

## A wavelet frame method with shape prior for ultrasound video segmentation

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**Abstract.** Ultrasound video segmentation is a challenging task due to low contrast, shadow effects, complex noise statistics, and the needs of high precision and efficiency for real time applications such as operation navigation and therapy planning. In this paper, we propose a wavelet frame based video segmentation framework incorporating different noise statistics and sequential distance shape priors. The proposed individual frame nonconvex segmentation model is solved by a proximal alternating minimization algorithm and the convergence of the scheme is established based on the recently proposed Kurdyka-Lojasiewicz property. The performance of the overall method is demonstrated through numerical results on two real ultrasound video data sets. The proposed method is shown to achieve better results compared to the related level sets models and edge indicator shape priors, in terms of both segmentation quality and computational time.

**Keywords:** Variational model; wavelet frames; shape-prior; alternating proximal methods; Gaussian and Poisson noise; real-time tracking.

**1. Introduction.** Image segmentation is a classical and important task in medical image analysis. It is the process of dividing the image domain into sub-domains, typically used to locate regions of interests (ROI) and extract boundaries (lines, curves, etc.), which allows a more meaningful and simplified representation of images. In this paper, we are interested in ultrasound images and videos segmentation within a reasonable computation time. There are many potential applications for ultrasound video segmentation in clinical diagnosis, monitoring and treatment and operation navigation. Due to low contrast and complex noise statistics and the needs of efficient implementation for real time applications, it is still challenging to develop a fast and stable segmentation algorithm for ultrasound videos. One may refer to [31] and the references therein for a survey of methods for ultrasound image segmentation and its applications. Many methods from different perspectives for image segmentation exist in the literature. Here, we mainly focus on variational methods for their modeling flexibility, along the line of the seminal work [30, 14, 41]. Variational models such as active contours models [14, 41] have been widely used for edge detection and motion tracking. In addition, shape priors with geometry information are extensively studied in literature such as [15, 6, 39, 33]. For example, a set of hand drawn shapes are incorporated in active contour level set model in [15] for cardiac ultrasound video segmentation. In [37], topological information is also incorporated in the variational model. In [16], Cremers proposed a level set based image sequence segmentation model incorporating a statistical shape priors using an auto-regressive model for the dynamic deformation and transformation of the shapes. From computational prospective, many variants of convex relaxation and numerical algorithms are developed and analyzed (cf. e.g. [13, 7, 43, 28, 34, 12, 3]). Some recent work, such as [37, 40] attempted to

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investigate the impact of different noise models for ultrasound segmentation under variational framework. Some related variational image segmentation models will be reviewed in Section 2.

**1.1. Motivations and contributions.** Despite that variational image segmentation models have been successfully applied to solve various image segmentation problems, it is still necessary to investigate the application of variational methods to video segmentation efficiently, especially when the videos are of low quality, such as ultrasound video. Recently, tight frame based segmentation models have been proposed in [38, 19] where promising results are shown for segmenting complex tubular structures in low contrast medical images with fine structures. The wavelet frame, constructed from multi-resolution analysis (MRA) can be generated by the unitary extension principle (UEP) of [36], because of its multi-resolution property and redundancy which is beneficial for both algorithm efficiency and sparse representation of images [17, 22]. The wavelet frame based models [38, 19] and the variational models of [14, 41, 3] are closely related, as rigorously studied in the series of recent work [20, 8, 9]. It is shown that wavelet frame transforms are discretizations of differential operators in variational, PDE and piecewise smooth Mumford-Shah framework. Numerous experiments in image restoration and image segmentation show that the discretization by wavelet frame transforms has advantages over standard finite difference discretization. Wavelet frames based method can extract more details than existing variational methods especially when the image contains features of different scales [38]. Meanwhile, there are some recent developments on efficient convex minimization algorithms for sparsity related problems, which also motivates us to consider a wavelet frame based approach.

Our proposed model aims to segment an ultrasound video collectively and automatically, by exploring the temporal continuity of the video. Our main contribution is to propose a wavelet frame based variational video segmentation model on incorporating shape priors for single image segmentation, and sequential shape priors are computed automatically for the following frame segmentation. Such strategy yields an efficient implementation, as required in most of real time operation navigation scenarios. In addition, as studied in [37], different noise statistics, in particular Gaussian and Poisson noise are investigated under this framework. It should be pointed out the proposed model is different from the previous shape prior based models. For example, compared to [16], the current model does not train a set of hand-drawn shapes for fitting the shape regression models, and there is no specific deformation model for the shapes transition between frames. The shape priors are calculated directly from previous segmented shapes and used as an edge factor for the current image segmentation. Furthermore, a region based labeling function are used instead of level set functions. Together with recently developed efficient first-order optimization algorithm, this leads to an efficient implementation for the image sequences segmentation. The proposed individual image segmentation nonconvex model, under the two types of noise assumptions, is solved by a proximal alternating minimization algorithm [5], and the convergence is established based on Kurdyka-Lojasiewicz (KL) property. Numerical results show that the use of distance function based shape priors for wavelet frame video segmentation is beneficial for the accuracy and efficiency of extracting edges on real ultrasound data.

**1.2. Outline of the paper.** The rest of the paper is organized as follows. In Section 2, some related variational segmentation model, mainly based on bounded variation, are present. In Section 3, we present the distance function based shape prior wavelet frame video segmentation model. In Section 4, the algorithm for solving the proposed single image segmentation model is present in detail and in Section 5, a convergence analysis for this algorithm is provided. In Section 6, numerical experiments are performed on two real ultrasound video data sets and the paper is concluded in Sections 7.

**2. Variational segmentation models.** Let  $\Omega \subset \mathbb{R}^2$  be an open and bounded image domain,  $f$  be the given image defined on  $\Omega$ . The segmentation problem consists in finding a decomposition of the region  $\Omega = (\bigcup_{i=1, \dots, K} \Omega_i) \cup \Gamma$  where  $\Gamma$  is the closed edge set and  $\Omega_i$  are disjoint open sets representing different regions of interests. The celebrated Mumford-Shah segmentation model [30] is to minimize the following functional:

$$E_{\text{MS}}(u, \Gamma) := \mathcal{H}^{d-1}(\Gamma) + \lambda \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \mu \int_{\Omega} (u - f)^2 dx \quad (2.1)$$

where  $\mathcal{H}^{d-1}(\Gamma)$  is Hausdorff measure in  $\Omega$ , and  $u$  is the piecewise smooth approximation of  $f$ , and the positive constants  $\lambda$  and  $\mu$  balance the terms. The functional in (2.1) is to establish an optimality criterion for segmenting an image into sub-regions by forcing the approximation  $u$  to be piecewise smooth except along the edges  $\Gamma$  and the total length of the edges to be as short as possible. More details on the theoretical aspects of the Mumford-shah and its approximation, one may refer to [30, 41]. The Mumford-Shah model is not easy to solve, and a convenient simplification is Chan-Vese active contour model [14, 41], by taking  $u$  as a piecewise-constant function for each open region  $\Omega_i$ . In particular, for the two regions case on extracting a open region of interest  $\Sigma \subset \Omega$  and  $\Gamma = \partial\Sigma$ , we minimize

$$E_{\text{CV}}(c_1, c_2, \Sigma) = \mathcal{H}^{d-1}(\partial\Sigma) + \mu \int_{\Sigma} (c_1 - f)^2 dx + \mu \int_{\Omega \setminus \Sigma} (c_2 - f)^2 dx \quad (2.2)$$

where  $c_1$  and  $c_2$  represent the constant estimation of  $\Sigma$  (foreground) and  $\Omega \setminus \Sigma$  (background). It is easy to see that for a fixed  $\Sigma$ , the constants  $c_1$  and  $c_2$  can be obtained as the mean intensity of each region. By representing the edge  $\partial\Sigma$  as a Lipschitz level set function [32]  $\phi : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{cases} \partial\Sigma = \{x \in \Omega : \phi(x) = 0\}; \\ \Sigma = \{x \in \Omega : \phi(x) > 0\}; \\ \Omega \setminus \Sigma = \{x \in \Omega : \phi(x) < 0\}, \end{cases} \quad (2.3)$$

and the length term in the energy functional (2.1) can be expressed as

$$\begin{aligned} \mathcal{H}^{d-1}(\partial\Sigma) &= \int_{\Omega} |\nabla H(\phi)| dx \\ &= \int_{\Omega} \delta_0(\phi) |\nabla \phi| dx. \end{aligned} \quad (2.4)$$

where  $H(z)$  is the Heaviside function

$$H(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0, \end{cases} \quad (2.5)$$

and the one-dimensional Dirac measure  $\delta_0(z) = \frac{d}{dz}H(z)$ . The overall Chan-Vese functional (2.2) can be written as

$$\begin{aligned} E_{\text{CV-LS}}(c_1, c_2, \phi) &= \int_{\Omega} \delta_0(\phi) |\nabla \phi| dx + \frac{\mu}{2} \int_{\Omega} (c_1 - f)^2 H(\phi) dx + \frac{\mu}{2} \int_{\Omega} (c_2 - f)^2 (1 - H(\phi)) dx. \end{aligned} \quad (2.6)$$

Numerically, we need to replace (2.6) with a regularized functional

$$\begin{aligned} E_{\text{CV-LS}_\epsilon}(c_1, c_2, \phi) &= \int_{\Omega} \delta_\epsilon(\phi) |\nabla \phi| dx + \frac{\mu}{2} \int_{\Omega} (c_1 - f)^2 H_\epsilon(\phi) dx + \frac{\mu}{2} \int_{\Omega} (c_2 - f)^2 (1 - H_\epsilon(\phi)) dx. \end{aligned} \quad (2.7)$$

where

$$H_\epsilon(z) = \begin{cases} 1, & z > \epsilon; \\ 0, & z < -\epsilon; \\ \frac{1}{2} \{1 + \frac{z}{\epsilon} + \frac{1}{\pi} \sin(\frac{\pi z}{\epsilon})\}, & |z| \leq \epsilon. \end{cases} \quad \delta_\epsilon(z) = \begin{cases} 0, & |z| > \epsilon; \\ \frac{1}{2\epsilon} \{1 + \cos(\frac{\pi z}{\epsilon})\}, & |z| \leq \epsilon. \end{cases} \quad (2.8)$$

For fixed  $c_1$  and  $c_2$ , the minimization of  $E_{\text{CV-LS}_\epsilon}$  with respect to  $\phi$  can be obtained by solving the following evolution equation

$$\begin{cases} \frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi) [\mathbf{div}(\frac{\nabla \phi}{|\nabla \phi|}) - \mu(c_1 - f)^2 + \mu(c_2 - f)^2] & \text{in } (0, \infty) \times \Omega, \\ \phi(0, x, y) = \phi_0(x) & \text{in } \Omega, \\ \frac{\delta_\epsilon(\phi)}{|\nabla \phi|} \frac{\partial \phi}{\partial \bar{n}} = 0 & \text{on } \partial \Omega. \end{cases} \quad (2.9)$$

The Chan-Vese model is also relaxed as a continuous field binary labeling problem in [35, 13]

$$E_{\text{CV}}(c_1, c_2, u) = \int_{\Omega} |Du| dx + \frac{\mu}{2} \int_{\Omega} u(c_1 - f)^2 dx + \frac{\mu}{2} \int_{\Omega} (1 - u)(c_2 - f)^2 dx \quad (2.10)$$

where

$$u(x) = \begin{cases} 1, & \text{if } x \in \Sigma \\ 0, & \text{if } x \in \Omega \setminus \Sigma \end{cases} \quad (2.11)$$

and the first formula is due to the co-area formula [1, 40]

$$\mathcal{H}^{d-1}(\partial \Sigma) = \int_{\Omega} |D\chi_{\Sigma}(x)| dx. \quad (2.12)$$

Many convex relaxation of the model (2.10) and its relation to the global solutions are studied in [13, 7, 43, 28, 34, 12, 3]. In addition, edge indicator function can be also used as shape prior as proposed in [15, 6, 39, 33]:

$$E_{\text{CV}}(c_1, c_2, u) := \int_{\Omega} |\varphi \cdot \nabla u| dx + \mu \int_{\Omega} u(c_1 - f)^2 dx + \mu \int_{\Omega} (1 - u)(c_2 - f)^2 dx. \quad (2.13)$$

The function  $\varphi$  is a positive and regular function depending on the edges of  $f$  satisfying  $\lim_{t \rightarrow \infty} \varphi(t) = 0$ . The use of shape priors allows to find boundaries that are close to the given

shape priors, which are extremely useful for extracting ROIs and moving boundaries. For instance in [11, 6], the following edge indicator function

$$\varphi(x) = g(|\nabla f(x)|) = \frac{1}{1 + |\nabla \tilde{f}(x)|^2}, \quad (2.14)$$

where  $\tilde{f}(x) = (G_\sigma * f)(x)$  is a smoothed version of  $f$  and  $G_\sigma$  is a Gaussian kernel with variance  $\sigma$ , is used to drive the partition to embed into the pre-defined edges.

**3. Wavelet frame video segmentation model with shape prior.** In this section, we consider a wavelet frame segmentation model in a discrete setting. Given a video sequences  $\{f_t\}$ , for  $t \in 1, \dots, T$  and each frame  $f_t$  is defined on a discrete grid domain  $\Omega \subset \mathbb{R}^2$ , our objective is to find a meaningful partition of the pixel index set  $\Omega$ . We start with introducing a segmentation model for each individual frame  $f_t$ . For simplicity, we will drop the index  $t$  for the presentation of single image segmentation  $f(x)$  for  $x \in \Omega$  and we only consider a two-region approximation of  $f$  as

$$z(x) = z_1(x)u(x) + z_2(x)(1 - u(x)), \quad \text{for } x \in \Omega$$

where

$$u(x) = \begin{cases} 1, & \text{if } x \in \Sigma \\ 0, & \text{if } x \in \Omega \setminus \Sigma \end{cases} \quad (3.1)$$

and  $\Sigma$  denotes the foreground ROI and  $\Omega \setminus \Sigma$  denotes the background,  $z_1(x)$  and  $z_2(x)$  denote the approximation of the original noise free image on  $\Sigma$  and  $\Omega \setminus \Sigma$  with appropriate extension respectively.

The fast tight frame transform or decomposition operator is denoted as  $\mathcal{W}$ , where the operator  $\mathcal{W}$  is consisted of  $J + 1$  sub-filtering operators  $\mathcal{W}_0, \mathcal{W}_1, \dots, \mathcal{W}_J$ , and  $\mathcal{W}^T \mathcal{W} = I$ , i. e. ,  $\mathcal{W}^T \mathcal{W} u = u$  for any image  $u$ . For more details on wavelet frame transform and its application in imaging, one may refer to [21]. The proposed wavelet frame based segmentation model takes the general form

$$\min_{z_1, z_2, u \in \{0,1\}} E(z_1, z_2, u) := \mu D(z_1, z_2, u) + \sum_{i=1}^2 R(z_i) + |\varphi \cdot \mathcal{W} u|_1 \quad (3.2)$$

where  $D(z_1, z_2, u) := -\log p(f|z)$  denotes the negative log likelihood function related to data misfitting,  $R(z_i)$  denotes a regularization term related to approximation  $z_i$  and the last term is the weighted sparse regularization on the characteristic function  $u$  with predefined  $\varphi(x)$ . We note that the weight function is applied to all the subbands and this term is related to the regularity of the edge set  $\Sigma$ . The weight function  $\varphi$  allows to incorporate shape priors, which we will discuss later. The idea of recovering a piecewise smooth function with a general smoothness regularization and general singularity set, including high order jump discontinuities has been studied in a recent work [9], and it is rigorously proven that the wavelet frame approach converge to a variational approach on the space of piecewise smooth functions as image resolution goes to infinity. Instead of recover edge sets  $\partial \Sigma$  directly, we propose to recover the region characteristic function, which allows stable and fast algorithm for

a convex relaxed function. Meanwhile, this energy can also be interpreted from the prospective of maximum a posterior (MAP) estimation, as proposed in [37].

Under the assumption of independence of noise at each pixel  $x$ , we have

$$p(f|z) \propto \prod_{x \in \Omega} p(f(x)|z(x)) = \prod_{x \in \Sigma} p(f(x)|z_1(x)) \cdot \prod_{x \in \Omega \setminus \Sigma} p(f(x)|z_2(x)),$$

and  $D(z_1, z_2, u)$  reduces to

$$D(z_1, z_2, u) = - \sum_{x \in \Sigma} \log(p(f(x)|z_1(x))) - \sum_{x \in \Omega \setminus \Sigma} \log(p(f(x)|z_2(x))) \quad (3.3)$$

$$= \langle u, -\log p(f|z_1) \rangle + \langle \mathbf{1} - u, -\log p(f|z_2) \rangle \quad (3.4)$$

Here  $\langle \cdot, \cdot \rangle$  denotes the usual inner product of two vectors and  $\mathbf{1}$  denotes all-one vector.

In the following, we further simplify the models under the assumption of piecewise constant approximation of  $f$ , i.e.  $z_i(x) = c_i$ . For the noise statistics, we discuss Gaussian and Poisson noise present in the data  $f$ , for such simplification is enough for ultrasound image segmentation.

- When the noise of data is assumed to be Gaussian, which is the usual choice in many segmentation formulations, we have

$$p(f(x)|z_i(x)) \propto \exp^{-\frac{1}{2\sigma^2}(c_i - f(x))^2} \quad (3.5)$$

and the data term reduces to

$$D(c_1, c_2, u) = \langle u, (f - c_1)^2 \rangle + \langle \mathbf{1} - u, (f - c_2)^2 \rangle; \quad (3.6)$$

- If  $f$  is corrupted by Poisson noise, the distribution of  $f$  is in the form of

$$p(f|c_i) = \frac{c_i^f}{f!} e^{-c_i}, \quad \text{for } i \in \{1, 2\} \quad (3.7)$$

and the data term becomes

$$D(c_1, c_2, u) = \langle u, c_1 - f \log c_1 \rangle + \langle \mathbf{1} - u, c_2 - f \log c_2 \rangle + \langle \mathbf{1}, \log(f!) \rangle. \quad (3.8)$$

Furthermore, for a convenient convergent analysis of the proposed algorithm in the following section, we consider  $R(c_i) = \chi_{[c_l, c_u]}(c_i)$ , where  $\chi$  is the set indicator function, i.e. we consider a constraint for  $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in [c_l, c_u]$  where  $c_l$  and  $c_u$  are the lower and upper bound for the intensity. Without loss of generality, we assume  $0 < c_l < c_u$ , and then we propose the model with binary constraint as

$$\min_{u \in \{0,1\}, c_l \leq \vec{c} \leq c_u} |\varphi \cdot \mathcal{W}u|_1 + \mu D(\vec{c}, u). \quad (3.9)$$

It is well-known that it is hard to solve the problem (3.9) with binary constraint  $u \in \{0, 1\}$ . We relax this constraint as  $u \in [0, 1]$  as in some previous work of variational image segmentation models. Finally, we propose the following model

$$\min_{0 \leq u \leq 1, c_l \leq \vec{c} \leq c_u} |\varphi \cdot \mathcal{W}u|_1 + \mu D(\vec{c}, u) \quad (3.10)$$

for single image segmentation.

If the global minimum of (3.10) is binary, then it is naturally a global minimum of the nonconvex model (3.9). In the following, we also analyze the equivalence of the binary solution and relaxed solution for a fixed  $\vec{c}$ . We rewrote the Fenchel conjugate form of (3.10) as

$$\max_{|p(x)| \leq \varphi(x)} \min_{0 \leq u \leq 1} E(u, p) := \langle \mathcal{W}^T p + I_1 - I_2, u \rangle \quad (3.11)$$

where  $p$  is a dual variable with the same dimension as  $\mathcal{W}u$ , and  $I_i = \begin{cases} (c_i - f)^2, & \text{Gaussian noise,} \\ c_i - f \log c_i, & \text{Poisson noise,} \end{cases}$  for  $i = 1, 2$ . Let  $(u^*, p^*)$  be the primal dual solution pair of (3.11), then  $p^*$  is the optimal solution of the dual problem  $\max_{|p(x)| \leq \varphi(x)} E_D(p) := \min_{0 \leq u \leq 1} E(u, p)$ . The strong duality theory leads to

$$u^* = \arg \min_{0 \leq u \leq 1} E(u, p^*)$$

This implies that the optimal solution

$$u^*(x) = \begin{cases} 1, & \text{if } [\mathcal{W}^T p^* + I_1](x) < I_2(x), \\ 0, & \text{if } [\mathcal{W}^T p^* + I_1](x) > I_2(x), \\ [0, 1], & \text{otherwise} \end{cases}$$

It is easy to see that even though an optimal solution  $u^*$  is not binary everywhere, one can transform those non-binary points into 0 or 1 and it still satisfies the optimality of the convex relaxed problem. This binary solution is thus a global solution for the binary nonconvex model. In practice, the convex problem (3.11) is solved by a primal dual algorithm, which alternatively gives the approximation of  $p^*$  and  $u^*$ .

In the following, we will present the algorithm for solving  $c$  and  $u$  alternatively. First of all, combing with the Gaussian and Poisson noise distribution present in (3.6) and (3.8), the above model (3.10) can be rewritten as

$$\min_{0 \leq u \leq 1, c_l \leq \vec{c} \leq c_u} |\varphi \cdot \mathcal{W}u|_1 + \mu \langle u, \xi(\vec{c}) \rangle + \mu \langle \mathbf{1}, \psi(\vec{c}) \rangle \quad (3.12)$$

where

- Gaussian noise:

$$\begin{cases} \xi(\vec{c}) &= (c_1 - f)^2 - (c_2 - f)^2 \\ \psi(\vec{c}) &= (c_2 - f)^2 \end{cases}; \quad (3.13)$$

- Poisson noise:

$$\begin{cases} \xi(\vec{c}) &= (c_1 - c_2) - f(\log c_1 - \log c_2) \\ \psi(\vec{c}) &= c_2 - f \log c_2. \end{cases} \quad (3.14)$$

For the weighted sparse regularization term  $|\varphi \cdot \mathcal{W}u|$  in (3.12), a usual choice for  $\varphi$  is edge indicator of  $f$  (2.14), as proposed in many references [6, 3, 22]. However, this is not a good choice for automatically tracking the moving boundary of the region of interest in sequential frames, as shown in Section 6, when we consider segmentation of the video sequence  $f_t$  for  $t = 1, \dots, T$ . In [15], manually drawn contours for each frame of a video is also proposed to be used as a shape prior. However, it is not practical to trace all the frames, especially when online application is required. In order to extract the moving boundary  $\Sigma_t$  automatically from  $f_t$ , we adopt a shape prior function  $\varphi_t$  defined directly from previous segmentation results. In fact, it is reasonable to assume that the relative changes of the boundaries in the neighboring frames are minor. The weight term  $|\varphi_t \cdot \mathcal{W}u_t|_1$  should be able to force the shape of underlying segmentation  $u_t$  close to  $u_{t-1}$  or to even more previous segmentation. In other words, the edge of the resulting partition is dominated partially by the approximated shape prior and the regularity of the current frame. More specially, the weight function  $\varphi_t(x)$  at time  $t$  is chosen as

$$\varphi_t(x) = \max_{1 \leq j \leq M} \{\omega(d_{t-j}(x))\} \quad (3.15)$$

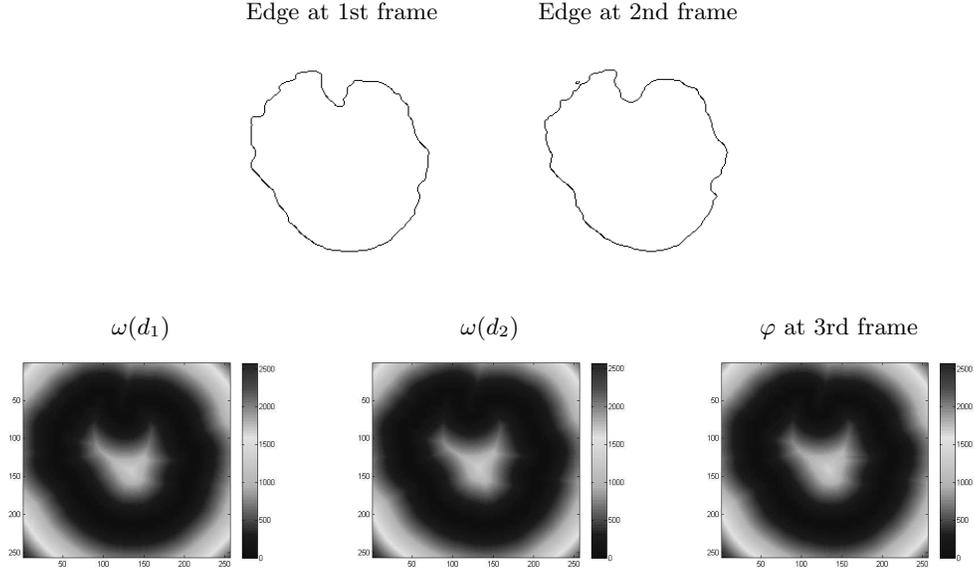
where  $d_{t-j}(x)$  denotes the distance function to the boundary  $\partial\Sigma_{t-j}$  found at the previous  $j$ -th frame,  $\omega(y)$  is a monotone increasing function with respect to  $y$  generally. Such function enforces the current regularized shape is close to the previous shape, as bigger weights will result in zero wavelet coefficients, when one moves away from the shape prior. Some functions such as power function, logarithm can be utilized. In fact the choice of the function type is not truly important, while the parameter choices are more important to result in a similar weight curve. In this paper, a power function  $w(y) = ay^b$  for some positive  $a$  and  $b$  will be used in the numerical experiments. A good shape function should value relatively small near the boundary (represent by the distance function) and increases quickly when one moves away from the shape. The weight can be viewed as a smoothed narrow band function. The parameters  $a$  and  $b$  of the power function  $\varphi$  determine the width of narrow band. Empirically, we find that the max function over time leads to a more stable result compared to the mean one through the experiments. An example of the weight function of two sequential frames and the resulted weight function defined as (3.15) is shown in Figure 3.1.

**4. Algorithm.** In general, alternating minimization scheme with Gauss-Seidel fashion (or block coordinate descent) is adopted to solve the problems as (3.12), as done in many literature (e.g. [19, 14, 41]). The convergence of such scheme can not be guaranteed in general. Inspired by recent work on the so-called Kurdyka-Łjasiewicz (KL) property [5, 42, 18, 4], we adopt the proximal alternating minimization (PAM) proposed in [5], and we will verify the global convergence of the proposed scheme for the model (3.12) in Section 5.

The PAM algorithm for solving (3.12) is given as

$$\begin{cases} u^{k+1} = \operatorname{argmin}_{0 \leq u \leq 1} |\varphi \cdot \mathcal{W}u|_1 + \mu \langle u, \xi(\vec{c}^k) \rangle + \frac{1}{2\sigma} \|u - u^k\|^2 \\ \vec{c}^{k+1} = \operatorname{argmin}_{c_l \leq \vec{c} \leq c_u} \mu \langle \mathbf{1}, \psi(\vec{c}) \rangle + \mu \langle u^{k+1}, \xi(\vec{c}) \rangle + \frac{1}{2\gamma} \|\vec{c} - \vec{c}^k\|^2 \end{cases} \quad (4.1)$$

where  $k$  denotes the iteration number,  $\sigma$  and  $\gamma$  are two positive numbers.



**Figure 3.1.** An example of the shape prior, corresponding  $w(d_i)$  and weight function  $\varphi$ .

The subproblem for solving  $u^{k+1}$  is convex and can be solved by the split Bregman method [26, 10] which is also equivalent to the well-known alternating direction of multiplier method (cf. e.g. [25, 24, 23]). This subproblem is reformulated as

$$\min_{0 \leq u \leq 1} |\varphi \cdot \mathcal{W}u|_1 + \mu \langle u, \tilde{\xi} \rangle + \frac{1}{2\sigma} \|u - \tilde{u}\|^2 \quad (4.2)$$

where we set  $\tilde{\xi} = \xi(c_1^k, c_2^k)$  and  $\tilde{u} = u^k$ .

By introducing the variable  $v = \mathcal{W}u$ , the split Bregman algorithm for solving (4.2) is as follows

$$\begin{cases} u^{q+1} = \operatorname{argmin}_{0 \leq u \leq 1} \mu \langle u, \tilde{\xi} \rangle + \frac{\rho}{2} \|\mathcal{W}u - v^q + b^q\|_2^2 + \frac{1}{2\sigma} \|u - \tilde{u}\|^2 \\ v^{q+1} = \operatorname{argmin}_v |\varphi \cdot v|_1 + \frac{\rho}{2} \|\mathcal{W}u^{q+1} - v + b^q\|_2^2 \\ b^{q+1} = b^{q+1} - (v^{q+1} - \mathcal{W}u^{q+1}) \end{cases} \quad (4.3)$$

where the outer iteration superscript  $k$  is omit, and superscript  $q$  denotes the inner iteration.

It is now standard to get the solution of the second subproblem in (4.3) by soft thresholding. Its formulation is

$$v^{q+1} = \operatorname{shrink}(\mathcal{W}u^{q+1} + b^q, \frac{\varphi}{\rho}) \quad (4.4)$$

where the pointwise soft shrinkage operator *shrink* is defined as

$$\operatorname{shrink}(y, \nu) = \operatorname{sgn}(y) \cdot \max\{|y| - \nu, \mathbf{0}\} \quad (4.5)$$

**Lemma 4.1.** The solution to the first subproblem of (4.3) is given by

$$u^{q+1} = \Pi_{[0, 1]} \left( \frac{\sigma \rho \mathcal{W}^T (v^q - b^q) - \sigma \mu \tilde{\xi} + \tilde{u}}{\sigma \rho + 1} \right) \quad (4.6)$$

where  $\Pi_C$  is the projection operator onto the convex set  $C$ .

*Proof.* Using the Karush-Kuhn-Tucker optimality conditions for the first subproblem in (4.3), we have

$$\begin{cases} -u \leq \mathbf{0}, u - 1 \leq \mathbf{0} \\ \lambda_1 \geq \mathbf{0}, \lambda_2 \geq \mathbf{0} \\ \lambda_1 \cdot (-u) = \mathbf{0}, \lambda_2 \cdot (u - 1) = \mathbf{0} \\ \mu\xi + \rho\mathcal{W}^T(\mathcal{W}u - v^q + b^q) + \frac{1}{\sigma}(u - \tilde{u}) - \lambda_1 + \lambda_2 = \mathbf{0} \end{cases} \quad (4.7)$$

By using  $\mathcal{W}^T\mathcal{W} = I$ , the last equation yields

$$u^{q+1} = \frac{\sigma\rho\mathcal{W}^T(v^q - b^q) - \sigma\mu\xi + \tilde{u}}{\sigma\rho + 1} + \frac{\sigma}{\sigma\rho + 1}(\lambda_1 - \lambda_2) \quad (4.8)$$

$$:= \bar{u} + \frac{\sigma}{\sigma\rho + 1}(\lambda_1 - \lambda_2) \quad (4.9)$$

where we denote  $\bar{u} = \frac{\sigma\rho\mathcal{W}^T(v^q - b^q) - \sigma\mu\xi + \tilde{u}}{\sigma\rho + 1}$ . Thus for  $\forall x \in \Omega$ , we have the following pointwise relation

- If  $0 \leq \bar{u}(x) \leq 1$ , then

$$\lambda_1(x) = 0, \lambda_2(x) = 0, u^{q+1}(x) = \bar{u}(x);$$

- If  $\bar{u}(x) < 0$ , then

$$\lambda_1(x) = -\frac{\sigma\rho + 1}{\sigma}\bar{u}(x), \lambda_2(x) = 0, u^{q+1}(x) = 0;$$

- If  $\bar{u}(x) > 1$ , then

$$\lambda_1(x) = 0, \lambda_2(x) = \frac{\sigma\rho + 1}{\sigma}(\bar{u}(x) - 1), u^{q+1}(x) = 1.$$

It follows that  $u^{q+1}$  is given as (4.6). ■

In summary, the subproblem for  $u$  in (4.1) can be updated by the following scheme

$$\begin{cases} u^{q+1} = \Pi_{[0, 1]} \left( \frac{\sigma\rho\mathcal{W}^T(v^q - b^q) - \sigma\mu\xi + \tilde{u}}{\sigma\rho + 1} \right) \\ v^{q+1} = \text{shrink}(\mathcal{W}u^{q+1} + b^q, \frac{\rho}{\sigma}) \\ b^{q+1} = b^q + (\mathcal{W}u^{q+1} - v^{q+1}) \end{cases} \quad (4.10)$$

For the  $\tilde{c}^{k+1}$  subproblem, we need to solve for

$$\underset{c_1 \leq \tilde{c} \leq c_u}{\operatorname{argmin}} \mu \langle \mathbf{1}, \psi(\tilde{c}) \rangle + \mu \langle \tilde{u}, \xi(\tilde{c}) \rangle + \frac{1}{2\gamma} \|\tilde{c} - \tilde{c}\|^2 \quad (4.11)$$

Here we abuse the notations by setting  $\tilde{u} := u^{k+1}$  and  $\tilde{c} := \tilde{c}^k$ .

Recall that the terms  $\xi(\tilde{c})$  and  $\psi(\tilde{c})$  are given as (3.13) and (3.14) for the case of Gaussian and Poisson noise respectively. It is not hard to see that (4.11) is convex for both  $c_1$  and  $c_2$  for both cases. In fact, the closed-form solutions can be obtained for both cases.

For Poisson noise, the function  $\xi(\vec{c})$  and  $\psi(\vec{c})$  are defined in (3.14). A closed form solution for solving (4.11) is given by the following Lemma.

**Lemma 4.2.** *For  $\xi(\vec{c})$  and  $\psi(\vec{c})$  defined as (3.14), the solution to (4.11) is given as*

$$\begin{cases} c_1^{k+1} = \Pi_{[c_l, c_u]} \left( \frac{-\gamma\mu(\mathbf{1}, \tilde{u}) + \tilde{c}_1 + \sqrt{(\gamma\mu(\mathbf{1}, \tilde{u}) - \tilde{c}_1)^2 + 4\gamma\mu\langle f, \tilde{u} \rangle}}{2} \right) \\ c_2^{k+1} = \Pi_{[c_l, c_u]} \left( \frac{-\gamma\mu(\mathbf{1}, \mathbf{1} - \tilde{u}) + \tilde{c}_2 + \sqrt{(\gamma\mu(\mathbf{1}, \mathbf{1} - \tilde{u}) - \tilde{c}_2)^2 + 4\gamma\mu\langle f, \mathbf{1} - \tilde{u} \rangle}}{2} \right) \end{cases} \quad (4.12)$$

*Proof.*

Recall that  $\xi(\vec{c}) = (c_1 - c_2) - f(\log c_1 - \log c_2)$ ,  $\psi(\vec{c}) = c_2 - f \log c_2$ . The Karush-Kuhn-Tucker optimality conditions of problem (4.11) are

$$\begin{cases} -\vec{c} + c_l \leq \mathbf{0}, \quad \vec{c} - c_u \leq \mathbf{0} \\ \lambda_1 \geq \mathbf{0}, \quad \lambda_2 \geq \mathbf{0} \\ \lambda_1 \cdot (-\vec{c} + c_l) = 0, \quad \lambda_2 \cdot (\vec{c} - c_u) = 0 \\ \mu(\langle \tilde{u}, \mathbf{1} \rangle - \frac{1}{c_1} \langle \tilde{u}, f \rangle) + \frac{1}{\gamma}(c_1 - \tilde{c}_1) - \lambda_{11} + \lambda_{12} = 0 \\ \mu(\langle \mathbf{1} - \tilde{u}, \mathbf{1} \rangle - \frac{1}{c_2} \langle \mathbf{1} - \tilde{u}, f \rangle) + \frac{1}{\gamma}(c_2 - \tilde{c}_2) - \lambda_{12} + \lambda_{22} = 0 \end{cases} \quad (4.13)$$

where  $\lambda_1 = \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix}$  and  $\lambda_2 = \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix}$  are Lagrangian multipliers. Denote the solution of (4.11) as  $(c_1^*, c_2^*)$ . From the fourth equation, we have

$$\frac{1}{\gamma c_1} (c_1^2 + \alpha c_1 + \beta) = \lambda_{11} - \lambda_{12}$$

where  $\alpha = \gamma\mu(\mathbf{1}, \tilde{u}) - \tilde{c}_1$  and  $\beta = -\gamma\mu\langle f, \tilde{u} \rangle$ . For  $c_1^2 + \alpha c_1 + \beta = 0$ , the positive solution is  $\bar{c}_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} = \frac{-\gamma\mu(\mathbf{1}, \tilde{u}) + \tilde{c}_1 + \sqrt{(\gamma\mu(\mathbf{1}, \tilde{u}) - \tilde{c}_1)^2 + 4\gamma\mu\langle f, \tilde{u} \rangle}}{2}$ .

- If  $0 < c_l \leq \bar{c}_1 \leq c_u$ , we have  $\frac{\bar{c}_1^2 + \alpha\bar{c}_1 + \beta}{\gamma\bar{c}_1} = 0$ ,  $\lambda_{11} = 0$  and  $\lambda_{12} = 0$ , then the solution is  $c_1^* = \bar{c}_1$ .
- If  $\bar{c}_1 < c_l$ , we have  $\frac{c_l^2 + \alpha c_l + \beta}{\gamma c_l} > 0$ ,  $\lambda_{11} > 0$  and  $\lambda_{12} = 0$ . Thus  $c_1^* = c_l$ .
- If  $\bar{c}_1 > c_u > 0$ , we have  $\frac{c_u^2 + \alpha c_u + \beta}{\gamma c_u} < 0$ ,  $\lambda_{11} = 0$  and  $\lambda_{12} > 0$ . Thus  $c_1^* = c_u$ .

Thus the solution  $c_1^* = \Pi_{[c_l, c_u]}(\bar{c}_1)$ . An similar argument can be applied to  $c_2$  and it follows that the solutions are given as (4.12). ■

For Gaussian noise with  $\xi(\vec{c}) = (c_1 - f_t)^2 - (c_2 - f)^2$ ,  $\psi(\vec{c}) = (c_2 - f)^2$ , it is easy to see that the solution of (4.11) is given as

$$\begin{cases} c_1^{k+1} = \Pi_{[c_l, c_u]} \left( \frac{2\mu\gamma\langle f, \tilde{u} \rangle + \tilde{c}_1}{2\mu\gamma\langle \mathbf{1}, \tilde{u} \rangle + 1} \right) \\ c_2^{k+1} = \Pi_{[c_l, c_u]} \left( \frac{2\mu\gamma\langle f, \mathbf{1} - \tilde{u} \rangle + \tilde{c}_2}{2\mu\gamma\langle \mathbf{1}, \mathbf{1} - \tilde{u} \rangle + 1} \right). \end{cases} \quad (4.14)$$

The overall scheme for individual image segmentation is summarized in Algorithm 1.

We now present the algorithm for processing the whole video sequence. In particular, we incorporate the weight function  $\varphi(x)$ , as defined in (3.15) using the distance function to a

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**Algorithm 1** Proximal Alternating Minimization Algorithm based individual image segmentation algorithm

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**Step 1.** Set the initial value with  $u^{0,0}$ ,  $c_1^{0,0}$ ,  $c_2^{0,0}$  and set tolerance value of the stop tolerance  $\delta_1, \delta_2$ ;

**Step 2.**

**while**  $\|u^k - u^{k-1}\|_2 > \delta_1$  or  $K = 1$  **do**

Set  $u^{k+1,0} := u^k$

**while**  $\|u^{k+1,q} - u^{k+1,q-1}\|_2 > \delta_2$  or  $k = 1$  **do**

Update  $(u^{k+1,q+1}, v^{k+1,q+1}, b^{k+1,q+1})$  with (4.10), where  $\tilde{u} := u^k$ ;

$q = q + 1$ ;

**end while**

Set  $u^{k+1} := u^{k+1,q}$ ;

Set  $\tilde{u} := u^{k+1}$  and  $\tilde{c} := \tilde{c}^k$ ;

For Gaussian noise model, update  $\tilde{c}^{k+1}$  with (4.14).

For Poisson noise model, update  $\tilde{c}^{k+1}$  with (4.12).

$k = k + 1$ ;

**end while**

**Step 3.** Get the final segmentation  $u = \mathbf{1}_{u^k > 0.5}$ .

---

predefined shape. For the first frame  $f_1$ , we usually adopt a hand-drawn initial shape and compute the distance function. After we compute current segmentation  $u_t$  by Algorithm 1, and we obtain  $\Sigma_t := \{u_t(x) = 0\}$ . The fast sweeping method [44] is adopted here to obtain the distance function by solving the partial differential equation  $|\nabla d(x)| = 1$  with the boundary condition  $d(x) = 0$ ,  $x \in \partial\Sigma_t$ . For the completeness of presentation, we present the details in Algorithm 2. A distance function in our numerical results is illustrated in Figure 3.1.

Finally, the whole procedure for video segmentation is present in Algorithm 3 as followed.

**5. Convergence analysis.** The convergence of the proximal alternating scheme (4.1) (Algorithm 1) is established based on the framework presented in [2].

Consider a class of nonconvex and nonsmooth problems of the form

$$L(x, y) = f(x) + Q(x, y) + g(y) \quad (5.1)$$

where

$$(H_1) : \begin{cases} f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}, g : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\} \text{ are proper l.s.c.}; \\ Q : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \text{ is a } C^1 \text{ function}; \\ \nabla Q \text{ is Lipschitz continuous on bounded subsets of } \mathbb{R}^n \times \mathbb{R}^m. \end{cases} \quad (5.2)$$

This class of optimization problems are arisen in a variety of applications, such as compressed sensing, matrix factorization, dictionary learning, sparse approximations of signals and images, and blind decomposition etc. The proximal alternating minimization (PAM)

**Algorithm 2** Fast sweeping method for computing distance function**Step 1:** Discretize using upwind difference.

$$[(d_{i,j} - d_{xmin})^+]^2 + [(d_{i,j} - d_{ymin})^+]^2 = h^2 \quad (4.15)$$

where  $d_{xmin} = \min(d_{i-1,j}, d_{i+1,j})$ ,  $d_{ymin} = \min(d_{i,j-1}, d_{i,j+1})$ **Step 2:** Initialize  $d$  to enforce  $d(x) = 0, x \in \Gamma$ .

- (1) Assign exact values at grid points in or near  $\Gamma$  which will be fixed in later computation.
- (2) Assign large positive value at all other points.

**Step 3:** Gauss-Seidel iteration using different sweeping order:

$$\begin{aligned} (1) \quad & i = 1 : I, j = 1 : J; \quad (2) \quad i = I : 1, j = 1 : J; \\ (3) \quad & i = I : 1, j = J : 1; \quad (4) \quad i = 1 : I, j = J : 1. \end{aligned} \quad (4.16)$$

$$d_{i,j}^{new} = \min(d_{i,j}^{old}, \bar{d})$$

where  $d$  is the solution to (4.15) using current  $d_{i-1,j}, d_{i+1,j}, d_{i,j-1}, d_{i,j+1}$ .**Algorithm 3** Video segmentation algorithm**Step 0.** Given  $\{f_1, \dots, f_T\}$  and a (hand-drawn) initial shape  $\Sigma_0$  for  $f_1$ .**Step 1.** For  $t = 1, \dots, T$ **Step 1.0** Set  $u^0 = u_{t-1}$ ,  $c_1^{0,0}, c_2^{0,0}$  as the mean intensity of initial region determined by  $u_{t-1}$ .**Step 1.1** Compute the distance function  $d_{t-1}$  and  $\varphi^k$  with (3.15);**Step 1.2** Apply Algorithm 1 to get final segmentation  $u_t$  and  $\Sigma_t$ .

algorithm has been proposed in [5, 2] for solving this broad class of problems (5.1)

$$\begin{cases} x^{k+1} = \operatorname{argmin} L(x, y^k) + \frac{1}{2\theta^k} \|x - x^k\|^2, \\ y^{k+1} = \operatorname{argmin} L(x^{k+1}, y) + \frac{1}{2\eta^k} \|y - y^k\|^2. \end{cases} \quad (5.3)$$

A standard assumption for the algorithm is as followed :

$$(H_2) : \begin{cases} \inf_{\mathbb{R}^n \times \mathbb{R}^m} L > -\infty, \\ \text{the function } L(\cdot, y_0) \text{ is proper,} \\ \text{for some positive } l < u, \text{ the sequences of stepsize } l < \theta_k, \eta_k < u, \text{ for all } k \geq 0. \end{cases} \quad (5.4)$$

An important tool to demonstrate the convergence of the sequence is based on the so-called Kurdyka-Lojasiewicz (KL) property. We first list some useful definitions and results, and for more details on the definitions and properties one may refer to the series of work [2, 5] and references therein.

**Definition 5.1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper l.s.c. function.

1. The domain of  $f$  is defined as  $\operatorname{dom}(f) := \{x \in \mathbb{R}^n : f(x) < \infty\}$ .

2. For each  $x \in \text{dom}(f)$ , the Fréchet subdifferential of  $f$  at  $x$  is defined as follows:

$$\hat{\partial}f(x) = \{s : \liminf_{y \rightarrow x, y \neq x} \frac{1}{\|x - y\|} (f(y) - f(x) - \langle s, y - x \rangle) \geq 0\} \quad (5.5)$$

If  $x \notin \text{dom } f$ , we set  $\hat{\partial}f(x) = \emptyset$ .

3. The (limiting-) subdifferential of  $f$  at  $x \in \text{dom}(f)$  is defined as follows:

$$\partial f(x) = \{x^* : \exists x_n \rightarrow x, \text{ s.t. } f(x_n) \rightarrow f(x), s_n \in \hat{\partial}f(x_n) \rightarrow x^*\} \quad (5.6)$$

Denote the domain of  $\partial f$  is defined as  $\text{dom}(\partial f) = \{x : \partial f \neq \emptyset\}$ .

4. A point  $x$  is called a critical point of  $f$  if  $0 \in \partial f(x)$ .

**Definition 5.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper l.s.c. function.  $f$  is said to have the Kurdyka-Lojasiewicz (KL) property at  $\bar{x} \in \text{dom}(\partial f)$  if there exist  $\eta > 0$ , a neighborhood  $U$  of  $\bar{x}$  and a continuous concave function  $\phi : [0, \eta] \rightarrow \mathbb{R}_+$  such that

- $\phi(0) = 0$
- $\phi$  is  $C^1$  on  $(0, \eta)$
- for all  $s \in (0, \eta)$ ,  $\phi'(s) > 0$
- for all  $x \in U \cap [f(\bar{x}) < f(x) < f(\bar{x}) + \eta]$ , the KL inequality holds

$$\phi'(f(x) - f(\bar{x}) \text{dist}(0, \partial f(x))) > 1$$

where  $\text{dist}(0, \partial f(x)) := \min\{\|y\| : y \in \partial f(x)\}$ .

This property was introduced in [29] and [27] and it is used in some recent work for establishing the global convergence of the proximal alternating scheme, see for instance [2, 5, 42]. However it is not trivial to verify the conditions listed above, thus there are some special classes of functions which are proved to be KL functions, such as real analytic functions, semi-algebraic functions etc.

**Definition 5.3.** A smooth function  $\phi(t)$  on  $\mathbb{R}$  is analytic if  $(\frac{\phi^{(k)}(t)}{k!})^{1/k}$  is bounded for all  $k$  and on any compact set  $D \subset \mathbb{R}$ . For a real function  $\psi(x)$  on  $\mathbb{R}^n$ , one can verify the analyticity of  $\psi$  is analytic by checking the analyticity of  $\phi(t) := \psi(x + ty)$  for any  $x, y \in \mathbb{R}^n$ .

**Definition 5.4.** A subset  $S$  of  $\mathbb{R}^n$  is called the semi-algebraic set if there exists a finite number of real polynomial functions  $g_{ij}, h_{ij}$  such that

$$S = \bigcup_j \bigcap_i \{x \in \mathbb{R}^n : g_{ij}(x) = 0, h_{ij}(x) < 0\}. \quad (5.7)$$

A function  $f(x)$  is called the semi-algebraic function if its graph  $\{(x, t) \in \mathbb{R}^n \times \mathbb{R}, t = f(x)\}$  is a semi-algebraic set in  $\mathbb{R}^{n+1}$ .

**Lemma 5.5 (Lemma 5 in [2]).** Under assumptions (5.2) and (5.4), the sequences  $(x_k, y_k)$  in (5.3) are correctly defined. Moreover, the following estimation holds

(i) The sequence  $L(x_k, y_k)$  is non-increasing and

$$L(x_k, y_k) + \frac{1}{2\theta_k} \|x_k - x_{k-1}\|^2 + \frac{1}{2\mu_{k-1}} \|y_k - y_{k-1}\|^2 \leq L(x_{k-1}, y_{k-1}), \quad \forall k \geq 1,$$

(ii) (Square summable)  $\sum_{k=1}^{\infty} (\|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2) < \infty$ , hence

$$\lim_{k \rightarrow \infty} \|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2 \rightarrow 0.$$

**Theorem 5.6** (Theorem 9 in [2]). *Assume that (5.2) and (5.4) hold. The sequence  $(x^{(k)}, y^{(k)})$  generated by the proximal alternating minimization algorithm (5.3) converges to a critical point of (5.1), if the following conditions hold:*

(i)  $L(x, y)$  is a KL function;

(ii)  $(x^k, y^k), k = 1, 2, \dots$  is a bounded sequence.

We will demonstrate the convergence of the alternating algorithm (4.1) by verifying the conditions listed above. Recall the single image segmentation model (3.12)

$$L(u, \vec{c}) = |\varphi \cdot \mathcal{W}u|_1 + \chi_{0 \leq u \leq 1} + \chi_{c_l \leq \vec{c} \leq c_u} + \langle u, \xi(c_1, c_2) \rangle + \langle \mathbf{1}, \psi(\vec{c}) \rangle \quad (5.8)$$

We rewrite it as a separable form

$$L(u, \vec{c}) = g(u) + Q(u, \vec{c}) + h(\vec{c})$$

where

$$g(u) = |\varphi \cdot \mathcal{W}u|_1 + \chi_{0 \leq u \leq 1}; \quad (5.9)$$

$$h(\vec{c}) = \langle \mathbf{1}, \psi(\vec{c}) \rangle + \chi_{c_l \leq \vec{c} \leq c_u}; \quad (5.10)$$

$$Q(u, \vec{c}) = \langle u, \xi(\vec{c}) \rangle \quad (5.11)$$

We will verify the properties of the objective function  $L(u, \vec{c})$  to justify the convergence of the consequence  $(u^K, \vec{c}^K)$  under the above framework. We first note that

$$\text{dom}(L) = \text{dom}(\partial L) = [0, 1]^n \times [c_l, c_u]^2$$

is bounded and compact.

**Lemma 5.7.** *For  $Q(u, \vec{c})$  defined as in (5.11), its gradient is Lipschitz continuous on any bounded set  $\subset \text{dom}(L)$ .*

*Proof.* Recall that  $Q(u, \vec{c}) = \langle u, \xi(c_1, c_2) \rangle$ , and its gradient is given as

$$\nabla Q(u, c) = \begin{bmatrix} \xi(c_1, c_2) \\ \langle u, \frac{\partial \xi(c_1, c_2)}{\partial c_1} \rangle \\ \langle u, \frac{\partial \xi(c_1, c_2)}{\partial c_2} \rangle \end{bmatrix} \quad (5.12)$$

where

$$\frac{\partial \xi(c_1, c_2)}{\partial c_1} = \begin{cases} 2(c_1 - f) & \text{Gaussian noise} \\ 1 - \frac{f}{c_1} & \text{Poisson noise} \end{cases}; \quad \frac{\partial \xi(c_1, c_2)}{\partial c_2} = \begin{cases} -2(c_2 - f) & \text{Gaussian noise} \\ -1 + \frac{f}{c_2} & \text{Poisson noise} \end{cases} \quad (5.13)$$

The corresponding Hessian matrices are

$$H = \begin{bmatrix} 0 & 2(c_1 - f) & -2(c_2 - f) \\ 2(c_1 - f)^\top & 2\langle u, \mathbf{1} \rangle & 0 \\ -2(c_2 - f)^\top & 0 & -2\langle u, \mathbf{1} \rangle \end{bmatrix} \quad (5.14)$$

for Gaussian noise and

$$H = \begin{bmatrix} 0 & 1 - \frac{f}{c_1} & -1 + \frac{f}{c_2} \\ 1 - \frac{f}{c_1} & \langle u, \frac{f}{c_1^2} \rangle & 0 \\ -1 + \frac{f}{c_2} & 0 & -\langle u, \frac{f}{c_2^2} \rangle \end{bmatrix} \quad (5.15)$$

for Poisson noise respectively.

Since  $\text{dom}(L)$  is bounded and  $c_l > 0$ , we can see that both (5.14) and (5.15) are bounded on any bounded subset. Namely,  $\nabla Q(u, \vec{c})$  in (5.8) is Lipschitz continuous on any bounded set.

■

**Proposition 5.8.** *Let  $(u^k, \vec{c}^k)$  be the updated sequence generated by (4.1), then*

(i) (Energy non-increasing) *The sequence  $L(u^k, \vec{c}^k)$ , where  $L(u, \vec{c})$  is defined as (5.8), is non-increasing.*

(ii) (Square summable)  $\lim_{k \rightarrow \infty} \|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2 \rightarrow 0$ .

*Proof.* By Lemma 5.5, we only need to check if the assumptions (5.2) and (5.4) are satisfied. Both  $g(u)$  and  $h(\vec{c})$  are proper l.s.c and  $Q(u, \vec{c})$  is  $C^1$ . By Lemma 5.7,  $Q(u, \vec{c})$  has Lipschitz continuous gradient. Thus Assumption (5.2) holds. Since the constraint sets for  $L(u, \vec{c})$  are bounded, it is easy to see that the first two conditions in (5.4) holds. The third condition in (5.4) can be satisfied by choosing the algorithm parameters properly. ■

**Lemma 5.9.** *The objective function  $L(u, \vec{c})$  defined in (5.8) is a KL function on  $\text{dom}(\partial L)$ .*

*Proof.* We show all the three terms in  $L(u, \vec{c})$  are real-analytic functions by [2]. Note that  $h(\vec{c}) = \langle \mathbf{1}, \psi(\vec{c}) \rangle + \mathcal{C}_{c \leq \vec{c} \leq c_u}$ . The indicator function is analytic. For Gaussian noise and Poisson case, it is easy to see that  $\psi(\vec{c})$  is real analytic on  $\text{dom}(\partial L)$ . For  $Q(u, \vec{c}) = \langle u, \xi(c_1, c_2) \rangle$ , we discuss the two cases. For Gaussian noise,  $\xi(\vec{c}) = (c_1 - f)^2 - (c_2 - f)^2$ , and it is easy to see that  $Q(u, \vec{c})$  is a semi-algebraic function. For Poisson noise,  $\xi(\vec{c}) = (c_1 - c_2) - f \log \frac{c_1}{c_2}$ , the function is real analytic on  $\text{dom}(L)$ . Finally, the term  $|\varphi \cdot \mathcal{W}u|_1$  is also semi-algebraic functions. Finally we conclude that  $L(u, \vec{c})$  is a KL function on  $\text{dom}(\partial L)$ . ■

**Theorem 5.10 (Sequence convergence).** *The sequence  $(u^k, \vec{c}^k)$  generated by (4.1) converges to a critical point of  $L(u, \vec{c})$ .*

*Proof.* By the proof in Proposition 5.8, we can see that the function  $L(u, \vec{c})$  (5.8) satisfies the assumptions (5.2) and (5.4). Due to the bounded constraints on  $u$  and  $c$ , the sequences are bounded. Together with Lemma 5.9 and Theorem 5.6, the theorem is proved. ■

## 6. Numerical experiments.

### 6.1. Methods in comparisons.

For the reason of comparison, we also consider the level set implementation for a similar variation model for single image segmentation.

$$\min_{\phi, c_l \leq \vec{c} \leq c_u} |\varphi \cdot \nabla H(\phi)|_1 + \mu \langle H(\phi), \xi(\vec{c}) \rangle + \mu \langle \mathbf{1}, \psi(\vec{c}) \rangle \quad (6.1)$$

where  $\phi$  is the usual level set function for defining the boundary and  $\xi, \psi$  are defined as (3.13) and (3.14) for Gaussian and Poisson noise respectively. For  $\vec{c}$  fixed, a gradient flow scheme as [14, 32] for solving (6.1) is given as followed

$$\begin{cases} \frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi)[\varphi \mathbf{div}(\frac{\nabla \phi}{|\nabla \phi|}) + \mu \xi(c_1, c_2)] & \text{in } (0, \infty) \times \Omega \\ \phi(0, x, y) = \phi_0(x) & \text{in } \Omega \\ \frac{\delta_\epsilon(\phi)}{|\nabla \phi|} \frac{\partial \phi}{\partial \vec{n}} = 0 & \text{on } \partial \Omega \end{cases} \quad (6.2)$$

Once  $\phi$  is fixed, the region  $\Sigma$  is obtained as  $\{\phi > 0\}$  and it is easy to see that the mean intensity for each region yields the optimal solution of (6.1) with respect to  $\vec{c}$  for both noise cases, as in classical Chan-Vese model.

We thus compare the above level set formulation (6.1) with the proposed wavelet frame model (3.12). For the two models, we solve with the above gradient flow (6.2) and the proposed scheme (4.1) (Algorithm 1) respectively. Combining with different shape priors  $\varphi$  based on edge indicator (2.14) and distance function (3.15), and different noise statistics: Poisson (3.14) and Gaussian (3.13), we investigate the five methods for individual image segmentation, as listed in Table 6.1.

**Table 6.1**  
*Methods in comparisons*

Methods	Model	Noise	Shape prior
M1	level set (6.1)	Poisson	(2.14)
M2	level set (6.1)	Poisson	(3.15)
M3	wavelet frame (3.12)	Poisson	(2.14)
M4	wavelet frame (3.12)	Gaussian	(3.15)
M5	wavelet frame (3.12)	Poisson	(3.15)

**6.2. Multi-level implementation.** In the implementation of the wavelet frame method for single image segmentation, we use the piecewise linear B-spline wavelets with 2 levels of decomposition for  $\mathcal{W}$ . The corresponding filter masks chosen here are

$$h_0 = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}], h_1 = [-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}], h_2 = [\frac{\sqrt{2}}{4}, 0, -\frac{\sqrt{2}}{4}].$$

We also use the coarse-to-fine multilevel implementation to accelerate the computation speed. The procedure is to perform segmentation on a downsampled smoothed image of each frame  $f_t$  and then the low-resolution segmentation results are upsampled and used as the initial  $u_0$  for higher resolution image segmentation. For instance, each image of our video dataset is with  $512 \times 512$  pixels, and we first run the segmentation for the resized image  $64 \times 64$ , and then resize the segmentation  $128 \times 128$  as initial  $u_0$ , etc. Finally, we get the segmentation result for the fine scale image. An illustration is shown in Figure 6.1.

**6.3. Results on real data sets.** In this section, numerical experiments on real ultrasound video data are presented for different methods listed in Table 6.1. The computation are performed on a PC with an intel i5-3570 3.4GHZ CPU and 8GB memory.

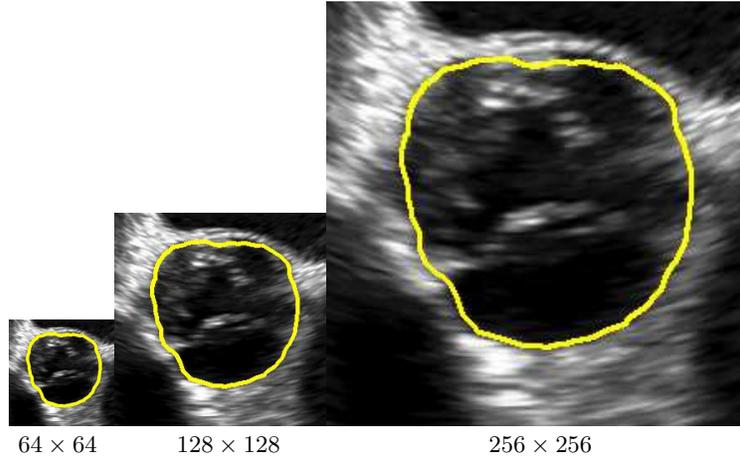


Figure 6.1. The coarse-to-fine method

The size of the first ultrasound video is  $455 \times 505 \times 97$  and some frames are shown in Figure 6.2. For this data set, an initial shape prior  $\Sigma_0$  used in the Algorithm for the first frame  $f_1$  is hand-drawn contour by an expert and then the sequential shape priors for other frames are computed according to (2.14) or (3.15).

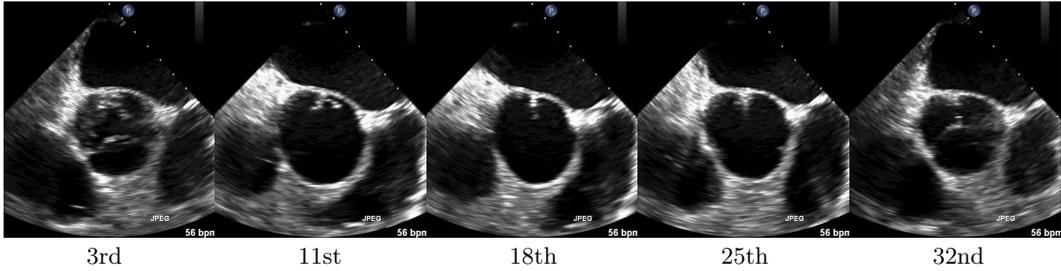


Figure 6.2. Some frames of Dataset 1

For the distance function prior, we use the segmentation result of two previous frames and empirically set the function  $\varphi$  as

$$\varphi = \max\{s(d_{t-1})^{1.5}, s(d_{t-2})^{1.5}\}$$

where  $s$  is a parameter for a balance of efficiency and quality. In our tests, we find out if the data set is of high quality, only one previous frame use can works sufficiently well. In fact, more information of previous frame are utilized, the model will be more stable, while the computation cost will increase. Thus the choice of sequential shape prior is of a balance of efficiency and quality.

In Figure 6.3, we show the segmentation results for some selected frames, for which where hand-drawn contours by a physician with expertise are available. These contours are considered as the ground truths, as shown in the first row; In the 2nd to 6th row, we list the corresponding results with M1 to M 5. Note that we have optimized the parameters for each method. We can see that both methods with edge function (2.14) as shape priors (M1 and

**M3**) fail to track the regions of interests, and **M2** and **M4** may introduce some undesirable regions during the whole procedure of video segmentation. Once those regions appeared, it will affect the sequential frames. The method **M5**, using the wavelet frame method together with Poisson noise and distance prior appears to be more stable for the contour tracking.

To quantitatively measure the segmentation quality, we use the Dice coefficient defined as

$$R = \frac{|\{x \in \Omega : u(x) = u_e(x)\}|}{|\Omega|} \quad (6.3)$$

where  $u_e(x)$  is the ground truth segmentation. The numerical results of this quantity for different methods are listed in Table (6.3), and we can see that the accuracy by **M5** is higher for most of the frames.

**Table 6.2**  
Segmentation accuracy (%) of Dataset 1

Method	3rd	11th	18th	25th	32nd
<b>M1</b>	67.70	64.49	62.07	63.67	62.44
<b>M2</b>	95.23	92.38	<b>94.48</b>	93.74	93.87
<b>M3</b>	63.47	67.44	64.81	66.10	64.40
<b>M4</b>	96.22	92.99	94.05	93.23	91.41
<b>M5</b>	<b>96.23</b>	<b>93.01</b>	94.20	<b>94.07</b>	<b>94.54</b>

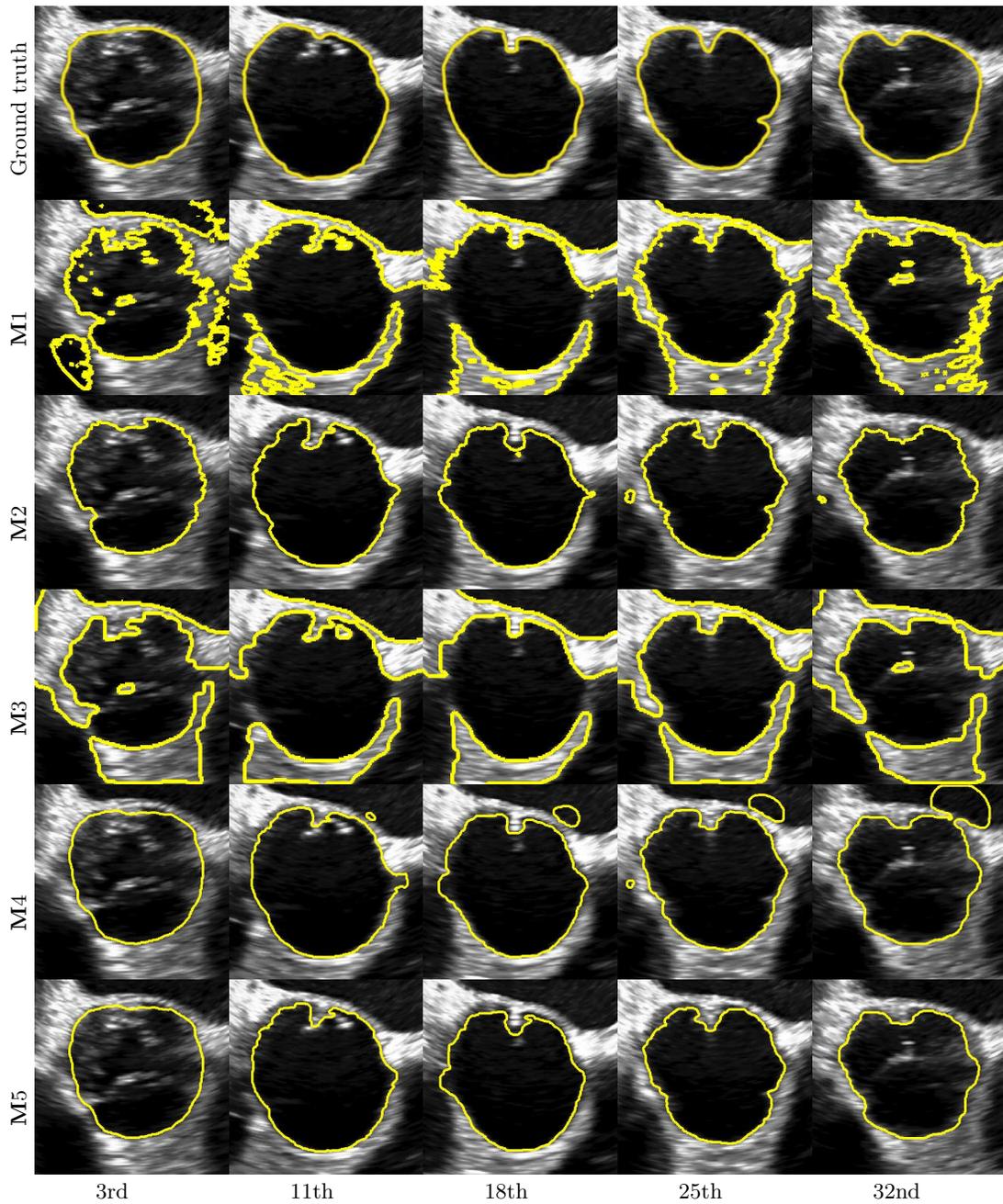
To illustrate the relation between the solution of the relaxed model and the binarized solution, we show the results in Figure 6.4 Method **M5** without proceeding the Step 3 in Algorithm 1 .

A more challenging data set of size  $455 \times 505 \times 63$  is also tested and shown in Figure 6.5. The numerical results with different methods on this data set are shown in Figure 6.6. The contours tend to leak out and it is harder to keep the topology for this data set. We have similar observations for the five methods in comparison as for the data set 1. We can see that the tight frame method with Poisson/Gaussian statistics with distance prior achieve the best performance and the Poisson one achieves slightly better result for some frames. Further quantitative comparison is present in Table 6.3.

**Table 6.3**  
Segmentation accuracy (%) of Dataset 2

Method	1st	9th	15th	22nd	29th
<b>M1</b>	35.21	33.39	35.19	34.37	34.90
<b>M2</b>	96.38	95.91	95.16	95.08	93.56
<b>M3</b>	36.67	35.14	37.56	36.81	36.88
<b>M4</b>	96.36	96.58	<b>96.45</b>	<b>96.17</b>	95.03
<b>M5</b>	<b>96.41</b>	<b>96.59</b>	96.34	96.06	<b>95.19</b>

Finally, we present the computation time of the five methods for the whole video sequence. We notice that the tight frame models **M3**, **M4** and **M5** reduce the computation time significantly compared to level set formulations. The multi-level implementation also reduce the



**Figure 6.3.** First row, the ground truths drawn by a physician with expertise; Second row,  $M1$  with  $\mu = 2.3$ ; Third row,  $M2$  with  $s = 2.4$  and  $\mu = 30.6$ ; Fourth row,  $M3$  with  $\mu = 2.3$ ; Fifth row,  $M4$  with  $s = 2.4$  and  $\mu = 20.4$ ; Sixth row,  $M5$  with  $s = 2.4$  and  $\mu = 30.6$ .

computation time in a factor of 2. Thus this model can be potentially applied for a real time tracking for operation navigation in practice.

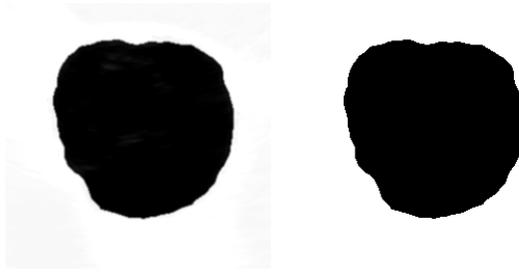


Figure 6.4. The solution before and after the thresholding Step 3 in Algorithm 1

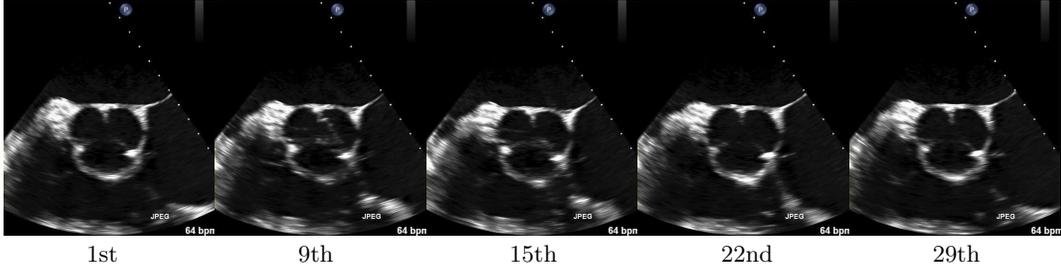


Figure 6.5. Some frames of Dataset 2

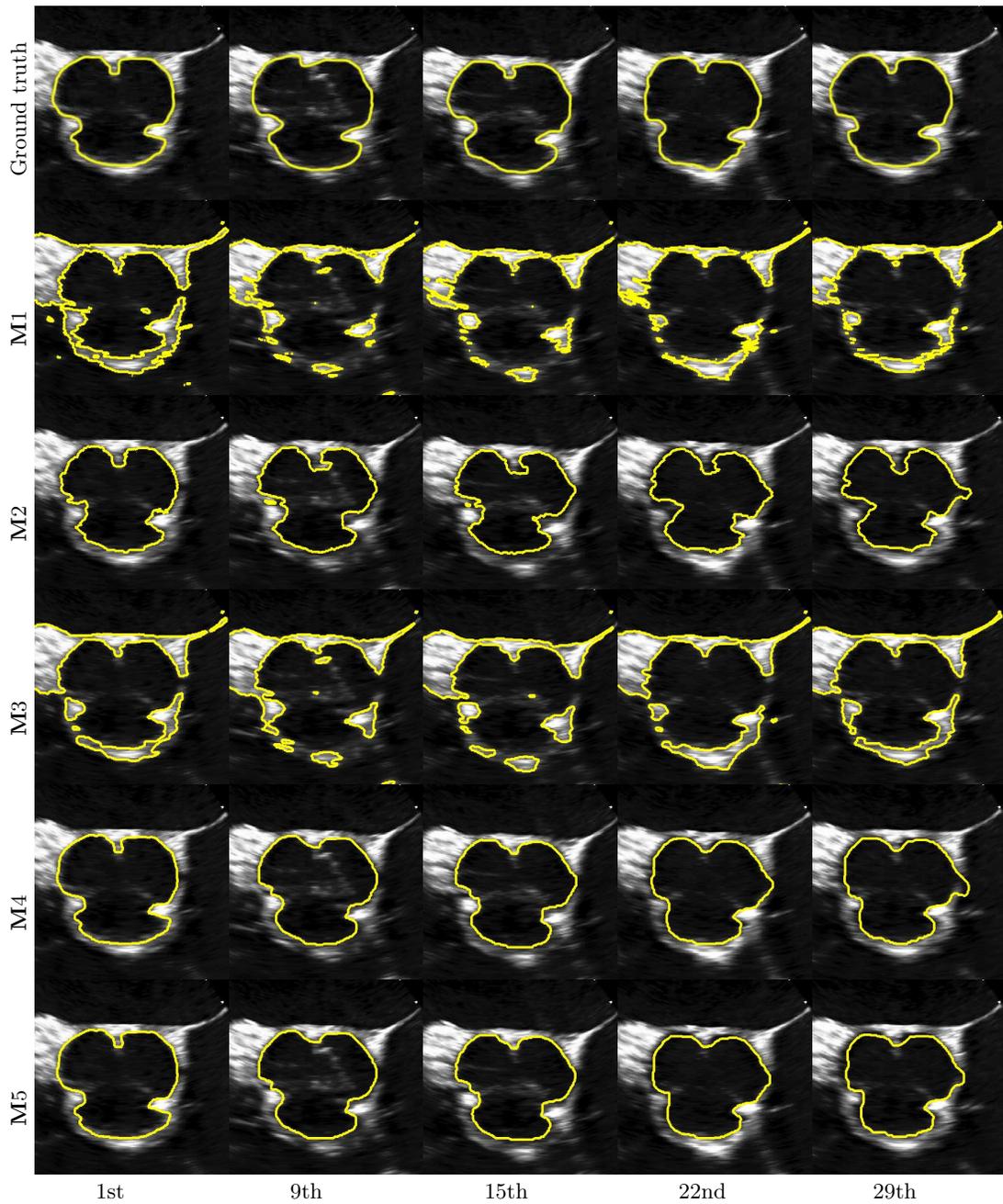
Table 6.4

Computation time for whole video segmentation( in second)

Method	Dataset 1	Dataset 2
M1	604.5	447.3
M2	511.5	365.4
M3	122.4	59.8
M4	80.7	55.0
M5	82.3	55.4

**7. Conclusion.** We have proposed a general wavelet frame based model for tracking the regions of interest. The proposed model allows a general assumption for the noise present in the data and incorporate the shape prior by using weighted wavelet frame regularization. The numerical results on real ultrasound data sets demonstrate that the proposed wavelet frame model with distance prior can track the regions of interest effectively. Furthermore, the proposed model for single image segmentation can be solved by the proximal alternating minimization algorithm and the convergence of the iterative sequence is established. Compared to the level set method, the proposed scheme is significantly faster. Additionally, the use of Poisson noise fidelity term can slightly improve the performance for ultrasound segmentation compared to the usual Gaussian noise assumption.

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**Figure 6.6.** First row, the ground truths drawn by a physician with expertise; Second row,  $M1$  with  $\mu = 2.2$ ; Third row,  $M2$  with  $s = 2.0$  and  $\mu = 30.6$ ; Fourth row,  $M3$  with  $\mu = 2.2$ ; Fifth row,  $M4$  with  $s = 2.0$  and  $\mu = 20.4$ ; Sixth row,  $M5$  with  $s = 2.0$  and  $\mu = 30.6$ .

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