EXAMPLES OF REFINABLE COMPONENTWISE POLYNOMIALS

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ABSTRACT. This short note presents four examples of compactly supported symmetric refinable componentwise polynomial functions: (i) a componentwise constant interpolatory continuous refinable function and its derived symmetric tight wavelet frame; (ii) a componentwise constant continuous orthonormal and interpolatory refinable function and its associated symmetric orthonormal wavelet basis; (iii) a differentiable symmetric componentwise linear polynomial orthonormal refinable function; (iv) a symmetric refinable componentwise linear polynomial which is interpolatory and differentiable.

This note presents four examples of compactly supported symmetric refinable functions with some special properties such as the componentwise polynomial property, which is defined to be

Definition. We say that a function $\phi : \mathbb{R} \mapsto \mathbb{C}$ is a *componentwise polynomial* if there exists an open set G such that the Lebesgue measure of $\mathbb{R} \backslash G$ is zero and the restriction of ϕ on every connected component of G coincides with a polynomial. Of courses, on different components ϕ may coincide with different polynomials.

It is clear that a compactly supported piecewise polynomial (i.e. the open set G has only finitely many connected components), which is called a spline, is a componentwise polynomial. Therefore, although a componentwise polynomial is generally not a spline, it is closely related to a spline and generalizes the concept of a spline. The difference between a componentwise polynomial and a spline lies in that it can have infinitely many "pieces" and the "knots" could consist of a compact set, which may have cluster points and therefore, not knots any more in the sense of the theory of splines. For example, a nontrivial compactly supported componentwise constant could be continuous, as shown in Example 1. Componentwise polynomials were first introduced in [1, 10] under the name of local polynomials. Some basic properties of componentwise polynomials can be found in [1, 10]. It was shown in [1, 10] that a compactly supported refinable componentwise polynomial has an analytic form. In particular, an iteration formula is given in [1, Lemma 2] to compute the polynomial on each component.

We say that a function ϕ is *interpolatory* if ϕ is continuous and satisfies $\phi(0) = 1$ and $\phi(k) = 0$ for all $k \in \mathbb{Z} \backslash \{0\}$. We say that ϕ is *orthonormal* if $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system (sequence) in $L_2(\mathbb{R})$. It is proven in [7] that a compactly supported refinable spline whose shifts form a Riesz system must be a B-spline function, up to an integer shift. So, the only refinable orthonormal spline is $\chi_{[0,1]}$, the discontinuous characteristic function of $[0, 1]$. The only spline interpolatory refinable function ϕ is the hat function, which is not differentiable. Extending the concept of piecewise polynomials (that is, splines) to componentwise polynomials, we are able to construct four interesting examples: the first one is a compactly supported refinable componentwise constant which is symmetric and continuous. This immediately leads to an example of shortly supported symmetric tight wavelet frame such that each framelet is continuous. The second one is a compactly supported refinable componentwise constant which is symmetric, continuous, interpolatory and orthonormal, plus whose mask has rational coefficients. This immediately leads to a componentwise constant symmetric orthonormal wavelet basis which is

continuous. The third example is a compactly supported refinable componentwise linear polyno-
mial which is symmetric, differentiable and orthonormal. The last one is a compactly supported
refinable componentwise linear polynomial which is symmetric, differentiable and interpolatory.

A function ϕ is *M-refinable* if it satisfies $\widehat{\phi}(M\xi) = H(\xi)\widehat{\phi}(\xi)$, where the *mask H* is a 2π-periodic
trigonometric polynomial and $\widehat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-i\xi t}\, dt$ for $f \in L_1(\mathbb{R})$. In other words, a compactly
supported (normalized) M-refinable function (or distribution) ϕ with mask H is obtained by
$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} H(M^{-j}\xi)$, $\xi \in \mathbb{R}$. If a mask H is given by

$$(1) \quad H(\xi) = (1 + e^{-i\xi} + \cdots + e^{-i(M-1)\xi})^N Q_r(\xi) \quad \text{with} \quad H(0) = 1, \quad Q_r(\xi) := \sum_{k=0}^{r} q(k)e^{-ik\xi},$$

where N is a positive integer and $0 < r < M - 1$, then it has been proved in [1, Theorem 1'] and
[10, Theorem 2.12.1] that ϕ is a componentwise polynomial and the degree of the polynomial
on each component is no more than $N - 1$. In this note, we are particularly interested in the
mask H taking the form of (1) so that the corresponding refinable function ϕ is a componentwise
polynomial with some desirable properties such as interpolation and orthogonality properties.

For $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$, we say that $f \in \text{Lip}(\alpha, L_p(\mathbb{R}))$, if there is a constant C such
that $\|f - f(\cdot - h)\|_{L_p(\mathbb{R})} \leq Ch^\alpha$ for all $h > 0$. The smoothness of a function ϕ is measured by

$$(2) \qquad \nu_p(\phi) := \sup\{n + \alpha \; : \; n \in \mathbb{N} \cup \{0\}, 0 < \alpha \leq 1, \phi^{(n)} \in \text{Lip}(\alpha, L_p(\mathbb{R}))\}.$$

In order to discuss interpolatory and orthonormal M-refinable functions, let us recall a quantity
$\nu_p(H, M)$ from [4]. For a 2π-periodic trigonometric polynomial H with $H(0) = 1$, we can write
$H(\xi) = (1 + e^{-i\xi} + \cdots + e^{-i(M-1)\xi})^N Q(\xi)$ for some 2π-periodic trigonometric polynomial Q such
that $\sum_{\mu=1}^{M-1} |Q(2\pi\mu/M)| \neq 0$. As in [4, Page 61 and Proposition 7.2], we define

$$(3) \qquad \nu_p(H, M) := 1/p - 1 - \log_M[\limsup_{n\to\infty} \|Q_n\|_{\ell_p(\mathbb{Z})}^{1/n}], \quad 1 \leq p \leq \infty,$$

where $\|Q_n\|_{\ell_p(\mathbb{Z})}^p := \sum_{k\in\mathbb{Z}} |Q_n(k)|^p$ and $\sum_{k\in\mathbb{Z}} Q_n(k)e^{-ik\xi} := Q(M^{n-1}\xi)Q(M^{n-2}\xi)\cdots Q(M\xi)Q(\xi)$.
It was proved in [4, Theorem 4.3] that the cascade algorithm with mask H converges in $L_p(\mathbb{R})$
(as well as $C(\mathbb{R})$ when $p = \infty$) if and only if $\nu_p(H, M) > 0$. Let ϕ be the compactly supported
normalized M-refinable function with mask H. In general, we have $\nu_p(H, M) \geq \nu_\infty(H, M)$ ([4,
(4.7)]) and $\nu_p(H, M) \leq \nu_p(\phi)$. If the shifts of ϕ form a Riesz system, then $\nu_p(H, M) = \nu_p(\phi)$. The
quantity $\nu_p(H, M)$ plays an important role in the study of the convergence of cascade algorithms
and smoothness of refinable functions, (see e.g. [4] and references therein). It is well known (see
e.g. [3, 4, 5, 6, 8]) that ϕ is an interpolatory function if and only if (i) the cascade algorithm
with mask H converges in $C(\mathbb{R})$, i.e., $\nu_\infty(H, M) > 0$; (ii) its mask H is interpolatory, i.e.,
$\sum_{\mu=0}^{M-1} H(\xi + 2\pi\mu/M) = 1, \xi \in \mathbb{R}$. Similarly, an M-refinable function ϕ is orthonormal if and only
if (i) the cascade algorithm with mask H converges in $L_2(\mathbb{R})$, i.e., $\nu_2(H, M) > 2$; (ii) the mask H is
orthogonal, i.e., $\sum_{\mu=0}^{M-1} |H(\xi + 2\pi\mu/M)|^2 = 1, \xi \in \mathbb{R}$. Assume that $\nu_\infty(H, M) > 0$ and H is either
interpolatory or orthogonal, then ϕ is interpolatory or orthonormal and $\nu_\infty(\phi) = \nu_\infty(H, M)$. If
a mask H takes the form of (1) with $0 \leq r \leq M - 1$, then by [3, Corollary 2.2],

$$(4) \qquad \nu_\infty(H, M) = -1 - \log_M \max(|q(0)|, \ldots, |q(r)|).$$

In all our examples, the mask H is constructed so that it satifies either interpolatory or orthogonal
(or both). Then, we compute $\nu_\infty(H, M)$ by (4) which turns out always larger than zero. Hence,
we conclude that the corresponding refinable function is interpolatory or orthonormal (or both).

Example 1. Let ϕ be the 3-refinable function with an interpolatory mask

$$H(\xi) = (1 + e^{-i\xi} + e^{-i2\xi})(c + (1 - c)e^{i\xi})/3, \qquad c \in \mathbb{R}.$$

By (4), $\nu_\infty(H, 3) = -\log_3 \max(|1 - c|, |c|) \leq \log_3 2 \approx 0.630930$. The equality holds if and only if
$c = 1/2$. By [1, Theorem 1'] and [10, Theorem 2.12.1], it is a componentwise constant. For $c = 0$,

it is just the characteristic function $\chi_{(-1/2,1/2)}$. For $c = 1/2$, since $\nu_\infty(H, 3) = \log_3 2 > 0$, ϕ is interpolatory and $\nu_\infty(\phi) = \nu_\infty(H, 3)$. Moreover, ϕ is supported on $[-1/2, 1]$ and $\phi(1/2 - \cdot) = \phi$. Using the unitary extension principle in [9], we obtain a tight wavelet frame whose wavelet masks are given by

$$\frac{\sqrt{2}}{6}\left(e^{-2i\xi} - e^{-i\xi} - 1 + e^{i\xi}\right), \quad \frac{\sqrt{3}}{6}\left(e^{-2i\xi} - e^{i\xi}\right), \quad \frac{\sqrt{6}}{6}\left(e^{-i\xi} - 1\right).$$

See Figure 1 for graphs of the interpolatory refinable function ϕ and its tight wavelet frame.
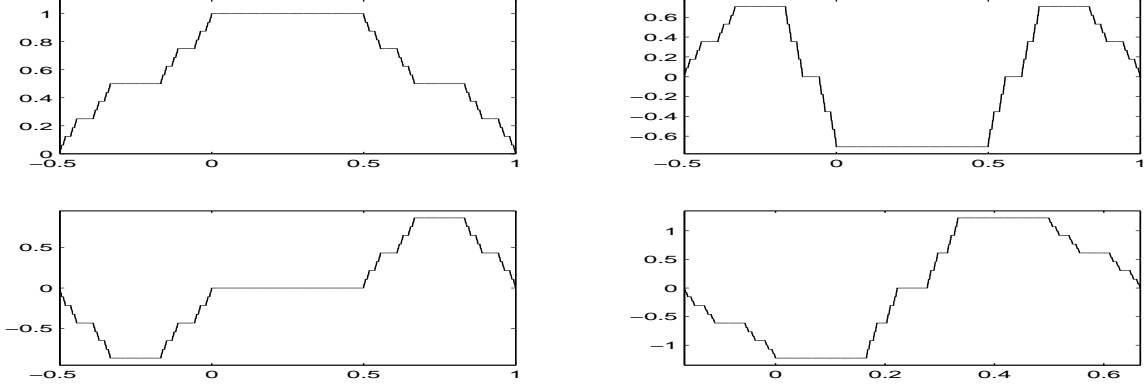

FIGURE 1. The refinable function (top left corner) and the three generators of the tight wavelet frame in Example 1.

An iteration formula is given in [1, Lemma 2] to compute the polynomial on each component. To illustrate the structure of the above componentwise polynomial, we give the analytic form of ϕ of above example. For this, we need to present the analytic form of ϕ on every connected component of an open set G, where $G \subseteq \operatorname{supp}(\phi)$ and $\operatorname{supp}(\phi) \backslash G$ has measure zero. Assume that $c \neq 0, 1$. Then $\operatorname{supp}(\phi) = [-1/2, 1]$. The refinement equation in time domain becomes

$$(5) \qquad \phi(x) = (1 - c)\phi(3x + 1) + \phi(3x) + \phi(3x - 1) + c\phi(3x - 2)$$

and by the partition unity of ϕ, we have that

$$(6) \qquad \phi(x) + \phi(x + 1) = 1 \quad \forall x \in (-1/2, 1/2); \quad \phi(x) = 1, \forall x \in (0, 1/2).$$

Then, for any given $k \geq 1$ and $\epsilon_j \in \{0, 1\}, 1 \leq j \leq k$, define the open intervals

$$A_{(\epsilon_1,\dots,\epsilon_k)} := \left(\sum_{j=1}^{k} 3^{-j}\epsilon_j + 2^{-1}3^{-k} - 2^{-1} \ , \ \sum_{j=1}^{k} 3^{-j}\epsilon_j + 3^{-k} - 2^{-1}\right).$$

Let $\mathcal{O} := \cup_{k=1}^{\infty} \cup_{\epsilon_j \in \{0,1\}, 0 \leq j \leq k-1, \epsilon_k=0} A_{(\epsilon_1,\dots,\epsilon_k)}$. Then $\mathcal{O} \subseteq (-1/2, 0)$. Set $G := \mathcal{O} \cup (0, 1/2) \cup (\mathcal{O}+1)$. Then $[-1/2, 1] \backslash G$ has measure zero. Now we compute the values of ϕ on G. First, we note that $\phi(x) = 1$ on $(0, 1/2)$. Next, it is clear that $\phi(x) = 1 - c$, $x \in A_{(0)}$. Since ϕ is constant on the interval $A_{(0)}$, we simply write it as $\phi(A_{(0)}) = 1 - c$. Similarly, $\phi(A_{(1)}) = 1$. For other intervals in $\mathcal{O}$, the values of ϕ are defined iteratively by

$$(7) \qquad \phi(A_{(0,\epsilon_1,\dots,\epsilon_k)}) = (1 - c)\phi(A_{(\epsilon_1,\dots,\epsilon_k)}), \qquad \phi(A_{(1,\epsilon_1,\dots,\epsilon_k)}) = (1 - c) + c\phi(A_{(\epsilon_1,\dots,\epsilon_k)}).$$

Finally, the values of ϕ on $\mathcal{O} + 1$ can be defined by (6) from the values of ϕ on $\mathcal{O}$.

Example 2. Let ϕ be the 6-refinable function with an orthogonal and interpolatory mask

$$H(\xi) = e^{i5\xi}(1 + e^{-i\xi} + \dots + e^{-i5\xi})\left[-(1 + e^{-i4\xi}) + 3(e^{-i\xi} + e^{-i3\xi}) + e^{-i2\xi}\right]/30.$$

By (4), $\nu_\infty(H, 6) = -\log_6(3/5) \approx 0.285097$. By [1, 10], ϕ is a componentwise constant polynomial. Since $\nu_\infty(H, 6) > 0$ and H is interpolatory and orthogonal, ϕ is both interpolatory and orthonormal with $\nu_\infty(\phi) = \nu_\infty(H, 6)$. Moreover, ϕ is supported on $[-1, 4/5]$ and $\phi(-1/5 - \cdot) = \phi$.

Note that the mask H has rational coefficients. We also obtain five symmetric orthonormal wavelets as given in Figure 2 (together with ϕ) with the wavelet masks given below.

$$\frac{\sqrt{3}}{6}\big[(e^{i\xi}-1)\big], \qquad \frac{\sqrt{15}}{30}\big[(e^{i3\xi}-e^{-i2\xi})+2(e^{i2\xi}-e^{-i\xi})\big], \qquad \frac{\sqrt{15}}{30}\big[(e^{i5\xi}-e^{-i4\xi})-2(e^{i4\xi}-e^{-i3\xi})\big],$$

$$\frac{\sqrt{42}}{84}\big[(e^{i3\xi}+e^{-i2\xi})+2(e^{i2\xi}+e^{-i\xi})-3(e^{i\xi}+1)\big],$$

$$\frac{\sqrt{14}}{420}\big[14(e^{i5\xi}+e^{-i4\xi})-28(e^{i4\xi}+e^{-i3\xi})+3(e^{i3\xi}+e^{-i2\xi})+6(e^{i2\xi}+e^{-i\xi})+5(e^{i\xi}+1)\big].$$
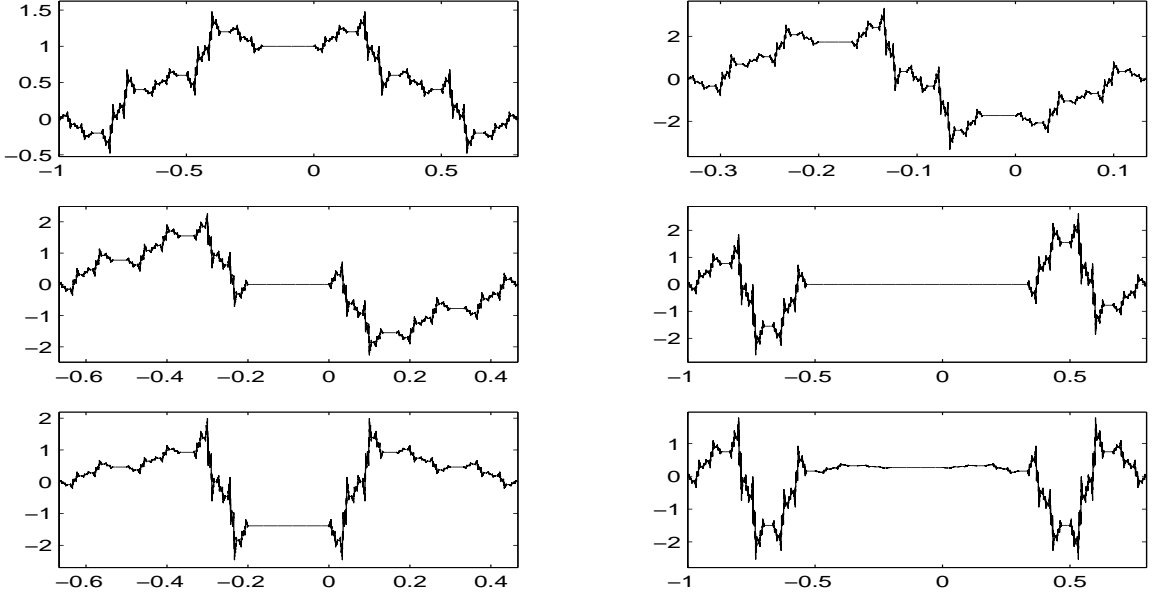

FIGURE 2. The symmetric, continuous, orthonormal and interpolatory refinable componentwise constant polynomial ϕ (top left corner) and the five associated orthonormal and symmetric wavelet functions in Example 2.

A few examples of refinable functions that are both interpolatory and orthonormal were constructed in [3, 6], but none of them are componentwise polynomials and their supports are relatively large. In general, for the construction of interpolatory or orthonormal refinable functions in one variable, one always sets it to be the convolution of a B-spline with a distribution. The B-spline component normally provides the smoothness of the resulting refinable function while the distribution part helps to obtain the required interpolation or orthogonality property. The distribution part takes away the smoothness from the B-spline, hence, the corresponding refinable function normally is not as smooth as the spline component. The examples provided here are different. The distribution part (which is a Cantor measure) not only helps to obtain the required interpolation or orthogonality property, it also improves the smoothness of the refinable function obtained from the convolution of the distribution with the spline component.

Next, we give two examples of symmetric and differentiable componentwise linear polynomials which are either orthonormal or interpolatory.

Example 3. Let ϕ be the 8-refinable function with an orthogonal mask

$$H(\xi) = (1+e^{-i\xi}+\cdots+e^{-i7\xi})^2\big[(\sqrt{403}-58)(1+e^{-i6\xi})+(53-2\sqrt{403})(e^{-i\xi}+e^{-i5\xi})$$
$$+(58-\sqrt{403})(e^{-i2\xi}+e^{-i4\xi})+(4\sqrt{403}-58)e^{-i3\xi}\big]/3072.$$

By (4), $\nu_\infty(H,8)=1-\log_8(29/24-\sqrt{403}/48)\approx 1.11329$. By [1, 10], ϕ is a componentwise linear polynomial. Since $\nu_\infty(H,8)>0$ and H is orthogonal, ϕ is orthonormal and $\nu_\infty(\phi)=\nu_\infty(H,8)$. ϕ is supported on $[0,20/7]$ and $\phi(5/7-\cdot)=\phi$. The refinable function ϕ is given in Figure 3 (left)

Example 4. Let ϕ be the 6-refinable function with an interpolatory mask

$$H(\xi) = e^{i7\xi}(1 + e^{-i\xi} + \cdots + e^{-i5\xi})^2 \big[-(1 + e^{-i4\xi}) + 2(e^{-i\xi} + e^{-i3\xi}) + 2e^{-i2\xi}\big]/144.$$

By (4), $\nu_\infty(H, 6) = 1 - \log_6(1/2) \approx 1.38685$. By [1, 10], ϕ is a componentwise linear polynomial. Since $\nu_\infty(H, 6) > 0$ and H is interpolatory, ϕ is interpolatory and $\nu_\infty(\phi) = \nu_\infty(H, 6)$. ϕ is supported on $[-7/5, 7/5]$ and $\phi(-\cdot) = \phi$. The refinable function ϕ is given in Figure 3 (right).
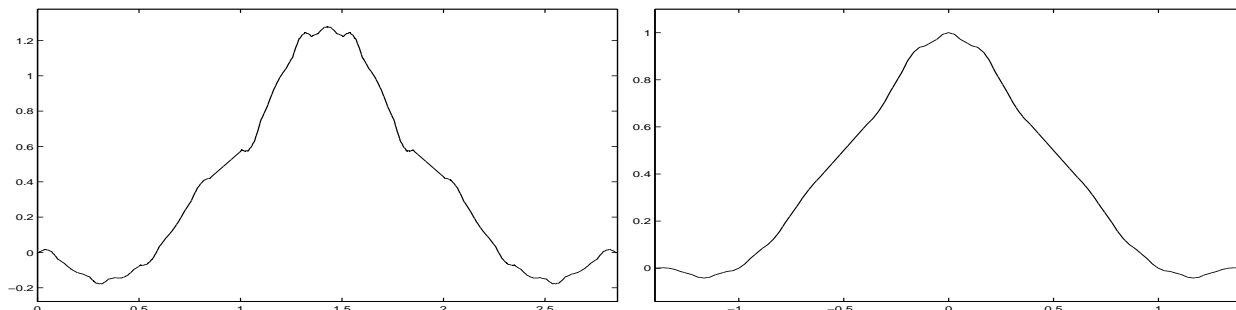

FIGURE 3. Left is the symmetric, orthonormal and differentiable refinable componentwise linear polynomial ϕ in Example 3. Right is the symmetric, interpolatory and differentiable refinable componentwise linear polynomial in Example 4.

One may notice that all the above four examples have dilation factor $M > 2$. In fact, it is proven in [2] that for dilation $M = 2$, a compactly supported refinable componentwise polynomial must be a B-spline function. So, for dilation $M = 2$, the only compactly supported orthonormal refinable componentwise polynomial is the Haar function $\chi_{[0,1]}$. The only interpolatory refinable componentwise polynomial ϕ must be the hat function. The above examples illustrate that for dilation $M > 2$, we have refinable functions with some extra interesting properties such as the componentwise polynomial property, symmetry, orthogonality and interpolation.

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