MA4255: Problem Sheet 1

AY 2022/23

Q 1 *Surrounding Picard's theorem*

(i) Show that the IVP $y'(x) = e^{-x^2} \arctan(y(x))$, $y(1) = 5$ has a unique continuously differentiable solution $y : [1, \infty) \to \mathbb{R}$.

(ii) Let $x_0, y_0 \in \mathbb{R}$ be fixed. Show that the IVP $y'(x) = 2(1 + e^{-|x|})\frac{y(x)}{1+(y(x))^2}$, $y(x_0) = y_0$ has a unique continuously differentiable solution $y : [x_0, \infty) \to \mathbb{R}$.

(iii) Let $m \in \mathbb{N}$ be fixed and write $d := \frac{2m}{2m+1}$. For any $X_M > 0$, show that the IVP $y'(x) = (y(x))^d$, $y(0) = 0$ has infinitely many continuously differentiable solutions defined on $[0, X_M]$. Show that this does not contradict Picard's theorem.

Q 2 *Error analysis of a one-step method*

We consider the IVP

$$y'(x) = f(x, y(x)) \quad \text{for } x \in (0, 1), \qquad y(0) = 0, \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, z) := \frac{1}{2\pi} \arctan(x) - \ln(1 + z^2)$. For a fixed $N \in \mathbb{N}_{\geq 2}$, we set $h := \frac{1}{N}$ and define $x_n := nh$ for $n \in \{0, 1, \ldots, N\}$. We consider the one-step method

$$y_{n+1} = y_n + h\Phi(x_n, y_n, h) \quad \text{for } n \in \{0, 1, \ldots, N-1\}, \qquad y_0 = 0, \tag{2}$$

where $\Phi : \mathbb{R} \times \mathbb{R} \times (0, 1) \to \mathbb{R}$, $\Phi(x, z, h) := \frac{1}{2} f(x, z) + \frac{1}{2} f(x + h, z + hf(x, z))$.

(i) Show that the IVP (1) has a unique continuously differentiable solution $y : [0, 1] \to \mathbb{R}$.

(ii) Write (2) as a Runge–Kutta method and define the corresponding consistency error T_n for $n \in \{0, 1, \ldots, N-1\}$. Show that $|T_n| = \mathcal{O}(h^2)$.

(iii) Assume that $C > 0$ is a known constant such that $\max_{n\in\{0,1,\ldots,N-1\}} |T_n| \leq Ch^2$. First, prove

$$|\Phi(x, z_1, h) - \Phi(x, z_2, h)| \leq \frac{3}{2}|z_1 - z_2| \qquad \forall (x, z_1, h), (x, z_2, h) \in \mathbb{R} \times \mathbb{R} \times (0, 1),$$

and use this result to show that

$$|e_n| \leq \frac{2C}{3}\left(e^{\frac{3}{2}} - 1\right) h^2 \quad \forall n \in \{0, 1, \ldots, N\}, \quad \text{where} \quad e_n := y(x_n) - y_n.$$

Find $N_0 \in \mathbb{N}$, expressed in terms of C, such that $\max_{n\in\{0,1,\ldots,N-1\}} |e_n| \leq 10^{-6}$ if $N \geq N_0$.

Q 3 *Accuracy of an implicit one-step method*

For $\alpha, \beta \in \mathbb{R}$, consider the one-step method

$$y_{n+1} = y_n + \frac{h}{3}\left(\alpha f(x_n, y_n) + \beta f(x_{n+1}, y_{n+1}) + \frac{h}{2} f_x(x_n, y_n) + \frac{h}{2} f(x_n, y_n) f_z(x_n, y_n)\right),$$

where f_x, f_z denote the first-order partial derivatives of $f = f(x, z)$. Show that there exist $\alpha, \beta \in \mathbb{R}$ such that the order of accuracy of this method is at least 3.
(Optional: Can $\alpha, \beta \in \mathbb{R}$ be chosen such that the method is fourth-order accurate?)

Q 4 *MATLAB: one-step methods in practice*

Write a MATLAB program which approximates the solution of the IVP $y'(x) = \sin(x^2)y(x)$ for $x \in [0, 10]$, $y(0) = 1$ (optional: show that this IVP has a unique solution) using

(i) the explicit Euler method with $h = \frac{1}{10}$,

(ii) the implicit Euler method with $h = \frac{1}{10}$,

(ii) the implicit midpoint rule with $h = \frac{1}{10}$.

(You may find it helpful to first perform one step of each of these methods by hand.)

Q 5 *Surrounding Runge–Kutta methods*

(i) Show that the explicit Euler, implicit Euler, and implicit midpoint rule methods are Runge–Kutta methods.

(ii) We have seen in the lectures that when a two-stage second-order explicit Runge–Kutta method is applied to the IVP $y'(x) = \lambda y(x)$, $y(0) = y_0$, where $\lambda < 0$ and $y_0 \neq 0$, we have that $y_n = (1 + \lambda h + \frac{1}{2}\lambda^2 h^2)^n y_0$ for $n \in \mathbb{N}_0$. Show that if $\lambda h \in (-2, 0)$, then

$$|y(x_n) - y_n| \leq \frac{1}{6}(-\lambda)^3 h^2 x_n |y_0| \qquad \forall n \in \mathbb{N}_0,$$

where $x_n := nh$ for $n \in \mathbb{N}_0$. (Hint: First, show that $|a^n - b^n| \leq n|a - b|$ for any $a, b \in [-1, 1]$ and $n \in \mathbb{N}_0$.)

Q 6 *Accuracy of Runge–Kutta methods*

(i) For $\alpha, \beta, \gamma \in \mathbb{R}$, consider the Runge–Kutta method $y_{n+1} = y_n + h\gamma k_3$ with

$$k_1 = f(x_n, y_n), \quad k_2 = f(x_n + h\alpha, y_n + h\alpha k_1), \quad k_3 = f(x_n + h\beta, y_n + h\beta k_2).$$

Show that there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that the order of accuracy of this method is at least 2.

(Optional: Can $\alpha, \beta, \gamma \in \mathbb{R}$ be chosen such that the method is third-order accurate?)

(ii) Consider the Runge–Kutta method $y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3)$ with

$$k_1 = f(x_n, y_n), \quad k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \quad k_3 = f(x_n + h, y_n - hk_1 + 2hk_2).$$

Show that the order of accuracy of this method is at least 3.