) Show that a LMM with $\sigma(1) \neq 0$ has order of accuracy of at least p iff

$$\rho(e^h) - h\sigma(e^h) = \mathcal{O}(h^{p+1}).$$

Q 6 *Stiff IVPs*

Solve the following IVPs and decide whether they are stiff.

(i) $y'(x) = \left(10^5 e^{-10^4 x} + 1\right)(1 - y(x))$, $x \in [0, 1]$, $y(0) = 2$.

(ii) $\mathbf{y}'(x) = \begin{pmatrix} -0.5 & 0.501 \\ 0.501 & -0.5 \end{pmatrix} \mathbf{y}(x)$, $x \in [0, 1]$, $\mathbf{y}(0) = \begin{pmatrix} 1.1 \\ -0.9 \end{pmatrix}$. (Here, $\mathbf{y} : [0, 1] \to \mathbb{R}^2$.)

Q 7 *Convergence, order of accuracy, and A-stability of LMMs*

We consider the linear 2-step method

$$y_{n+2} - y_n = h\left(-\gamma f_{n+2} + 2(1 + \gamma) f_{n+1} - \gamma f_n\right),$$

where $\gamma \in \mathbb{R}$ is some parameter.

(i) Show that the method is convergent for any $\gamma \in \mathbb{R}$.

(ii) For each $\gamma \in \mathbb{R}$, determine the order of accuracy of the method and the error constant.

(iii) Find all values of $\gamma \in \mathbb{R}$ for which the method is A-stable.