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DIMENSION OF KERNELS OF LINEAR OPERATORS†

By RONG-QING JIA, SHERMAN RIEMENSCHNEIDER, and ZUOWEI SHEN

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1. Introduction. The basic question addressed in this paper is one of expressing the dimension of the intersection of kernels of linear operators that arise naturally in multivariate approximation theory in terms of the more easily computable dimensions of some basic building blocks. The theory, as it has progressed, connects concepts arising in multivariate approximation theory with ideas from general algebra and algebraic geometry. We shall first describe the present setting for the problem and then describe its development and our motivation from the point of view of approximation theory.

Throughout this paper, G will denote a semigroup of commuting linear operators on a linear space S over a field k with the group operation taken as composition of linear operators. An important property for our study is the *s*-**dimensional additivity** of such a semigroup G. This concept appears, for example, in algebraic geometry as the additivity of the intersection index (see e.g. the book of Shafarevich [19, p. 185] and Section 3 below) and can be described as follows: From the two subsets of linear operators

$$F_1 = \{l_1, \ldots, l_j, \ldots, l_s\} \quad \text{and} \quad F_2 = \{l_1, \ldots, \tilde{l}_j, \ldots, l_s\}$$

in G, a new subset F is formed by

$$F := \{l_1, \ldots, \tilde{l}_j l_j, \ldots, l_s\}.$$

We say that G has *s*-**dimensional additivity** if for arbitrary F_1, F_2 and F as above:

$$\dim K(F) = \dim K(F_1) + \dim K(F_2),$$

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where, for any subset F^* of s operators from G, $K(F^*)$ is the intersection of kernel spaces

$$K(F^*) := \bigcap_{l \in F^*} \ker l, \qquad \ker l := \{f \in S : lf = 0\}. \tag{1.1}$$

The kernels of interest to us arise when an external structure is imposed on a subset of G through the structure of a finite index set. Let X be an index set with cardinality $|X| < \infty$. A matroid structure is imposed on X by a collection of "independent" subsets, $\mathcal{T}$, satisfying: i) The empty set is in $\mathcal{T}$. ii) If $V \in \mathcal{T}$, then any subset of V is in $\mathcal{T}$. iii) For arbitrary U, $V \in \mathcal{T}$ with $|U| = |V| + 1$, there exists $x \in U \backslash V$ such that $V \cup \{x\} \in \mathcal{T}$. (See, for example, the book of Welsh [22] for a detailed account of matroids.) Of course, there can be many matroid structures on a given X, but we shall simply say the matroid X to refer to X with some fixed matroid structure and only specify the structure if it has some importance. One exception is that for an arbitrary subset Y of a matroid X, we will always assume that the submatroid structure is imposed.

For any matroid there is a rank function, $\rho : 2^X \to \mathbf{Z}_+$, defined on subsets $V \subseteq X$ by

$$\rho(V) := \max\{|Y| : Y \subseteq V,\ Y \in \mathcal{T}\}.$$

A maximal independent subset of X is called a base for the matroid X. Every base of X has the same cardinality, $\rho(X)$, which is called the rank of X. We deal mainly with matroids of rank s (this number is connected with the s-dimensional additivity of G).

The collection $\mathcal{B}(X)$ of all bases for the matroid X is described as

$$\mathcal{B}(X) := \{B \subseteq X : |B| = \rho(B) = \rho(X)\}.$$

For a subcollection $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$, we define $\mathcal{A}(X, \mathcal{B}_1^{(X)})$ to be all the subsets of X which intersect all bases in $\mathcal{B}_1^{(X)}$; i.e.,

$$\mathcal{A}(X, \mathcal{B}_1^{(X)}) := \{V \subseteq X : V \cap B \neq \emptyset\ \forall B \in \mathcal{B}_1^{(X)}\}.$$

We are now in a position to describe the kernel spaces of interest: For a matroid X with rank $\rho(X) = s$, and the commutative semigroup

G of linear operators on S, we take L_X to be the image of some mapping $X \to G$ that associates a linear operator $l_x \in G$ to each $x \in X$. For any subset $V \subseteq X$, we set

$$L_V = \{l_x : x \in V\},$$

and define

$$l_V = \prod_{x \in V} l_x,$$

to be the composition (in any order) of the operators from L_V.

The main problem addressed in this paper is to describe the dimension of the kernel spaces

$$K(L_X, \mathcal{B}_1^{(X)}) := \{f \in S : l_V f = 0, \forall V \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\}, \tag{1.2}$$

in terms of the dimensions of the kernels $K(L_B)$ given in (1.1) for the s linear operators L_B, $B \in \mathcal{B}_1^{(X)}$.

How did such a question arise from approximation theory and why would its answer be interesting? Kernels of the type (1.2) appear in de Boor and Höllig's paper [3], the first one dealing extensively with the properties of box splines. Without going into details, for a given set of nonzero vectors X that span $\mathbf{R}^s$ (with the natural matroid structure on X), the box spline is a compactly supported piecewise polynomial function with the polynomial pieces from $D(X)$ which is the kernel in (1.2) with $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$ and the operators $l_x = D_x$, the directional derivative in the direction of x, $x \in X$. The dimension formula

$$\dim D(X) = |\mathcal{B}(X)| \tag{1.3}$$

was first shown by Dahmen and Micchelli [9]. Its importance is derived from the fact that when the directions X are in $\mathbf{Z}^s$, the polynomials in the linear space spanned by the integer translates of the box spline are precisely the functions in $D(X)$, and this plays an essential role in both de Boor and Höllig's and Dahmen and Micchelli's studies of the algebraic and approximation properties of such spaces ([3], & [8]–[10]).

There is a natural relation between the directional derivatives D_x and the difference operators $\nabla_x : f \mapsto f - f(\cdot - x)$, $x \in \mathbf{R}^s$. The

corresponding kernel space $\nabla(X)$ given as in (1.2) with $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$ and $l_x = \nabla_x$, and the formula ([10]) for its dimension

$$\dim \nabla(X) = \sum_{B \in \mathcal{B}(X)} |\det B|, \tag{1.4}$$

were also crucial in Dahmen and Micchelli's studies of the algebraic properties of box splines.

As a natural extension of the box spline, Ron [18] introduced the exponential box splines for a set of directions X and complex numbers $c_X = \{c_x\}_{x \in X}$. The kernels $D_{c_X}(X)$ and $\nabla_{c_X}(X)$ defined as in (1.2) for the differential operators $D_{x,c_x} := D_x - c_x$, $x \in X$, and the difference operators $\nabla_{x,c_x} : f \mapsto f - \exp(c_x) f(\cdot - x)$, $x \in X$, played the same type of essential role in the studies of exponential box splines by Dahmen and Micchelli [4, 5], Ron [18], and Ben-Artzi and Ron [1]. In particular, the dimension formulas for $D_{c_X}(X)$ and $\nabla_{c_X}(X)$ are the same as given in (1.3) and (1.4) respectively. This (nontrivial) extension precipitated two separate but related lines of study into the deeper algebraic ideas behind the box spline theory.

Dahmen and Micchelli realized in their investigations of exponential box splines [12] that the problem could be formulated for $\mathcal{B}(X)$ in terms of matroids and linear operators. Their results were announced previously in [11]. When $|X| = s$, the sums in (1.3) and (1.4) reduce to one summand and the definitions (1.1) and (1.2) agree. Therefore, both (1.3) and (1.4) are expressible as

$$\dim K(L_X, \mathcal{B}(X)) = \sum_{B \in \mathcal{B}(X)} \dim K(L_B). \tag{1.5}$$

Dahmen and Micchelli proved that the inequality

$$\dim K(L_X, \mathcal{B}(X)) \le \sum_{B \in \mathcal{B}(X)} \dim K(L_B) \tag{1.6}$$

always holds if X has a matroid structure [12, Theorem 3.1], and gave a sufficient condition [12, Theorem 3.3] for equality based on the solvability of certain systems of operator equations [12, Theorem 3.2].

In another paper [13], Dahmen and Micchelli investigated the dimensions of certain spline spaces and the relationship of these questions to syzygies and the kernels of systems of differential equations. In par-

ticular, they proved that if the linear operators in G are the partial differential operators given by homogeneous polynomials on $\mathbf{R}^s$, then (1.5) holds for $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$ in some very special cases and they conjectured that it would always hold. This conjecture initiated Shen's study that concluded with the following

(1.7) THEOREM. [20, Theorem 2.4] *The semigroup G has s-dimensional additivity if and only if for any matroid X of rank $\rho(X) = s$, and any $L_X \subseteq G$ associated with X,*

$$\dim K(L_X, \mathcal{B}(X)) = \sum_{B \in \mathcal{B}(X)} \dim K(L_B). \tag{1.8}$$

Along other lines, de Boor, Dyn, and Ron ([14], [15], [4]–[6], and [2]) were taking advantage of polynomial ideals, their varieties and codimensions to gain new insights and results for various problems in multivariate interpolation, approximation and spline theory. They also encountered dimension problems of the type considered here, usually in the context of partial differential operators given by polynomials, but sometimes free from any matroid structure. For example, in the case when G consists of differential operators given by affine polynomials, de Boor and Ron [5, Theorem 6.6] give the lower bound

$$\dim K(L_X, \mathcal{B}_1^{(X)}) \geq \left|\mathcal{B}_1^{(X)}\right|$$

for arbitrary $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$, in contrast to the upper bound in (1.6). They also prove that equality holds for these special operators if $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$ [5, Theorem 6.9].

The goal of this paper is to partially unite these two lines of study by extending the above theorem of Shen. In Section 2, we extend Shen's theorem to an order closed subset $\mathcal{B}_1^{(X)}$ of $\mathcal{B}(X)$; i.e., de Boor and Ron's theorem is extended to arbitrary linear operators from a semigroup G with s-dimensional additivity. It should be noted that although we have an underlying matroid structure, it is only the total order on X that matters (in fact, only the elements of X appearing in some $B \in \mathcal{B}_1^{(X)}$) since the trivial matroid structure given by the independent sets being all subsets of cardinality $\leq s$ can be used. In Section 3, we observe the s-dimensional additivity for partial differential operators given by polynomials while in Section 4 the s-dimensional additivity is observed for

difference operators given by polynomials. In both cases we give some examples where simple explicit formulas exist for dim $K(L_B)$.

2. Dimension formula for an order closed set. We wish to determine some conditions on $\mathcal{B}_1^{(X)}$ under which the dimension formula in Theorem 1.7 will still hold. For a matroid X, we say that a set of linear operators $L_X \subseteq G$ is **well associated** with $(X, \mathcal{B}_1^{(X)})$, if $\dim K(L_B) < \infty$ for any $B \in \mathcal{B}_1^{(X)}$. If, for any $L_X \subseteq G$, the formula

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B \in \mathcal{B}_1^{(X)}} \dim K(L_B)$$

holds, then G is said to be **excellently associated** with $(X, \mathcal{B}_1^{(X)})$.

Let us recall that $\mathcal{B}(X)$ is the collection of all bases for the matroid X. For a subcollection $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$, we define $\mathcal{A}(X, \mathcal{B}_1^{(X)})$ to be all the minimal subsets of X which intersect all bases in $\mathcal{B}_1^{(X)}$; i.e.,

$$\mathcal{A}(X, \mathcal{B}_1^{(X)}) := \{V \subseteq X : V \cap B \neq \emptyset \ \forall B \in \mathcal{B}_1^{(X)} \text{ and}$$

$$\forall y \in V \ \exists\, B \in \mathcal{B}_1^{(X)} \text{ such that } (V \backslash \{y\}) \cap B = \emptyset\}.$$

We also set

$$\mathcal{M}(X, \mathcal{B}_1^{(X)}) := \{X \backslash V : V \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\}.$$

Equivalently, $\mathcal{M}(X, \mathcal{B}_1^{(X)})$ is all the maximal subsets of X such that $X \backslash M$ intersect all $B \in \mathcal{B}_1^{(X)}$, or, the maximal subsets of X that do not contain any elements of $\mathcal{B}_1^{(X)}$. When $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$, $\mathcal{M}(X, \mathcal{B}_1^{(X)}) =: \mathcal{H}(X)$ is the set of "hyperplanes" for the matroid structure.

Note that we have changed the definition of $\mathcal{A}(X, \mathcal{B}_1^{(X)})$, but this does not affect the definition of the space $K(L_X, \mathcal{B}_1^{(X)})$.

In this section we are concerned with an order closed subset of $\mathcal{B}(X)$ as introduced by de Boor and Ron [5]. Suppose that a total order on X is given. This order induces a partial order on $\mathcal{B}(X)$;

$$B = (x_1, \ldots, x_s) \preceq \tilde{B} = (\tilde{x}_1, \ldots, \tilde{x}_s) \Leftrightarrow x_j \leq \tilde{x}_j, \qquad j = 1, \ldots, s,$$

where the elements of each sequence are arranged in an increasing order. We say that $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$ is an **order closed subset** of $\mathcal{B}(X)$, if

$$B_1 \in \mathcal{B}_1^{(X)}, \qquad B_2 \in \mathcal{B}(X), \qquad \text{and} \quad B_2 \leq B_1 \Rightarrow B_2 \in \mathcal{B}_1^{(X)}.$$

Our aim in this section is to extend Shen's Theorem 1.7 to an order closed subset of $\mathcal{B}(X)$. We shall prove that if $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$, and G is a semigroup of linear operators with s-dimensional additivity, then G is excellently associated with $(X, \mathcal{B}_1^{(X)})$. Our proof is based on the following theorem, which gives a solvability criterion on a system of operator equations.

(2.1) THEOREM. *Let $X \cup \zeta$ be a matroid with rank $\rho(X \cup \zeta) = s$ and let $\mathcal{B}_1^{(X\cup\zeta)} \subseteq \mathcal{B}(X \cup \zeta)$. Suppose that $L_{X\cup\zeta} \subseteq G$ and $M \in \mathcal{M}(X, \mathcal{B}_1^{(X)})$ are given and satisfy the following conditions*:

i) *L_X and $L_{M\cup\zeta}$ are well associated to $(X, \mathcal{B}_1^{(X)})$ and $(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$ respectively*;
ii) *G is excellently associated to $(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$*;
iii) *For any $B \in \mathcal{B}_1^{(M\cup\zeta)}$ and $y \in X\backslash M$, $(B\backslash\{\zeta\}) \cup \{y\} \in \mathcal{B}_1^{(X)}$.*

Then the system

$$\begin{aligned} l_{X\backslash M} f &= \varphi \\ l_V f &= 0 \ \forall V \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\backslash\{X\backslash M\} \end{aligned} \tag{2.2}$$

is solvable for any $\varphi \in K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$.

Proof. To each $x \in M \cup \zeta$ we associate two linear operators $\hat{l}_x$ and $\bar{l}_x$ as follows:

$$\hat{l}_x = \begin{cases} l_\zeta l_{X\backslash M}, & \text{if } x = \zeta \\ l_x, & \text{otherwise,} \end{cases}$$

and

$$\bar{l}_x = \begin{cases} l_{X\backslash M}, & \text{if } x = \zeta \\ l_x, & \text{otherwise.} \end{cases}$$

Then we set

$$\tilde{L}_{M\cup\zeta} := \{\tilde{l}_x : x \in M \cup \zeta\}$$

and

$$\overline{L}_{M\cup\zeta} := \{\bar{l}_x : x \in M \cup \zeta\}.$$

Both $\tilde{L}_{M\cup\zeta}$ and $\overline{L}_{M\cup\zeta}$ are well associated with $(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$. Thus, for any $V \subseteq M \cup \zeta$, we have

$$\tilde{l}_V = \begin{cases} l_V, & \text{if } \zeta \notin V; \\ l_{X\setminus M} l_V, & \text{if } \zeta \in V. \end{cases}$$

Hence, $l_{X\setminus M}$ maps $K(\tilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ into $K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ with $K(\overline{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ as its kernel. Moreover, since G has s-dimensional additivity and is excellently associated to $M \cup \zeta$, we have

$$\begin{aligned} \dim K(\tilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) &= \sum_{B\in\mathcal{B}_1^{(M\cup\zeta)}} \dim K(\tilde{L}_B) \\ &= \sum_{B\in\mathcal{B}_1^{(M\cup\zeta)}} \dim K(\overline{L}_B) + \sum_{B\in\mathcal{B}_1^{(M\cup\zeta)}} \dim K(L_B) \\ &= \dim K(\overline{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) + \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}). \end{aligned}$$

Hence $l_{X\setminus M}$ is surjective, since the dimension of $K(\tilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ is finite.

Now that the image of the mapping $l_{X\setminus M}$ is $K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$, for any given $\varphi \in K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$, we can find a function $f \in K(\tilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ such that

$$l_{X\setminus M} f = \varphi.$$

We claim that this f also satisfies

$$l_W f = 0 \qquad \forall W \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\setminus\{X\setminus M\}.$$

Let $W \in \mathscr{A}(X, \mathscr{B}_1^{(X)})\backslash\{X\backslash M\}$. Then there is $y \in X\backslash M$, such that $y \notin W$. Otherwise, $W = X\backslash M$ by the minimal property of W. We want to show that W intersects any base in $\mathscr{B}_1^{(M\cup\zeta)}$. For this purpose, we pick $B \in \mathscr{B}_1^{(M\cup\zeta)}$. Then $\zeta \in B$, and $\tilde{B} := (B\backslash\{\zeta\}) \cup \{y\} \in \mathscr{B}_1^{(X)}$ by iii). Therefore,

$$W \cap B = W \cap (B\backslash\{\zeta\}) = W \cap \tilde{B} \neq \emptyset.$$

This shows that W intersects any base in $\mathscr{B}_1^{(M\cup\zeta)}$. Therefore, by the very definition of $\mathscr{A}(M \cup \zeta, \mathscr{B}_1^{(M\cup\zeta)})$, W must contain some $V \in \mathscr{A}(M \cup \zeta, \mathscr{B}_1^{(M\cup\zeta)})$. Since $\zeta \notin W$, we have $\zeta \notin V$, hence $l_V = \tilde{l}_V$. Now, $f \in K(\tilde{L}_{M\cup\zeta}, \mathscr{B}_1^{(M\cup\zeta)})$ and $V \in \mathscr{A}(M \cup \zeta, \mathscr{B}_1^{(M\cup\zeta)})$ imply

$$l_V f = \tilde{l}_V f = 0.$$

It follows that

$$l_W f = l_{(\cdots)} l_V f = 0.$$

Therefore, for the given φ, the system (2.2) is solvable. □

We will see that the hypothesis iii) of Theorem 2.1 is satisfied when $\mathscr{B}_1^{(X)}$ is order closed, but the theorem's proof did not depend on this property. Next we use the last theorem to prove the main result of this section.

(2.3) THEOREM. *Let X be a matroid with rank $\rho(X) = s$ and $\mathscr{B}_1^{(X)}$ be an order closed subset of $\mathscr{B}(X)$. If G is a semigroup of linear operators on S with s-dimensional additivity, then, for arbitrary $L_X \subseteq G$,*

$$\dim K(L_X, \mathscr{B}_1^{(X)}) = \sum_{B\in\mathscr{B}_1^{(X)}} \dim K(L_B).$$

Proof. If L_X is not well associated with $(X, \mathscr{B}_1^{(X)})$, then the equality holds simply because there exists a $B \in \mathscr{B}_1^{(X)}$ such that $\dim K(L_B) = \infty$ and $K(L_B) \subseteq K(L_X, \mathscr{B}_1^{(X)})$.

For the case that L_X is well associated with $(X, \mathscr{B}_1^{(X)})$, we will prove the equality by induction on $|X|$. When $\rho(X) = |X| = s$, the theorem is obviously true. Suppose now that the theorem holds for all X with s

$\leq |X| \leq n$ and $\rho(X) = s$. We want to establish it for $(Y, \mathcal{B}_1^{\{Y\}})$ with $|Y| = n + 1$, where $\mathcal{B}_1^{\{Y\}}$ is an order closed subset of $\mathcal{B}(Y)$ and $\rho(Y) = s$. Let

$$\zeta := \sup\{y \in Y : \rho(Y\backslash\{y\}) = s\}.$$

We write X for $Y\backslash\zeta$. Then $Y = X \cup \zeta$.

Let $\mathcal{P}$ be the mapping given by

$$\mathcal{P}f = (l_{X\backslash M} f)_{M \in \mathcal{M}(X, \mathcal{B}_1^{\{X\}})}, \qquad f \in K(L_{X\cup\zeta}, \mathcal{B}_1^{\{X\cup\zeta\}}).$$

Since the mapping $l_{X\backslash M}$ maps $K(L_{X\cup\zeta}, \mathcal{B}_1^{\{X\cup\zeta\}})$ to $K(L_{M\cup\zeta}, \mathcal{B}_1^{\{M\cup\zeta\}})$, the mapping $\mathcal{P}$ maps $K(L_{X\cup\zeta}, \mathcal{B}_1^{\{X\cup\zeta\}})$ to the Cartesian product

$$\prod_{M \in \mathcal{M}(X, \mathcal{B}_1^{\{X\}})} K(L_{M\cup\zeta}, \mathcal{B}_1^{\{M\cup\zeta\}}).$$

Note that $K(L_{M\cup\zeta}, \mathcal{B}_1^{\{M\cup\zeta\}}) = 0$ if $\mathcal{B}_1^{\{M\cup\zeta\}} = \emptyset$.

Observe that $\ker(\mathcal{P}) = K(L_X, \mathcal{B}_1^{\{X\}})$. Therefore,

$$\begin{aligned}(2.4)\quad &\dim K(L_{X\cup\zeta}, \mathcal{B}_1^{\{X\cup\zeta\}}) \\ &\leq \dim K(L_X, \mathcal{B}_1^{\{X\}}) + \sum_{M \in \mathcal{M}(X, \mathcal{B}_1^{\{X\}})} \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{\{M\cup\zeta\}}).\end{aligned}$$

Equality will hold in (2.4) if the mapping $\mathcal{P}$ is surjective. Before showing this, we pick out the nontrivial components in the image of $\mathcal{P}$: We claim

$$M \in \mathcal{M}(X, \mathcal{B}_1^{\{X\}}) \quad \text{and} \quad \mathcal{B}_1^{\{M\cup\zeta\}} \neq \emptyset \Rightarrow M \in \mathcal{H}(X).$$

For this purpose, we shall show $\rho(M) < s$. Suppose to the contrary that $\rho(M) = s$. Since $\mathcal{B}_1^{\{M\cup\zeta\}} \neq \emptyset$, there exists a base $B \in \mathcal{B}_1^{\{M\cup\zeta\}}$; it follows that $\zeta \in B$ and $\rho(B\backslash\zeta) = s - 1$. But $\rho(M) = s$; hence there exists some $y \in M$ such that $\rho((B\backslash\zeta) \cup y) = s$. By the very definition of ζ, we have $y < \zeta$. Therefore, $B_1 := (B\backslash\zeta) \cup y \in \mathcal{B}_1^{\{X\cup\zeta\}}$, because $\mathcal{B}_1^{\{X\cup\zeta\}}$ is order closed. Thus, M would contain a base B_1 in $\mathcal{B}_1^{\{X\}}$. This contradicts the choice of M. Hence, $\rho(M) < s$, so $M \subseteq H$ for some $H \in \mathcal{H}(X)$. But $(X\backslash H) \cap B \neq \emptyset$, for any $B \in \mathcal{B}(X)$; it follows that $H = M$ by the

maximality of M. In particular, this implies that the following union is a disjoint union:

$$\mathcal{B}_1^{(X\cup\zeta)} = \mathcal{B}_1^{(X)} \cup \left(\cup_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \mathcal{B}_1^{(M\cup\zeta)}\right).$$

This fact will be used in (2.6) below.

The mapping $\mathcal{P}$ is surjective if and only if for each $M \in \mathcal{M}(X, \mathcal{B}_1^{(X)})$ with $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$, the system (2.2) is solvable for any $\varphi \in K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$. Thus, it suffices to prove that Theorem 2.1 can be applied here. Condition i) of Theorem 2.1 holds since we have already restricted ourselves to the well associated case. Condition ii) holds by the induction hypothesis since $|M \cup \zeta| \leq |X|$. Condition iii) holds because of the following reason. Suppose that $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$. Let $B \in \mathcal{B}_1^{(M\cup\zeta)}$ and $y \in X\backslash M$. Note that $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$ implies $\rho(M \cup \zeta) = s$. Thus, if $y \in X\backslash M$, then $\rho(X \cup \zeta\backslash y) \geq \rho(M \cup \zeta) = s$. Hence, by the choice of ζ, we have $y < \zeta$. Since $\mathcal{B}_1^{(X\cup\zeta)}$ is order closed, for any $B \in \mathcal{B}_1^{(M\cup\zeta)}$ and $y \in X\backslash M$, we have $\tilde{B} := (B\backslash\zeta) \cup y \in \mathcal{B}_1^{(X)}$. Therefore, Theorem 2.1 can be applied and equality holds in (2.4):

$$\text{(2.5)}\quad \dim K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)}) = \dim K(L_X, \mathcal{B}_1^{(X)}) + \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}).$$

Finally, applying the induction hypothesis to X and to each $M \cup \zeta$, $M \in \mathcal{M}(X, \mathcal{B}_1^{(X)})$, we obtain

$$\begin{aligned}\text{(2.6)}\quad &\dim K(L_X, \mathcal{B}_1^{(X)}) + \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) \\ &= \sum_{B\in\mathcal{B}_1^{(X)}} \dim K(L_B) + \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \sum_{B\in\mathcal{B}_1^{(M\cup\zeta)}} \dim K(L_B) \\ &= \sum_{B\in\mathcal{B}_1^{(X\cup\zeta)}} \dim K(L_B).\end{aligned}$$

Substituting this into (2.5) gives the desired result:

$$\dim K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)}) = \sum_{B\in\mathcal{B}_1^{(X\cup\zeta)}} \dim K(L_B). \qquad \square$$

It should be remarked that the proof in [20, Theorem 2.4] shows that the s-dimensional additivity of G is a necessary condition for Theorem 2.3 to hold for arbitrary matroids of rank s.

3. Application to constant coefficient differential operators. The first part of this section consists of background material from algebraic geometry that is essentially known but not well known to our intended audience. It is meant to give a concise account of the material that is required for our purposes with appropriate references for the details.

Let k be an algebraically closed field, and k^s the s-dimensional affine space over k. Denote the ring of polynomials in s indeterminates over the field k by $k[Z] = k[Z_1, \ldots, Z_s]$. The ideal generated by $p_1, \ldots, p_m \in k[Z]$ will be denoted by $(p_1, \ldots, p_m)$. The codimension of an ideal I, denoted by $\operatorname{codim}(I)$, is the dimension of the quotient linear space $k[Z]/I$ over k.

For a multi-index $\alpha \in \mathbf{N}^s$, the formal differential operator D^α on $k[Z]$ is defined by

$$D^\alpha Z^\beta := \frac{\beta!}{(\beta - \alpha)!} Z^{\beta - \alpha}.$$

Here we make the convention that $Z^{\beta-\alpha} = 0$, if $\beta_j < \alpha_j$ for some j. For a polynomial $p(Z) = \sum_\alpha a_\alpha Z^\alpha$, the corresponding differential operator $p(D)$ is defined by $p(D) := \sum_\alpha a_\alpha D^\alpha$.

An ideal I of $k[Z]$ determines its variety

$$\operatorname{Var}(I) := \{a \in k^s : p(a) = 0 \ \forall p \in I\}.$$

Such a variety is a finite irredundant union of irreducible varieties (e.g., see [16, p. 122]). Suppose that $\theta = (\theta_1, \ldots, \theta_s)$ is an isolated zero of I; that is, $\{\theta\}$ is one of the irreducible components of $\operatorname{Var}(I)$. Let

$$I = \bigcap_{j=1}^{n} I_j$$

be a reduced primary decomposition, where I_j is P_j-primary for $j = 1, \ldots, n$, and the prime ideals $P_1, \ldots, P_n$ are all different. To the

component $\{\theta\}$ of $\mathrm{Var}(I)$ there corresponds one prime ideal, say P_1, such that $P_1 = (Z_1 - \theta_1, \ldots, Z_s - \theta_s)$. Then we have

$$\theta \notin \mathrm{Var}(P_j) = \mathrm{Var}(I_j), \qquad j = 2, \ldots, n,$$

for otherwise $\{\theta\}$ would not be a component of $\mathrm{Var}(I)$. In what follows we write I_θ for I_1.

The set

$$S_\theta := \{g \in k[Z] : g(\theta) \neq 0\}$$

is a multiplicative set of $k[Z] =: R$. Let $\mathcal{O}_\theta := S_\theta^{-1}R$ be the quotient ring of R by S_θ (the **localization** of R at S_θ); i.e.,

$$\mathcal{O}_\theta = \{f/g : f, g \in k[Z], g(\theta) \neq 0\}.$$

Thus, $\mathcal{O}_\theta$ is the local ring of the point θ (e.g. see [19, Chapter 2]). If I is an ideal of R, then $S_\theta^{-1}I$ is an ideal of $\mathcal{O}_\theta$.

LEMMA (3.1). $S_\theta^{-1}I = S_\theta^{-1}I_\theta$.

Proof. Since $\theta \notin \mathrm{Var}(I_j)$ for $j > 1$, we can find a polynomial $f_j \in k[Z]$ such that f_j vanishes on $\mathrm{Var}(I_j)$ but $f_j(\theta) \neq 0$. By Hilbert's Nullstellensatz, $f_j^m \in I_j$ for some positive integer m. Thus, we have

$$f_j^m \in S_\theta \cap I_j.$$

It follows that

$$1 \in S_\theta^{-1}I_j, \qquad j = 2, \ldots, n.$$

This gives

$$S_\theta^{-1}I = \cap_{j=1}^{n} (S_\theta^{-1}I_j) = S_\theta^{-1}I_\theta,$$

as desired. □

Let I be an ideal of $k[Z_1, \ldots, Z_s]$. If θ is an isolated zero of I, we define the **intersection index** of I at θ as follows:

$$\mathrm{ind}_\theta(I) := \dim(\mathcal{O}_\theta/(S_\theta^{-1}I)).$$

If θ is an isolated zero of I, then we also can talk about the multiplicity of I at θ. Let

$$P_{I,\theta} := \{p \in k[Z] : p(D)f(\theta) = 0, \forall f \in I\}.$$

The space $P_{I,\theta}$ is finite dimensional and is called the **multiplicity space** of I at θ. The dimension of $P_{I,\theta}$ is called the **multiplicity** of I at θ. It is known that (e.g. see [5, (3.12) Proposition])

$$I_\theta = \{f \in k[Z] : p(D)f(\theta) = 0, \forall p \in P_{I,\theta}\}.$$

The following theorem shows that the multiplicity of I at θ is just the intersection index of I at θ.

THEOREM (3.2). *Let I be an ideal of $k[Z_1, \ldots, Z_s]$. If θ is an isolated zero of I, then*

$$\dim(\mathbb{O}_\theta/(S_\theta^{-1}I)) = \dim P_{I,\theta}.$$

Proof. For $g \in \mathbb{O}_\theta$, the residue class of g in $\mathbb{O}_\theta/(S_\theta^{-1}I)$ will be denoted by $\overline{g}$. Consider the following bilinear function between $P_{I,\theta}$ and $\mathbb{O}_\theta/(S_\theta^{-1}I)$:

$$\langle p, \overline{g}\rangle := p(D)g(\theta).$$

This is well defined on $\mathbb{O}_\theta/(S_\theta^{-1}I)$, because for any $p \in P_{I,\theta}$ and $f \in S_\theta^{-1}I$, $p(D)f(\theta) = 0$. We want to show that the bilinear function is actually a scalar product. To this end, suppose that $\langle p, \overline{g}\rangle = 0$ for all $\overline{g} \in \mathbb{O}_\theta/(S_\theta^{-1}I)$. Then for any $f \in k[Z]$,

$$p(D)f(\theta) = \langle p, \overline{f}\rangle = 0.$$

It follows that $p = 0$.

On the other hand, suppose that $\langle p, \overline{g}\rangle = 0$ for all $p \in P_{I,\theta}$. Let $g = f/h$, with $f, h \in k[Z]$ and $h(\theta) \neq 0$. Since $P_{I,\theta}$ is D-invariant, by Leibnitz' formula, we have

$$p(D)f(\theta) = p(D)(gh)(\theta) = 0, \qquad \forall p \in P_{I,\theta}.$$

Hence, $f \in I_\theta$. This gives

$$g = \frac{f}{h} \in S_\theta^{-1} I_\theta = S_\theta^{-1} I;$$

i.e., $\bar{g} = 0$.

Since $\langle \quad , \quad \rangle$ is a scalar product between $P_{I,\theta}$ and $\mathcal{O}_\theta/(S_\theta^{-1} I)$, these two spaces must have the same dimension. □

The following additivity theorem plays an essential role in our study of kernels of differential and difference operators.

THEOREM (3.3). (Additivity). *If θ is an isolated common zero of $f_1, \ldots, f_s$, and if f_s is the product of two polynomials, $f_s = f_s' f_s''$, then*

$$\operatorname{ind}_\theta(f_1, \ldots, f_{s-1}, f_s)$$

$$= \operatorname{ind}_\theta(f_1, \ldots, f_{s-1}, f_s') + \operatorname{ind}_\theta(f_1, \ldots, f_{s-1}, f_s'').$$

Proof. This theorem can be proved as follows (see [19, Chapter IV, Section 1.3]). We denote the ring $\mathcal{O}_\theta/(S_\theta^{-1}(f_1, \ldots, f_{s-1}))$ by $\overline{\mathcal{O}}$, and the images of f_s' and f_s'' in $\overline{\mathcal{O}}$ under the canonical homomorphism by f' and f''. Then

$$\operatorname{ind}_\theta(f_1, \ldots, f_{s-1}, f_s) = \dim(\overline{\mathcal{O}}/(f'f'')),$$

$$\operatorname{ind}_\theta(f_1, \ldots, f_{s-1}, f_s') = \dim(\overline{\mathcal{O}}/(f')),$$

$$\operatorname{ind}_\theta(f_1, \ldots, f_{s-1}, f_s'') = \dim(\overline{\mathcal{O}}/(f'')).$$

Since the sequence

$$0 \to (f'')/(f'f'') \to \overline{\mathcal{O}}/(f'f'') \to \overline{\mathcal{O}}/(f'') \to 0$$

is exact, we have

$$\dim(\overline{\mathcal{O}}/(f'f'')) = \dim(\overline{\mathcal{O}}/(f'')) + \dim((f'')/(f'f'')).$$

It can be shown that f'' is not a zero divisor in $\overline{\mathcal{O}}$ (see [19, Chapter IV, Section 1.3, Lemma 1 and Lemma 2]). Then the homomorphism from

$\overline{\mathcal{O}}$ to $(f'')/(f'f'')$ given by $f \mapsto ff'' + (f'f'')$ is surjective and has (f') as its kernel. Hence, $\dim((f'')/(f'f'')) = \dim(\overline{\mathcal{O}}/(f'))$. □

The result above combined with those of Section 2 can be used to gain information about the kernels of linear partial differential operators defined on the ring of formal power series in s indeterminates, $k[[Z_1, \ldots, Z_s]] =: k[[Z]]$. We take the commutative semigroup G of linear operators to be the partial differential operators

$$G_{k[Z]}(D) := \{p(D) : p \in k[Z_1, \ldots, Z_s]\}$$

defined in the usual way. For polynomials $p_1, \ldots, p_m$, there is a relationship between the dimension of the kernel space,

$$K_{(p_1,\ldots,p_m)}(D) := \{f \in k[[Z]] : p_1(D)f = 0, \ldots, p_m(D)f = 0\},$$

and the cardinality of the variety $\mathrm{Var}(p_1, \ldots, p_m)$; namely,

$$\dim K_{(p_1,\ldots,p_m)}(D) < \infty \Leftrightarrow |\mathrm{Var}(p_1, \ldots, p_m)| < \infty,$$

see [13, Proposition 2.1] and [5, Corollary 3.21]. In fact, [5] gives an explicit formula

$$\dim K_{(p_1,\ldots,p_m)}(D) = \mathrm{codim}(p_1, \ldots, p_m) = \sum_{\theta \in \mathrm{Var}(p_1,\ldots,p_m)} \mathrm{ind}_\theta(p_1, \ldots, p_m). \tag{3.4}$$

These results were proved for $\mathbf{C}$, but hold equally well for any algebraically closed field k (the exponential function is defined by its formal power series).

As an immediate consequence of Theorem 3.3 and (3.4), we have

COROLLARY (3.5). *If $p_s = p_s' p_s''$, then the kernel spaces for the ideals*

$$I = (p_1, \ldots, p_{s-1}, p_s), \qquad I' = (p_1, \ldots, p_{s-1}, p_s'),$$

$$\text{and} \quad I'' = (p_1, \ldots, p_{s-1}, p_s''),$$

satisfy the relation

$$\dim(K_I(D)) = \dim(K_{I'}(D)) + \dim(K_{I''}(D)).$$

In particular, $G_{k[Z]}(D)$ is a semigroup of linear operators with s-dimensional additivity.

For any matroid X and a collection of bases $\mathfrak{B}_1^{(X)}$, we can consider the kernel spaces (1.2) for arbitrary operators, $L_X \subset G_{k[Z]}(D)$, associated with X. By Corollary 3.5 and Theorem 2.3, we have

THEOREM (3.6). *If the matroid X has rank $\rho(X) = s$ and $\mathfrak{B}_1^{(X)}$ is an order closed subset of $\mathfrak{B}(X)$, then for $L_X \subset G_{k[Z]}(D)$,*

$$\dim K(L_X, \mathfrak{B}_1^{(X)}) = \sum_{B \in \mathfrak{B}_1^{(X)}} \dim K(L_B).$$

We now give a formula for the intersection index for some special polynomial ideals. Let $(p_1, \ldots, p_s)$ be the ideal generated by the homogeneous polynomials $p_1, \ldots, p_s \in k[Z_1, \ldots, Z_s]$. If zero is the only common zero of $p_1, \ldots, p_s$, then the intersection index of $p_1, \ldots, p_s$ at 0 is just $\operatorname{codim}(p_1, \ldots, p_s)$. Moreover,

$$\operatorname{codim}(p_1, \ldots, p_s) = \prod_{i=1}^{s} \deg p_i.$$

When $k = \mathbf{C}$, the complex field, Stiller [21] established a formula for $\operatorname{codim}(p_1, \ldots, p_s)$, while Dahmen and Micchelli [13] pointed out that his formula could be written in the above form. However, even for an algebraically closed field k, this result was known, see [19, p. 200].

This result can be extended a little further. Let $f_1, \ldots, f_s$ be polynomials in $k[Z]$, which are not necessarily homogeneous. Then each f_i can be written as a sum of its homogeneous components

$$f_i = \sum_{j=0}^{n_i} f_{i,j}, \qquad n_i = \deg f_i,$$

where $f_{i,j}$ is the homogeneous component of degree $n_i - j$ of f_i.

THEOREM (3.7). *If* $\operatorname{codim}(f_1, \ldots, f_s) < \infty$, *and if zero is the only common zero of the polynomials* $f_{1,0}, \ldots, f_{s,0}$, *then*

$$\operatorname{codim}(f_1, \ldots, f_s) = \prod_{i=1}^{s} \deg f_i.$$

Proof. Let $\tilde{f}_i$ be the homogenization of f_i at Z_0 for each $i = 1, \ldots, s$,

$$\tilde{f}_i(Z_0, Z_1, \ldots, Z_s) := \sum_{j=0}^{n_i} Z_0^j f_{i,j}(Z_1, \ldots, Z_s).$$

If $(a_0, a_1, \ldots, a_s) \in k^{s+1}\backslash\{0\}$ is a common zero of $\tilde{f}_1, \ldots, \tilde{f}_s$, then $a_0 \neq 0$; for otherwise, $f_{1,0}, \ldots, f_{s,0}$ would have a common zero other than zero. Since $\operatorname{codim}(f_1, \ldots, f_s) < \infty$, $\tilde{f}_1, \ldots, \tilde{f}_s$ have finitely many common zeros in $\mathbf{P}^s(k)$, the s-dimensional projective space over k, and the sum of the multiplicities of these zeros is $\Pi_{i=1}^s \deg \tilde{f}_i$ (see [19, p. 200]). Since any common zero $(a_0, a_1, \ldots, a_s)$ of $\tilde{f}_1, \ldots, \tilde{f}_s$ has $a_0 \neq 0$, hence all these common zeros are in the affine space k^s. We conclude that $\operatorname{codim}(f_1, \ldots, f_s) = \Pi_{i=1}^s \deg f_i$. □

The following result extends the result of Shen [20], who first confirmed the conjecture of Dahmen and Micchelli.

COROLLARY (3.8). *Under the assumption of Theorem* 3.6, *if the hypothesis of Theorem* 3.7 *is satisfied for* $\{p_b\}_{b\in B}$, $\forall\ B \in \mathcal{B}_1^{\{X\}}$, *then*

$$\dim K(L_X, \mathcal{B}_1^{\{X\}}) = \sum_{B\in\mathcal{B}_1^{\{X\}}} \prod_{b\in B} \deg p_b.$$

Another case in which an explicit formula can be given is discussed in the following example.

Example (3.9). Consider the semigroup $G_\Pi(D) \subset G_{k[Z]}(D)$ defined by products of linear polynomials in $k[Z]$; i.e.,

$$G_\Pi(D) := \left\{ p(D) : p(Z) = \prod_{j=1}^{m} (\lambda_j \cdot Z - c_j),\ \lambda_j \in k^s, \right.$$

$$\left. c_j \in k,\ j = 1, \ldots, m,\ m = m(p) \right\},$$

where $\lambda \cdot Z := \lambda(1)Z_1 + \cdots + \lambda(s)Z_s$ for $\lambda = (\lambda(1), \ldots, \lambda(s)) \in k^s$ and $Z = (Z_1, \ldots, Z_s)$. Let X be a matroid and $L_X \subset G_\Pi(D)$ be the associated operators. For each polynomial

$$p_b(Z) = \prod_{j=1}^{m_b} (\lambda_{b,j} \cdot Z - c_{b,j}), \qquad b \in X,$$

there is a corresponding set of elements from k^s given by

$$\Lambda_b := \{\lambda_{b,j} : j = 1, \ldots, m_b\}.$$

For any $B \in \mathcal{B}(X)$, we consider Λ_B to be the set of all possible matrices in $k^{s \times s}$ with columns indexed by $b \in B$ and with the bth column chosen from Λ_b. Let Ω_B be the set of all matrices in Λ_B of rank $< s$.

COROLLARY (3.10). *If the set $L_X \subset G_\Pi(D)$ is well associated to the matroid X, $\rho(X) = s$, and $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$, then*

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B \in \mathcal{B}_1^{(X)}} [\prod_{b \in B} \deg p_b - |\Omega_B|].$$

Proof. By Theorem 3.6, we only have to show that

$$\dim K(L_B) = [\prod_{b \in B} \deg p_b - |\Omega_B|], \qquad \forall\, B \in \mathcal{B}_1^{(X)}.$$

From given $B \in \mathcal{B}(X)$ and the associated L_B, we construct a matroid Y_B and a corresponding set of operators $\tilde{L}_{Y_B}$. The matroid Y_B consists of the elements b taken $\deg p_b$ times with the notion of independence inherited from B. For the operators $\tilde{L}_{Y_B}$, we take any one-to-one correspondence between $\{y \in Y_B : y = b\}$ and the linear factors of p_b. Thus, a basis $W \in \mathcal{B}(Y_B)$ corresponds to a selection of linear factors from p_b, $b \in B$, i.e. to a matrix $\Lambda(W) = [\lambda_{b,j(W)}]_{b \in B} \in \Lambda_B$, and the operators $\tilde{L}_W$ are just the linear partial differential operators $\{(\lambda_{b,j(W)} \cdot D - c_{b,j(W)})\}$.

With the above construction, $K(L_B) = K(\tilde{L}_{Y_B})$ and

$$\dim k(\tilde{L}_W) = \begin{cases} 1, & \text{if } \Lambda(W) \in \Lambda_B \backslash \Omega_B; \\ 0, & \text{if } \Lambda(W) \in \Omega_B. \end{cases}$$

Therefore, by Theorem 3.6 once again,

$$\dim K(L_B) = \dim K(\tilde{L}_{Y_B}) = \sum_{W \in \mathfrak{R}(Y_B)} \dim K(\tilde{L}_W)$$
$$= \prod_{b \in B} \deg p_b - |\Omega_B|. \qquad \square$$

When $k = \mathbf{C}$, $X = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbf{C}^{s \times n}$ has nonzero columns, and $L_X = \{D_\lambda - c_\lambda, \lambda \in X\}$, then Corollary 3.10 contains the result of de Boor and Ron [5, Theorem 6.9].

4. Application to linear difference operators. Let $\mathbf{Z}$ be the set of integers and s be a positive integer. As before k is an algebraically closed field. A mapping from $\mathbf{Z}^s$ to k is called an s-variate k-sequence, and we denote the linear space of all s-variate k-sequences by A. We wish to consider translation operators on A. These can be best described using the primitive translation operators τ_j given by

$$\tau_j f = f(\cdot + e_j), \qquad j = 1, \ldots, s, \qquad \text{for} \quad f \in A,$$

where

$$e_j = (0, \ldots, \underset{j\text{th}}{1}, \ldots, 0),$$

$j = 1, \ldots, s$, are the canonical unit vectors in $\mathbf{Z}^s$. For a multiindex $\alpha \in \mathbf{N}^s$, we define

$$\tau^\alpha := \tau_1^{\alpha_1} \cdots \tau_s^{\alpha_s}.$$

For a polynomial $p \in k[Z_1, \ldots, Z_s]$, $p(Z) = \sum a_\alpha Z^\alpha$, there is a corresponding translation operator

$$p(\tau) := \sum a_\alpha \tau^\alpha.$$

Similarly, for $\theta \in k^s$, we define

$$p(\theta\tau) := \sum a_\alpha \theta^\alpha \tau^\alpha.$$

If $q \in k[Z_1, \ldots, Z_s]$, then the sequence given by $\beta \mapsto q(\beta)$, $\beta \in \mathbf{Z}^s$ will also be simply denoted by q. Similarly, if Q is a subspace of $k[Z_1, \ldots, Z_s]$, then the sequence space $\{\beta \mapsto q(\beta) : q \in Q\}$ will again be denoted by Q. For any pair of polynomials $p, q \in k[Z]$, the notation $p(\tau)q$ means the sequence obtained by applying the difference operator $p(\tau)$ to the sequence $q : \beta \mapsto q(\beta)$, $\beta \in \mathbf{Z}^s$.

Given an ideal I of $k[Z]$, the kernel space $K_I(\tau)$ of all the difference operators $p(\tau)$, $p \in I$, is defined by

$$K_I(\tau) := \{f \in A : p(\tau)f = 0 \ \forall p \in I\}.$$

We wish to single out some special elements in $K_I(\tau)$. Let

$$(k\backslash\{0\})^s = \{(a_1, \ldots, a_s) \in k^s : a_1 \neq 0, \ldots, a_s \neq 0\}.$$

For any $\theta \in (k\backslash\{0\})^s$, we denote by $\theta^{(\ )}$ the sequence given by $\beta \mapsto \theta^\beta$, $\beta \in \mathbf{Z}^s$. It follows from the definition of $K_I(\tau)$ that

$$\theta^{(\ )} \in K_I(\tau) \Leftrightarrow \theta \in \mathrm{Var}(I).$$

THEOREM (4.1). *The dimension of $K_I(\tau)$ is finite if and only if the set $\mathrm{Var}(I) \cap (k\backslash\{0\})^s$ is finite. Moreover, in this case, to each $\theta \in \mathrm{Var}(I) \cap (k\backslash\{0\})^s$, there corresponds a translation invariant space $Q_{I,\theta}$ of polynomials such that*

$$K_I(\tau) = \bigoplus_{\theta \in \mathrm{Var}(I) \cap (k\backslash\{0\})^s} \theta^{(\ )} Q_{I,\theta}.$$

Proof. When $k = \mathbf{C}$, the proof of this theorem can be found from [17], [7], and [6]. Their proofs can be easily carried over for the case when k is an algebraically closed field. Let us find the spaces $Q_{I,\theta}$ explicitly. Observe that for $\alpha \in \mathbf{N}^s$ and $q \in A$, we have

$$\tau^\alpha(\theta^{(\ )}q) = \theta^{(\ )}(\theta^\alpha \tau^\alpha q).$$

It follows that for any $f \in I$,

$$f(\tau)(\theta^{(\ )}q) = \theta^{(\ )}(f(\theta\tau)q).$$

Thus,

$$\theta^{(\ )}q \in K_I(\tau) \Leftrightarrow f(\theta\tau)q = 0, \ \forall f \in I.$$

This shows that

$$Q_{I,\theta} = \{q \in k[Z] : f(\theta\tau)q = 0, \ \forall f \in I\}. \tag{4.2}$$ □

Recall that if θ is an isolated zero of I, then the multiplicity space, $P_{I,\theta}$, of I at θ is defined as

$$P_{I,\theta} := \{p \in k[Z] : p(D)f(\theta) = 0, \ \forall f \in I\}.$$

The following theorem shows that the spaces $Q_{I,\theta}$ and $P_{I,\theta}$ have the same dimension.

THEOREM (4.3). $\dim(Q_{I,\theta}) = \dim(P_{I,\theta})$.

Proof. To prove that $\dim(Q_{I,\theta}) = \dim(P_{I,\theta})$, it suffices to establish a linear isomorphism between the spaces $P_{I,\theta}$ and $Q_{I,\theta}$. For this purpose, we introduce

$$[Z]^\beta := [Z_1]^{\beta_1} \cdots [Z_s]^{\beta_s}, \qquad \beta \in \mathbf{N}^s,$$

with

$$[Z_j]^{\beta_j} := Z_j(Z_j - 1) \cdots (Z_j - \beta_j + 1).$$

Let Δ_j be the jth forward difference operator:

$$\Delta_j q := q(\cdot + e_j) - q, \qquad q \in A.$$

It is easy to verify that

$$\Delta^\alpha[\ \ ]^\beta = \frac{\beta!}{(\beta - \alpha)!}[\ \ ]^{\beta-\alpha}.$$

Here we make the convention that $[\ \ ]^{\beta-\alpha} = 0$ if $\beta_j < \alpha_j$ for some j.

To each $p \in k[Z]$, $p(Z) = \Sigma_\beta \, b_\beta Z^\beta$, we associate $q(Z) = \Sigma_\beta \, b_\beta \theta^{-\beta}[Z]^\beta$. The mapping $\sigma : p \mapsto q$ is a linear automorphism on $k[Z]$.

We want to show that σ is an isomorphism from $P_{I,\theta}$ to $Q_{I,\theta}$. For this purpose, we compute $p(D)f(\theta)$ and $f(\theta\tau)q$ as follows. Suppose $f(Z) = \Sigma_\alpha\, a_\alpha(Z - \theta)^\alpha$. Then

$$p(D)f(\theta) = \sum_\alpha a_\alpha b_\alpha \alpha!,$$

and

$$f(\theta\tau)q(Z) = (\sum_\alpha a_\alpha(\theta\Delta)^\alpha)(\sum_\beta b_\beta \theta^{-\beta}[Z]^\beta)$$

$$= \sum_{\alpha,\beta} a_\alpha b_\beta \frac{\beta!}{(\beta - \alpha)!}\, \theta^{-(\beta-\alpha)}[Z]^{\beta-\alpha}$$

$$= \sum_\gamma (\sum_\alpha a_\alpha b_{\alpha+\gamma}(\alpha + \gamma)!) \frac{\theta^{-\gamma}}{\gamma!}\, [Z]^\gamma.$$

Let $q \in Q_{I,\theta}$ and $p = \sigma^{-1}(q)$. Then for any $f \in I$, $f(\theta\tau)q = 0$; hence, $\Sigma_\alpha\, a_\alpha b_\alpha \alpha! = 0$ from the above formula. It follows that $p(D)f(\theta) = 0$, $\forall f \in I$; i.e., $p \in P_{I,\theta}$. Conversely, suppose $p \in P_{I,\theta}$ and $q = \sigma(p)$. Then $D^\gamma p \in P_{I,\theta}$ for any $\gamma \in \mathbb{N}^s$, since $P_{I,\theta}$ is D-invariant. We have

$$D^\gamma p = \sum_\beta b_\beta \frac{\beta!}{(\beta - \gamma)!}\, Z^{\beta-\gamma} = \sum_\beta b_{\beta+\gamma} \frac{(\beta + \gamma)!}{\beta!}\, Z^\beta.$$

Therefore,

$$0 = (D^\gamma p)(D)f(\theta) = \sum_\alpha a_\alpha \frac{(\gamma + \alpha)!}{\alpha!}\, b_{\alpha+\gamma}\alpha! = \sum_\alpha a_\alpha b_{\alpha+\gamma}(\alpha + \gamma)!.$$

This shows that $f(\theta\tau)q = 0$ for all $f \in I$; i.e., $q \in Q_{I,\theta}$. We conclude that σ is a linear isomorphism from $P_{I,\theta}$ to $Q_{I,\theta}$. □

Suppose now that I is generated by s polynomials $f_1, \ldots, f_s$ from $k[Z]$, and that $\mathrm{Var}(I) \cap (k\backslash\{0\})^s$ is a finite set. Then $\dim(K_I(\tau))$ is finite, so by Theorem 4.1 and Theorem 4.3, we have

$$\dim(K_I(\tau)) = \sum_{\theta \in \mathrm{Var}(I) \cap (k\backslash\{0\})^s} \dim(Q_{I,\theta})$$

$$= \sum_{\theta \in \mathrm{Var}(I) \cap (k\setminus\{0\})^s} \dim(P_{I,\theta})$$

$$= \sum_{\theta \in \mathrm{Var}(I) \cap (k\setminus\{0\})^s} \mathrm{ind}_\theta(f_1, \ldots, f_s).$$

THEOREM (4.4). *Suppose* $f_1, \ldots, f_{s-1}, f_s', f_s'' \in k[Z]$ *and* $f_s = f_s' f_s''$. *For the ideals*

$$I = (f_1, \ldots, f_{s-1}, f_s), \qquad I' = (f_1, \ldots, f_{s-1}, f_s'),$$

$$\text{and} \quad I'' = (f_1, \ldots, f_{s-1}, f_s''),$$

we have the relation

$$\dim(K_I(\tau)) = \dim(K_{I'}(\tau)) + \dim(K_{I''}(\tau)).$$

Proof. If one of the dimensions of $K_{I'}(\tau)$ or $K_{I''}(\tau)$ is infinite, then the dimension of $K_I(\tau)$ is also infinite, since $K_I(\tau)$ contains both $K_{I'}(\tau)$ and $K_{I''}(\tau)$. Suppose that both $K_{I'}(\tau)$ and $K_{I''}(\tau)$ are finite dimensional. Then $\mathrm{Var}(I') \cap (k\setminus\{0\})^s$ and $\mathrm{Var}(I'') \cap (k\setminus\{0\})^s$ are finite sets, hence $\mathrm{Var}(I) \cap (k\setminus\{0\})^s$ is finite as well. Thus, by Theorem 3.3 and the above results, we obtain

$$\begin{aligned}
\dim(K_I(\tau)) &= \sum_{\theta \in \mathrm{Var} \cap (k\setminus\{0\})^s} \mathrm{ind}_\theta(f_1, \ldots, f_{s-1}, f_s) \\
&= \sum_{\theta \in \mathrm{Var} \cap (k\setminus\{0\})^s} (\mathrm{ind}_\theta(f_1, \ldots, f_{s-1}, f_s') \\
&\qquad + \mathrm{ind}_\theta(f_1, \ldots, f_{s-1}, f_s'')) \\
&= \dim(K_{I'}(\tau)) + \dim(K_{I''}(\tau)).
\end{aligned}$$

□

Again we can combine this theorem with the results in Section 2 to obtain information about the kernels of partial difference equations on the sequence space A. The semigroup of commutative operators in this case will be

$$G_{k[Z]}(\tau) := \{p(\tau) : p \in k[Z_1, \ldots, Z_s]\}.$$

The statement of Theorem 4.4 is simply the s-dimensional additivity of $G_{k[Z]}(\tau)$.

THEOREM (4.5). *The semigroup $G_{k[Z]}(\tau)$ of difference operators has s-dimensional additivity. In particular, Theorem* 2.3 *holds for $G_{k[Z]}(\tau)$.*

A special case of this theorem has arisen previously in the study of the algebraic properties of box splines and exponential box splines. In that case an explicit formula for $\dim K(L_B)$, $B \in \mathfrak{B}_1^{(X)}$, can be given. Our next example extends these explicit formulae to a wider class of polynomials.

Example (4.6). Let $k = \mathbf{C}$ be the field of complex numbers. The polynomials in $\mathbf{C}[Z]$ to be associated with the matroid X in this example will be taken from the subset

$$\mathbf{Q} := \left\{ \prod_{j=1}^{m} (Z^{\lambda_j} - \mu_j) : \lambda_j \in \mathbf{N}^s \text{ and } \mu_j \in \mathbf{C}\backslash\{0\}, j = 1, \ldots, m \right\}.$$

The translation operators, $\mathbf{Q}(\tau) := \{p(\tau) : p \in \mathbf{Q}\}$, correspond to products of difference operators (with $\mu = \exp(c)$)

$$\begin{aligned} \Delta_{\lambda,c} f &:= f(\cdot + \lambda) - \exp(c) f(\cdot) \\ &=: \nabla_{\lambda,c} f(\cdot + \lambda) = \tau^{\lambda} \nabla_{\lambda,c} f. \end{aligned} \tag{4.7}$$

The difference operators $\nabla_{\lambda,c}$ arise quite naturally in the study of exponential box splines. Their relation to our situation is quite clear from (4.7): If $p_l(Z) = Z^{\lambda_l} - \exp(c_l)$, $l = 1, \ldots, s$, then

$$\begin{aligned} K_{(p_1,\ldots,p_s)}(\tau) &= \nabla_{c_1,\ldots,c_s}([\lambda_1, \ldots, \lambda_s]) \\ &:= \{f : \nabla_{\lambda_l,c_l} f = 0, l = 1, \ldots, s\} \end{aligned} \tag{4.8}$$

(in the notation of [12]). When $\dim \nabla_{c_1,\ldots,c_s}([\lambda_1, \ldots, \lambda_s]) < \infty$, an explicit formula for the dimension can be given:

$$\dim \nabla_{c_1,\ldots,c_s}([\lambda_1, \ldots, \lambda_s]) = |\det[\lambda_1, \ldots, \lambda_s]|. \tag{4.9}$$

This was proved in [1, Lemma 5.1] and [12, Lemma 6.1], when $\lambda_1, \ldots, \lambda_s \in \mathbf{Z}^s$ are linearly independent. If $\lambda_1, \ldots, \lambda_s$ are linearly dependent, then either $\dim(K_{(p_1,\ldots,p_s)}(\tau)) = 0$ or $\dim(K_{(p_1,\ldots,p_s)}(\tau)) = \infty$ depending on whether $p_1, \ldots, p_s$ have no common zeros in $(\mathbf{C}\backslash\{0\})^s$ or (necessarily) infinitely many common zeros in $(\mathbf{C}\backslash\{0\})^s$.

The formula (4.9) is a special case of the following setup: Let $X = \Lambda$ be a matrix of rank s in $\mathbf{Z}^{s\times n}$ with nonzero columns. To each $\lambda \in \Lambda$ we choose a $c_\lambda \in \mathbf{C}$ and associate the difference operator given by

$$\nabla_{\lambda,c_\lambda} f = f - \exp(c_\lambda) f(\cdot - \lambda).$$

For $V \subseteq \Lambda$, let $\nabla_{V,c_V} := \Pi_{v\in V} \nabla_{v,c_v}$. Then the kernel space

$$\nabla_{c_\Lambda}(\Lambda) := \{f : \nabla_{H,c_H} f = 0, \forall H \in \mathcal{H}(\Lambda)\}$$

has dimension given by

$$\dim(\nabla_{c_\Lambda}(\Lambda)) = \sum_{B\in\mathcal{B}(\Lambda)} |\det(B)|$$

(see [1, Theorem 1.1] and [12, Theorem 6.1]). As was done in Example 3.9, we wish to separate the matroid structure from the associated linear difference operators, using the matroid only as an index set for L_X (as opposed to using it to give the directions of the translations), and at the same time for each index we consider a product of difference operators (induced by the polynomials from $\mathbf{Q}$). In order to state the result, let Λ_B be the set of all possible matrices with columns indexed by $b \in B$ and with the bth column chosen from the exponents in p_b, namely, $\Lambda_b := \{\lambda_{b,j} : j = 1, \ldots, m_b\}$.

COROLLARY (4.10). *Let the set $L_X \subset \mathbf{Q}(\tau)$ be associated with a matroid X, $\rho(X) = s$, and suppose $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$. If for every $B \in \mathcal{B}_1^{(X)}$, the polynomials $\{p_b\}_{b\in B}$ have only finitely many common zeros in $(\mathbf{C}\backslash\{0\})^s$, then*

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B\in\mathcal{B}_1^{(X)}} \sum_{W\in\Lambda_B} |\det W|.$$

Proof. By Theorem 4.5, we only need to show that

$$\dim K(L_B) = \sum_{W\in\Lambda_B} |\det W|.$$

This can be done using the techniques in the proof of Corollary 3.10 together with (4.8) and (4.9) above. □

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