On Wilkie and Paris's notion of fullness

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10 December, 2018

Abstract

In their comprehensive study of the End-Extension Question, Wilkie and Paris devised a special notion of saturation called *fullness* for models of arithmetic (*Logic, Methodology and Philosophy of Science VIII*, North-Holland Publishing Company, Amsterdam, 1989, pages 143–161). In this paper, we take a closer look at their notion of fullness and refine some results in the original Wilkie–Paris paper. In particular, we characterize fullness in terms of the existence of recursively saturated end extensions. From this we see that every countable $\mathrm{I}\Delta_0$-full model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ is $(\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1)$-full.

1 Introduction

Fact 1.1 (folklore [34, Theorem 1])**.** If a model $M \models \mathrm{I}\Delta_0$ has a proper end extension $K \models \mathrm{I}\Delta_0$, then $M \models \mathrm{B}\Sigma_1$. $\square$

The proof of Fact 1.1 is a simple overspill argument. It is natural to ask whether some converse holds. This question has been known as the *End-Extension Question*. In the list of open problems edited by Clote and Krajíček [11], it is regarded as *the* fundamental problem in fragments of Peano arithmetic. There are two common formulations.

End-Extension Question (first formulation, Paris [26, Problem 1])**.** Does every countable model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ have a proper end extension to a model of $\mathrm{I}\Delta_0$?

End-Extension Question (second formulation, Wilkie–Paris [34, page 160f.])**.** Under what (natural) conditions does a countable model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ have a proper end extension to a model of $\mathrm{I}\Delta_0$?

2010 *Mathematics Subject Classification*: Primary 03C62; Secondary 03H15.
Key words and phrases: fullness, end-extension question, model theory of arithmetic.

The classic paper by Wilkie and Paris [34] provides a detailed analysis of the End-Extension Question. Amongst other clever ideas, they devised a peculiar saturation notion called *fullness*. As shown by Wilkie and Paris, every countable $\mathrm{I}\Delta_0$-full model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ has a proper end extension to a model of $\mathrm{I}\Delta_0$. Moreover, under the hypothesis that the Δ_0-hierarchy provably collapses in $\mathrm{I}\Delta_0$, the use of $\mathrm{I}\Delta_0$-fullness here is in a sense necessary. These constitute a partial answer to both formulations of the End-Extension Question.

It is not easy to see what Wilkie and Paris's notion of fullness means. This paper originated from the author's attempt to better understand fullness. Hopefully, this paper will convince the reader that fullness is not just an ad hoc notion manufactured by Wilkie and Paris to provide a partial answer to the End-Extension Question, as it may first seem.

In Section 2, we give a self-contained introduction to fullness. The main new theorem of this paper, Corollary 2.8, tells us that fullness is equivalent to the existence of recursively saturated end extensions. We also relate fullness to short Π_1 recursive saturation. Section 3 is a survey of the connections between definable satisfaction predicates for Δ_0 formulas and the End-Extension Question. In Section 4, we conclude the paper with some discussions on the results presented here. We put together many conditions that are equivalent to $\mathrm{I}\Delta_0$-fullness in Corollary 4.1. We pose several related questions that are hopefully easier to answer than the original End-Extension Question. Throughout the paper, we provide a generous number of pointers to the literature to give the reader some idea of the bigger picture.

The notation we use is mostly standard [20, 22]. By convention, all tuples have finite lengths. If $\mathcal{L}$ is a language and M is a set, then we denote by $\mathcal{L}(M)$ the language obtained from $\mathcal{L}$ by adding a new constant symbol m for every $m \in M$. Similarly, if Θ is a set of formulas in a language $\mathcal{L}$, then $\Theta(M)$ denotes the set of all $\mathcal{L}(M)$ formulas of the form $\theta(\bar{m})$, where $\theta(\bar{x}) \in \Theta$ and $\bar{m} \in M$. We often write $\mathcal{L}(\bar{m})$ for $\mathcal{L}(\{\bar{m}\})$ and $\Theta(\bar{m})$ for $\Theta(\{\bar{m}\})$.

The language for first-order arithmetic is $\mathcal{L}_\mathrm{A} = \{0, 1, +, \times, <\}$. The formula class Δ_0 is closed under Boolean operations and bounded quantification. Following Adamowicz–Kossak [8], we write Π_1^G for the closure of Δ_0 under bounded quantification and universal quantification. The theory $\mathrm{B}\Sigma_1$ includes $\mathrm{I}\Delta_0$. We denote by exp the usual axiom asserting the totality of exponentiation over $\mathrm{I}\Delta_0$.

2 Fullness

Roughly speaking, a Γ-full model satisfies all infinitary sentences of a certain form whose finite approximations are provable in the theory Γ. Wilkie and Paris [34] defined Γ-fullness only when Γ is a set of $\mathcal{L}_\mathrm{A}$ sentences, but their definition clearly also makes sense when Γ is in a larger language.

Definition (Wilkie–Paris [34]). Fix a recursive language $\mathcal{L}_\mathrm{A}^* \supseteq \mathcal{L}_\mathrm{A}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_\mathrm{A}(M) \cap \mathcal{L}_\mathrm{A}^* = \mathcal{L}_\mathrm{A}$. Let Γ be a set of $\mathcal{L}_\mathrm{A}^*(\bar{m})$ sentences. We say M is Γ-*full* if, for every recursive sequence $(\theta_i(x_0, x_1, \ldots, x_i))_{i \in \mathbb{N}}$ of

$\Delta_0(\bar{m})$ formulas,

$$\forall n{\in}\mathbb{N} \quad \Gamma \vdash \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i)$$

$$\Rightarrow \quad M \models \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge_{i=0}^{\infty} \theta_i(x_0, x_1, \ldots, x_i).$$

Remark 2.1. The sentences in the definition above can be rewritten as

$$\begin{aligned}&\forall x_0\ \big(\theta_0(x_0) \wedge \exists x_1{\leqslant}x_0\ \big(\theta_1(x_0, x_1)\\ &\wedge \forall x_2\ \big(\theta_2(x_0, x_1, x_2) \wedge \exists x_3{\leqslant}x_2\ \big(\theta_3(x_0, x_1, x_2, x_3)\\ &\quad \wedge \forall x_4\ \big(\theta_4(x_0, x_1, x_2, x_3, x_4) \wedge \cdots\big)\big)\big)\big)\big).\end{aligned}$$

Remark 2.2. In the definition of fullness, we restrict ourselves to sequences of formulas $(\theta_i(x_0, x_1, \ldots, x_i))_{i\in\mathbb{N}}$ in which the ith formula θ_i always mentions at most $(i+1)$-many free variables $x_0, x_1, \ldots, x_i$. This is purely for the sake of notational convenience. It is not hard to see that omitting this restriction in the definition does not change the notion of fullness.

One sees that if a structure M is Γ-full, where Γ is as above, then $M \models \Pi_1^{\mathrm{G}}\text{-Th}(\Gamma)$. Some potential ambiguity arises here: are we treating Γ as an $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ theory, or are we treating Γ as an $\mathcal{L}_{\mathrm{A}}^*(\bar{m}, \bar{m}')$ theory for some $\bar{m}' \in M \setminus \{\bar{m}\}$? Our first lemma says that this potential ambiguity does not cause any problem.

Lemma 2.3 (implicit in Wilkie–Paris [34, Proof of Theorem 4])**.** Let $\mathcal{L}_{\mathrm{A}}^*$ be a recursive language extending $\mathcal{L}_{\mathrm{A}}$, and $\mathcal{L}_{\mathrm{A}}^{**}$ be a recursive language extending $\mathcal{L}_{\mathrm{A}}^*$. Provided $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^{**} = \mathcal{L}_{\mathrm{A}}$, the following are equivalent for all $\bar{m}, \bar{m}' \in M \models \mathrm{I}\Delta_0$ and all sets of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences Γ.

(i) M is Γ-full, where Γ is viewed as a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences.

(ii) M is Γ-full, where Γ is viewed as a set of $\mathcal{L}_{\mathrm{A}}^{**}(\bar{m}, \bar{m}')$ sentences.

Proof. Suppose (i) holds. Without loss of generality, assume $\bar{m}' \notin \{\bar{m}\}$. Let $\bar{m}'$ be $m_1', m_2', \ldots, m_\ell'$. Take a recursive sequence $(\theta_i(x_0, x_1, \ldots, x_i, y_1, y_2, \ldots, y_\ell))_{i\in\mathbb{N}}$ of $\Delta_0(\bar{m})$ formulas such that

$$\forall n{\in}\mathbb{N} \quad \Gamma \vdash \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i, \bar{m}').$$

Since $\bar{m}'$ do not appear in Γ,

$$\begin{aligned}&\forall n{\in}\mathbb{N} \quad \Gamma \vdash \forall y_1\ \exists z_1{\leqslant}y_1\ \forall y_2\ \exists z_2{\leqslant}y_2\ \ldots\ \forall y_\ell\ \exists z_\ell{\leqslant}y_\ell\\ &\qquad\qquad \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i, \bar{y}).\end{aligned}$$

An application of (i) tells us

$$\begin{aligned}&M \models \forall y_1\ \exists z_1{\leqslant}y_1\ \forall y_2\ \exists z_2{\leqslant}y_2\ \ldots\ \forall y_\ell\ \exists z_\ell{\leqslant}y_\ell\\ &\qquad\qquad \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge_{i=0}^{\infty} \theta_i(x_0, x_1, \ldots, x_i, \bar{y}).\end{aligned}$$

So $M \models \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge_{i=0}^{\infty} \theta_i(x_0, x_1, \ldots, x_i, \bar{m}')$. □

What does the existence of a recursively saturated proper end extension tell us about a model of arithmetic, other than what we know from Fact 1.1? The next proposition provides an answer. It also shows that full models exist. For instance, the standard model $\mathbb{N}$ is Γ-full for every consistent set of sentences $\Gamma \supseteq Q$.

Proposition 2.4. Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences. If M has an end extension K which expands to a recursively saturated model $K^* \models \Gamma$, then M is Γ-full.

Proof. We follow the proof of Theorem 5(1) in Wilkie–Paris [34]. Take a recursive sequence $(\theta_i(x_0, x_1, \ldots, x_i))_{i\in\mathbb{N}}$ of $\Delta_0(\bar{m})$ formulas such that

$$\forall n{\in}\mathbb{N} \quad \Gamma \vdash \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots \bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i).$$

Pick any $a_0 \in M$. Since $K^* \models \Gamma$,

$$p_1(x_1) = \{x_1 \leqslant a_0\} \cup \Big\{\forall x_2\ \exists x_3{\leqslant}x_2\ \forall x_4\ \exists x_5{\leqslant}x_4\ \ldots \bigwedge_{i=0}^{n} \theta_i(a_0, x_1, x_2, \ldots, x_i) : n \in \mathbb{N}\Big\}$$

is a recursive type over K^*. Recursive saturation then gives us a_1 which realizes p_1 in K^*. Since $a_1 \leqslant a_0 \in M \subseteq_{\mathrm{e}} K$, we know $a_1 \in M$. From the definition of p_1, we see that $K \models \theta_0(a_0) \wedge \theta_1(a_0, a_1)$. So $M \models \theta_0(a_0) \wedge \theta_1(a_0, a_1)$ because $\theta_0, \theta_1 \in \Delta_0(\bar{m})$ and $M \subseteq_{\mathrm{e}} K$.

Now repeat the process: given any $a_2 \in M$, use recursive saturation to find $a_3 \in M$ which realizes

$$p_3(x_3) = \{x_3 \leqslant a_2\} \cup \Big\{\forall x_4\ \exists x_5{\leqslant}x_4\ \forall x_6\ \exists x_7{\leqslant}x_6\ \ldots \bigwedge_{i=0}^{n} \theta_i(a_0, a_1, a_2, x_3, x_4, \ldots, x_i) : n \in \mathbb{N}\Big\}$$

in K^*, etc. $\square$

A converse to Proposition 2.4 is true too, provided Γ is recursive. It follows that, in this case, the countable Γ-full models are precisely the initial segments of countable recursively saturated models of Γ. To show this converse, we need two lemmas. The heart of the proofs is a method of taking disjunctions and conjunctions of the relevant infinitary sentences that can be carried over to the finite approximations.

Lemma 2.5 (implicit in Wilkie–Paris [34, Proof of Theorem 4])**.** Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences and σ be an $\mathcal{L}_{\mathrm{A}}^*(M)$ sentence. If M is Γ-full, then M is either $(\Gamma \cup \{\sigma\})$-full or $(\Gamma \cup \{\neg\sigma\})$-full.

Proof. Suppose M is neither $(\Gamma \cup \{\sigma\})$-full nor $(\Gamma \cup \{\neg\sigma\})$-full. In view of Lemma 2.3, we may assume the tuple $\bar{m}$ already contains all the elements of M that appear in σ. Let $(\theta_i(x_0, x_1, \ldots, x_i))_{i\in\mathbb{N}}$ and $(\eta_i(y_0, y_1, \ldots, y_i))_{i\in\mathbb{N}}$ be recursive sequences of $\Delta_0(\bar{m})$ formulas such that for all $n \in \mathbb{N}$,

- $\Gamma \cup \{\sigma\} \vdash \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge\!\!\!\bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i)$;
- $\Gamma \cup \{\neg\sigma\} \vdash \forall y_0\ \exists y_1{\leqslant}y_0\ \forall y_2\ \exists y_3{\leqslant}y_2\ \ldots\ \bigwedge\!\!\!\bigwedge_{i=0}^{n} \eta_i(y_0, y_1, \ldots, y_i)$;
- $M \not\models \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \theta_i(x_0, x_1, \ldots, x_i)$; and
- $M \not\models \forall y_0\ \exists y_1{\leqslant}y_0\ \forall y_2\ \exists y_3{\leqslant}y_2\ \ldots\ \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \eta_i(y_0, y_1, \ldots, y_i)$.

Find families of functions $(f_{2i})_{i\in\mathbb{N}}$ and $(g_{2i})_{i\in\mathbb{N}}$ which make

$$\begin{aligned} M \models\ & \forall x_1{\leqslant}f_0\ \forall x_3{\leqslant}f_2(x_1)\ \ldots\ \neg \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \theta_i(f_0, x_1, f_2(x_1), x_3, \ldots) \\ & \wedge \forall y_1{\leqslant}g_0\ \forall y_3{\leqslant}g_2(y_1)\ \ldots\ \neg \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \eta_i(g_0, y_1, g_2(y_1), y_3, \ldots). \end{aligned}$$

By the choice of (θ_i) and (η_i),

$$\begin{aligned} \Gamma \vdash\ & \forall x_0\ \exists x_1{\leqslant}x_0\ \forall y_0\ \exists y_1{\leqslant}y_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \forall y_2\ \exists y_3{\leqslant}y_2\ \ldots \\ & \bigwedge\!\!\!\bigwedge_{i=0}^{n} \Big(\bigwedge\!\!\!\bigwedge_{e=0}^{i} \theta_e(x_0, x_1, \ldots, x_e) \vee \bigwedge\!\!\!\bigwedge_{e=0}^{i} \eta_e(y_0, y_1, \ldots, y_e) \Big) \end{aligned}$$

for every $n \in \mathbb{N}$. However, the choice of (f_{2i}) and (g_{2i}) implies

$$\begin{aligned} M \models\ & \forall x_1{\leqslant}f_0\ \forall y_1{\leqslant}g_0\ \forall x_3{\leqslant}f_2(x_1)\ \forall y_3{\leqslant}g_2(y_1)\ \ldots \\ & \neg \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \Big(\bigwedge\!\!\!\bigwedge_{e=0}^{i} \theta_e(f_0, x_1, f_2(x_1), x_3, \ldots) \vee \bigwedge\!\!\!\bigwedge_{e=0}^{i} \eta_e(g_0, y_1, g_2(y_1), y_3, \ldots) \Big). \end{aligned}$$

Hence M is not Γ-full. $\square$

Lemma 2.6 (Wilkie–Paris [34, Lemma 2])**.** Let $\mathcal{L}_{\mathrm{A}}^*$ be a recursive language extending $\mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. For every recursive set $\Gamma(\bar{w})$ of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ formulas with free variables $\bar{w}$, there is a recursive sequence $(\theta_i(x_0, x_1, \ldots, x_i, \bar{w}))_{i\in\mathbb{N}}$ of $\Delta_0(\bar{m})$ formulas such that

(1) for every $n \in \mathbb{N}$,

$$\Gamma(\bar{w}) \vdash \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge\!\!\!\bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i, \bar{w}),$$

where $\bar{w}$ are treated as fresh constant symbols; and

(2) for all $\bar{a} \in M \setminus \{\bar{m}\}$, the model M is $\Gamma(\bar{a})$-full if and only if

$$M \models \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots\ \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \theta_i(x_0, x_1, \ldots, x_i, \bar{a}).$$

Proof sketch. There are countably many recursive sequences of formulas that are relevant for Γ-fullness. Merge all these sequences into one. Since Γ is recursive, the resulting sequence can be made recursive too. In condition (2), we need to require the tuple $\bar{a}$ to be disjoint from the tuple $\bar{m}$ because otherwise $\Gamma(\bar{a})$ would prove many more formulas than a general $\Gamma(\bar{w})$; cf. page 150 in Wilkie–Paris [34]. $\square$

Definition. A recursive sequence of $\Delta_0(\bar{m})$ formulas $(\theta_i(x_0, x_1, \ldots, x_i, \bar{w}))_{i\in\mathbb{N}}$ satisfying conditions (1) and (2) in Lemma 2.6 is said to be *universal for* $\Gamma(\bar{w})$-*fullness*.

Wilkie and Paris [34, Theorem 4] showed that every countable $\mathrm{I}\Delta_0$-full model of $\mathrm{B}\Sigma_1$ has a proper end extension satisfying $\mathrm{I}\Delta_0$. In the following theorem, we are going to make this end extension recursively saturated. This gives the anticipated converse to Proposition 2.4. The construction by Wilkie and Paris uses the Omitting Types Theorem to ensure that no new element is added below an old element. To achieve recursive saturation, we need to realize types. In order to do both simultaneously, we unravel Wilkie and Paris's argument into a Henkin construction. Alternatively, one can use a full satisfaction class; cf. the proof of Theorem 15.8 in Kaye [22].

Notice that we do not assume $M \models \mathrm{B}\Sigma_1$ below. So, in view of Fact 1.1, we cannot require the end extension to be proper. As we will see in the proof of Corollary 2.8, it is not hard to get properness from $\mathrm{B}\Sigma_1$.

Theorem 2.7. Fix a recursive language $\mathcal{L}_\mathrm{A}^* \supseteq \mathcal{L}_\mathrm{A}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_\mathrm{A}(M) \cap \mathcal{L}_\mathrm{A}^* = \mathcal{L}_\mathrm{A}$. Let Γ be a recursive set of $\mathcal{L}_\mathrm{A}^*(\bar{m})$ sentences. If M is countable and Γ-full, then M has an end extension which expands to a recursively saturated model of Γ.

Proof. Let C be an infinite recursive set of constant symbols new to $\mathcal{L}_\mathrm{A}^*(M)$. We will build recursive sets of $\mathcal{L}_\mathrm{A}^*(M \cup C)$ sentences

$$\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$$

by recursion. At every step $s \in \mathbb{N}$, we inductively assume

(1) Γ_s mentions only finitely many elements of $M \cup C$; and

(2) M is Γ_s-full.

This construction is carried out so that $\Gamma_\omega = \bigcup_{s\in\mathbb{N}} \Gamma_s$ is complete, consistent, and Henkinized as an $\mathcal{L}_\mathrm{A}^*(M \cup C)$ theory. The $\mathcal{L}_\mathrm{A}$ reduct K of the Henkin model of Γ_ω will be the end extension we want. Notice the consistency of Γ_ω is guaranteed automatically by inductive assumption (2). Lemma 2.5 helps achieve completeness because it tells us that, for every $\mathcal{L}_\mathrm{A}^*(M \cup C)$ sentence σ, we can put either σ or $\neg\sigma$ into Γ_ω in the course of the construction.

Let us see how to ensure Henkinization and recursive saturation. Suppose Γ_s is found satisfying the inductive assumptions. Pick any recursive set of $\mathcal{L}_\mathrm{A}^*(M \cup C)$ formulas $p(\bar{v})$ with free variables $\bar{v}$ which mentions only those elements of $M \cup C$ that already appear in Γ_s. Let $(\zeta_j(\bar{v}))_{j\in\mathbb{N}}$ be a recursive enumeration of $p(\bar{v})$ and $\bar{c}$ be fresh constant symbols from C. Set

$$\Gamma_{s+1} = \Gamma_s \cup \Big\{ \exists\bar{v} \bigwedge_{j\leqslant\ell} \zeta_j(\bar{v}) \rightarrow \bigwedge_{j\leqslant\ell} \zeta_j(\bar{c}) : \ell \in \mathbb{N} \Big\}.$$

Then Γ_{s+1} is a conservative extension of Γ_s. Thus M is Γ_{s+1}-full by inductive assumption (2).

Finally, we show how to make K an end extension of M. Suppose Γ_s is found satisfying the inductive assumptions. Let $c \in C$ and $b \in M$ such that $\Gamma_s \vdash c \leqslant b$. List all elements of M mentioned in Γ_s in the sequence $m_0, m_1, \ldots, m_\ell$. If

M is $(\Gamma_s \cup \{c = m_j\})$-full for some $j \leqslant \ell$, then we are already done. So suppose not. In view of Lemma 2.5, we may assume $\Gamma_s \supseteq \{c \neq m_j : j \leqslant \ell\}$ without loss of generality. Take a recursive sequence $(\theta_i(x_0, x_1, \ldots, x_i, w))_{i\in\mathbb{N}}$ of $\Delta_0(\bar{m})$ formulas that is universal for $(\Gamma_s \cup \{c = w\})$-fullness. For every $n \in \mathbb{N}$,

$$\Gamma_s \vdash \forall u\ \exists w{\leqslant}u\ \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots \\ \Bigl(u = b \to \bigwedge_{j=0}^{\ell} w \neq m_j \wedge \bigwedge_{i=0}^{n} \theta_i(x_0, x_1, \ldots, x_i, w)\Bigr)$$

because $\Gamma_s \vdash c \leqslant b \wedge \bigwedge_{j=0}^{\ell} c \neq m_j$. Hence the Γ_s-fullness of M gives us $a \leqslant b$ in $M \setminus \{\bar{m}\}$ such that

$$M \models \forall x_0\ \exists x_1{\leqslant}x_0\ \forall x_2\ \exists x_3{\leqslant}x_2\ \ldots \bigwedge_{i=0}^{\infty} \theta_i(x_0, x_1, \ldots, x_i, a).$$

By the universality of (θ_i), we know M is $(\Gamma_s \cup \{c = a\})$-full. So we can set $\Gamma_{s+1} = \Gamma_s \cup \{c = a\}$. □

Wilkie and Paris made their end extension proper by changing the truth value of some sentence in the end extension [34, Lemma 3]. The clever trick that they used apparently works only when the set Γ in Theorem 2.7 is $\mathrm{I}\Delta_0$ or $\mathrm{B}\Sigma_1$. This was later generalized by Cornaros and Dimitracopoulos [14, Section 2] to a broader class of theories. Thanks to recursive saturation, we can avoid such a trick.

Corollary 2.8. Fix a recursive language $\mathcal{L}_\mathrm{A}^* \supseteq \mathcal{L}_\mathrm{A}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_\mathrm{A}(M) \cap \mathcal{L}_\mathrm{A}^* = \mathcal{L}_\mathrm{A}$ and M is countable. Let Γ be a recursive set of $\mathcal{L}_\mathrm{A}^*(\bar{m})$ sentences. The following are equivalent.

(i) M is Γ-full.

(ii) M has an end extension which expands to a recursively saturated model of Γ.

Furthermore, if $M \models \mathrm{B}\Sigma_1$ in addition, then the conditions above are equivalent to:

(iii) M has a proper end extension which expands to a recursively saturated model of Γ.

Proof. The equivalence between (i) and (ii) follows directly from Proposition 2.4 and Theorem 2.7.

Now suppose M satisfies $\mathrm{B}\Sigma_1$ and is Γ-full. Then Theorem 2.7 gives us an end extension of M which expands to a recursively saturated model of Γ. If this extension is proper, then we are already done. So suppose not. Then M itself expands to a recursively saturated model of Γ. By Solovay's self-embedding theorem [26, Theorem 4], the model M is isomorphic to a proper end extension K of itself. Being isomorphic to M, such K must expand to a recursively saturated model of Γ. □

Remark 2.9 (Bartosz Wcisło). Fullness, as formulated in the definition, is a Σ_2^1 property. Corollary 2.8 reduces the complexity to Σ_1^1.

Corollary 2.8 gives a natural characterization of those countable models of $\mathrm{B}\Sigma_1$ that have a proper end extension to a recursively saturated model of $\mathrm{I}\Delta_0$. It would answer the second formulation of the End-Extension Question *if*, for all countable models of $\mathrm{B}\Sigma_1$, having a proper end extension satisfying $\mathrm{I}\Delta_0$ is equivalent to having a recursively saturated one. Under the hypothesis that some satisfaction predicate for Δ_0 formulas is suitably definable over $\mathrm{I}\Delta_0$, this equivalence is known to hold — see Corollary 3.7. Without any extra hypothesis, the question whether this equivalence is true should be open.

For the sake of completeness, we give here a general saturation condition on a theory Γ under which any proper end extension satisfying Γ can be turned into a recursively saturated one. Observe that all topped proper initial segments of $\mathbb{N}$ are recursively saturated as relational structures.

Definition. Let Γ be a set of sentences in a language extending $\mathcal{L}_{\mathrm{A}}$. Then Γ is said to *prove topped recursive saturation* (with respect to $\mathcal{L}_{\mathrm{A}}$) if whenever b is an element of a countable $K^* \models \Gamma$, the initial segment of K^* with top b, equipped with the relational $\mathcal{L}_{\mathrm{A}}$ structure inherited from K^*, is recursively saturated.

As shown by Lessan [25, Proposition 4.1.5], the theory $\mathrm{I}\Delta_0 + \exp$ proves topped recursive saturation. His proof uses a suitable definable satisfaction predicate for Δ_0 formulas. However, even if such a formula exists over $\mathrm{I}\Delta_0$, it is still conceivable that $\mathrm{I}\Delta_0$ may not prove topped recursive saturation with respect to $\mathcal{L}_{\mathrm{A}}$. As shown by Cégielski, Mc Aloon and Wilmers [10, Théorème 3], with respect to the reduct $\{+\}$ of $\mathcal{L}_{\mathrm{A}}$, the theory $\mathrm{I}\Delta_0$ does prove topped recursive saturation.

In the proposition below, we restrict our attention to linearly ordered structures merely to avoid the trouble of determining what end extension should mean with respect to an arbitrary binary relation in this particular context.

Proposition 2.10. Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$ and M is countable. Let Γ be a recursive set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences which proves topped recursive saturation and $<$ is a linear order. Then the following are equivalent.

(i) M has a proper end extension that expands to a model of Γ.

(ii) M has a proper end extension that expands to a recursively saturated model of Γ.

Proof. Let K be a proper end extension of M that expands to a model of Γ. Without loss of generality, assume K is countable. Take any $b \in K \setminus M$. By topped recursive saturation, the initial segment of K with top b is recursively saturated. Now recall the following folkloric trick: one can view any extension (i.e., a structure with an embedding into it) of an infinite structure of the same cardinality as an expansion. As a result, since the initial segment of K with top b has an end extension that expands to a model of Γ, namely K, it must have one that expands to a recursively saturated such model by chronic resplendency [22, Theorem 15.8]. □

Proposition 2.10 gives many more examples of full models. For instance, it implies that if Γ is a recursive theory extending $\mathrm{I}\Sigma_1$ in a recursive language extending $\mathcal{L}_{\mathrm{A}}$, then, by a self-embedding result in Dimitracopoulos–Paris [17, Corollary 2.4] and Ressayre [29, Theorem 1.I(a)], the $\mathcal{L}_{\mathrm{A}}$ reduct of any countable model of Γ is Γ-full.

For each $k \in \mathbb{N}$, denote by Ω_k the usual axiom asserting the totality of the function ω_k over $\mathrm{I}\Delta_0$, where $\omega_0 \colon x \mapsto 2x$ and $\omega_{k+1} \colon x \mapsto 2^{\omega_k(\log x)}$. Adamowicz [4, page 4] noted that every model of $\mathrm{B}\Sigma_1 + \exp$ of countable cofinality has a proper end extension to a model of $\mathrm{B}\Sigma_1 + \{\Omega_k : k \in \mathbb{N}\}$. A proof can be found in Adamowicz [1] and Kołodziejczyk [23, bottom of page 634]. Therefore, every countable model of $\mathrm{B}\Sigma_1 + \exp$ is $(\mathrm{B}\Sigma_1 + \{\Omega_k : k \in \mathbb{N}\})$-full, because in this case one can apply Δ_0-overspill to find a sufficiently large recursively saturated topped initial segment to run the proof of Proposition 2.10; see Lessan [25, Proposition 4.1.5].

Wilkie and Paris [34, Theorem 5(1)] observed that all short Π_1-recursively saturated models of $\mathrm{B}\Sigma_1$ are $\mathrm{I}\Delta_0$-full. The next proposition refines this observation and provides a converse. Recall that for a set of $\mathcal{L}_{\mathrm{A}}$ formulas Θ, we say an $\mathcal{L}_{\mathrm{A}}$ structure M is *short Θ-recursively saturated* if it realizes all types of the form

$$p(v) = \{v \leqslant m_0\} \cup \{\theta_i(m_0, m_1, \ldots, m_\ell, v) : i \in \mathbb{N}\},$$

where $\bar{m} \in M$ and (θ_i) is a recursive sequence of elements of Θ.

Proposition 2.11. Let $M \models \mathrm{I}\Delta_0$ and $\mathcal{L}^*_{\mathrm{A}}$ be a recursive language extending $\mathcal{L}_{\mathrm{A}}$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}^*_{\mathrm{A}} = \mathcal{L}_{\mathrm{A}}$. The following are equivalent.

(i) M is short Π^{G}_1-recursively saturated.

(ii) M is Γ-full for every $\bar{m} \in M$ and every recursive set of $\mathcal{L}^*_{\mathrm{A}}(\bar{m})$ sentences Γ such that $M \models \Pi^{\mathrm{G}}_1(\bar{m})$-$\mathrm{Th}(\Gamma)$.

(iii) M is Γ-full for every $\bar{m} \in M$ and every recursive set of $\Pi^{\mathrm{G}}_1(\bar{m})$ sentences Γ satisfied in M.

Proof. The proof of (i) $\Rightarrow$ (ii) is similar to that of Proposition 2.4. In the special case when M is countable and satisfies $\mathrm{B}\Sigma_1$, one can actually apply Proposition 2.4 directly because, by a straightforward generalization of Theorem 4.20 in Enayat–Wong [19], if Γ is as in (ii), then M has a proper end extension which expands to a recursively saturated model of Γ.

The implication (ii) $\Rightarrow$ (iii) is obvious.

Now we show (iii) $\Rightarrow$ (i). Assume (iii) holds. Consider a type

$$p(v) = \{v \leqslant m\} \cup \{\theta_i(m, v) : i \in \mathbb{N}\}$$

over M, where $(\theta_i(u, v))_{i \in \mathbb{N}}$ is a recursive sequence of Π^{G}_1 formulas and $m \in M$. For each $i \in \mathbb{N}$, let $h(i) \in \mathbb{N}$ and $\eta_i \in \Delta_0$ such that

$$\begin{aligned} \theta_i(u, v) \quad = \quad & \forall x_0\ \exists x_1 {\leqslant} x_0\ \forall x_2\ \exists x_3 {\leqslant} x_2\ \ldots \\ & \qquad \forall x_{2h(i)-2}\ \exists x_{2h(i)-1} {\leqslant} x_{2h(i)-2}\ \eta_i(u, v, x_0, x_1, \ldots, x_{2h(i)-1}). \end{aligned}$$

Recursively define $e(0) = 0$ and $e(i+1) = e(i) + h(i)$ for every $i \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$,

$$\begin{aligned} M \models\ & \forall u\ \exists v {\leqslant} u\ \forall x_0\ \exists x_1 {\leqslant} x_0\ \forall x_2\ \exists x_3 {\leqslant} x_2\ \ldots \\ & \qquad \Big(u = m \to \bigwedge_{i=0}^{n} \eta_i(u, v, x_{2e(i)}, x_{2e(i)+1}, \ldots, x_{2e(i+1)-1})\Big). \end{aligned}$$

These $\Pi_1^{\mathrm{G}}(m)$ sentences form a recursive set Γ in which every member is satisfied in M. So M is Γ-full by (iii). This gives us $v \leqslant m$ which realizes $p(v)$ in M. $\square$

Paris [16, Remark 2.7] constructed a countable nonstandard model $M \models \mathrm{B}\Sigma_1 + \Pi_2\text{-Th(PA)}$ that is $\mathrm{I}\Sigma_k$-full for every $k \in \mathbb{N}$ but not PA-full. In view of Proposition 2.11, this model M cannot be short Π_1-recursively saturated. On the one hand, one can modify Paris's construction to find countable nonstandard models of $\mathrm{B}\Sigma_1 + \Pi_2\text{-Th}(\Gamma)$ that are not Γ-full for many recursive $\mathcal{L}_{\mathrm{A}}$ theories Γ. On the other hand, there are special situations when $\mathrm{I}\Delta_0$-fullness alone implies recursive saturation: see Corollary 4.5(1).

Wilkie and Paris also constructed non-full models.

Theorem 2.12 (Wilkie–Paris [34, Theorem 9])**.** Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$, and constant symbols $\bar{m}$ new to $\mathcal{L}_{\mathrm{A}}^*$. For every consistent recursive set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences $\Gamma \supseteq \mathrm{I}\Delta_0$, there is a countable model of $\mathrm{B}\Sigma_1 + \Pi_2(\bar{m})\text{-Th}(\Gamma)$ that is not Γ-full.

Proof sketch. Thanks to Theorem 2.7, we can simplify the original proof slightly. Without loss of generality, assume Γ is a set of $\mathcal{L}_{\mathrm{A}}(\bar{m})$ sentences. Let Ψ be a set of $\Sigma_1(\bar{m})$ sentences that is maximally consistent with respect to Γ, i.e.,

- $\Gamma + \Psi$ is consistent; and
- for every $\Sigma_1(\bar{m})$ sentence σ, if $\Gamma + \Psi + \sigma$ is consistent, then $\sigma \in \Psi$.

Follow pages 158f. in the Wilkie–Paris paper [34] to construct a countable model $M \models \mathrm{B}\Sigma_1 + \Pi_2(\bar{m})\text{-Th}(\Gamma) + \Psi$ in which $\Sigma_1\text{-tp}^M(\bar{m}) \notin \mathrm{SSy}(M)$.

Let $K \supseteq_{\mathrm{e}} M$ that expands to a model of Γ. By the maximality of Ψ, we know $\Sigma_1\text{-tp}^M(\bar{m}) = \Psi$ and $\Sigma_1\text{-tp}^K(\bar{m}) = \Sigma_1\text{-tp}^M(\bar{m}) \notin \mathrm{SSy}(M) = \mathrm{SSy}(K)$. Thus K cannot be recursively saturated. Since the end extension K of M was chosen arbitrarily, it follows from Theorem 2.7 that M is not Γ-full. $\square$

The proof of Theorem 2.12 relies heavily on the construction of a model of $\mathrm{B}\Sigma_1$ in which the Σ_1 theory is not coded. Variants of such a construction can be found in Adamowicz [4, 5], Cornaros–Dimitracopoulos [14, Theorem 25], and D'Aquino–Knight [15].

3 Definable satisfaction predicates

Definable satisfaction predicates for Δ_0 formulas play an important role in the study of weak fragments of arithmetic [27]. We isolate in the definition below those properties of satisfaction predicates used in the Wilkie–Paris paper [34]. It is our intention to eliminate the bound in the usual definition; cf. Definition 3 in Cornaros–Dimitracopoulos [14]. Note that $\mathrm{I}\Delta_0$ is already strong enough to evaluate all terms of standard shapes [20, Section V.5(b)].

Definition. Fix a language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences extending $\mathrm{I}\Delta_0$. A pair of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ formulas (Sat, δ) with free variables φ and ε is said to define a *partial satisfaction predicate for* Δ_0 *formulas* over Γ if Γ proves, for every standard Δ_0 formula $\varphi(x_1, x_2, \dots, x_\ell)$,

$$\forall\varepsilon\ \big(\delta(\varphi,\varepsilon) \to (\mathrm{Sat}(\varphi,\varepsilon) \leftrightarrow \varphi((\varepsilon)_1, (\varepsilon)_2, \dots, (\varepsilon)_\ell))\big).$$

An $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ formula Sat is said to define a *compositional satisfaction predicate for* Δ_0 *formulas* over Γ if Γ proves the compositional axioms for satisfaction for atoms, Boolean connectives, and bounded quantifiers, on all (standard and nonstandard) Δ_0 formulas; cf. Halbach [21, Chapter 8]. The pair of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ formulas (Sat, δ) defines a partial satisfaction predicate for Δ_0 formulas *with a nonstandard domain in* M if there is a nonstandard $\nu \in M$ such that

$$M \models \forall \varphi, \varepsilon \ \big(\varphi \leqslant \nu \to \delta(\varphi, \varepsilon) \wedge \delta(\neg\varphi, \varepsilon) \wedge (\neg\mathrm{Sat}(\varphi, \varepsilon) \vee \neg\mathrm{Sat}(\neg\varphi, \varepsilon))\big).$$

Notice the model M in our definition above is not required to satisfy any Tarski biconditional.

The usual definition of Δ_0 satisfaction [20, Section V.5(b)] gives formulas $\mathrm{Sat} \in \Pi_1$ and $\delta \in \Sigma_1$ such that (Sat, δ) defines a partial satisfaction predicate for Δ_0 formulas over $\mathrm{I}\Delta_0$, and Sat defines a compositional satisfaction predicate for Δ_0 formulas over $\mathrm{I}\Delta_0 + \exp$. As shown in Paris–Dimitracopoulos [27, Proposition 4], if the Δ_0-hierarchy collapses provably in $\mathrm{I}\Delta_0$, then there is a Π_1 formula that defines a compositional satisfaction predicate for Δ_0 formulas over $\mathrm{I}\Delta_0$.

The existence of a Π_1 formula that defines a compositional satisfaction predicate for Δ_0 formulas over $\mathrm{I}\Delta_0$ entails the solution to many open problems in arithmetic. For example, it directly implies the finite axiomatizability of $\mathrm{I}\Delta_0$ and $\mathrm{B}\Sigma_1$. It also easily implies the consistency of $\mathrm{I}\Delta_0 + \neg\exp + \neg\mathrm{B}\Sigma_1$ via standard arguments; a refinement and some related discussions can be found in Adamowicz–Kołodziejczyk–Paris [7]. Note that, as shown by Wilkie and Paris [34, page 161], a positive answer to the first formulation of the End-Extension Question also implies the consistency of $\mathrm{I}\Delta_0 + \neg\exp + \neg\mathrm{B}\Sigma_1$; see Adamowicz [2, page 886] and Cordón-Franco–Fernández-Margarit–Lara-Martín [13, end of §1] for an alternative proof. Some further connections between the End-Extension Question and complexity theory are investigated in Kołodziejczyk [24].

Definable satisfaction predicates are typically used to code a recursive sequence of formulas into one formula.

Example 3.1. Let M be a nonstandard model of $\mathrm{B}\Sigma_1$. Suppose some $\Pi_1(M)$ formula defines a compositional satisfaction predicate for Δ_0 formulas over the elementary diagram of M. Then

- M is short Π_1-recursively saturated if and only if $\mathbb{N}$ is not Π_1-definable in M; and
- M is Σ_1-recursively saturated if and only if $\mathbb{N}$ is not Σ_1-definable in M.

Since $\mathrm{B}\Sigma_1$ proves the least number principle for Δ_1-definable predicates [20, Lemma I.2.16], if M is not Σ_1-recursively saturated, then it must be short Π_1-recursively saturated. This is the main idea behind the proof of Theorem 5(4) in Wilkie–Paris [34].

The notion of fullness generally involves a recursive sequence of formulas. As in the example above, using a definable satisfaction predicate, one can code all these formulas into one. The resulting one-formula version of fullness is known as *provable overspill* in the literature.

Definition. Fix a language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences and Θ be a class of $\mathcal{L}_{\mathrm{A}}(\bar{m})$ formulas. We say that M satisfies Γ-*provable* Θ-*overspill* if for every $\theta(u) \in \Theta$,

$$\forall k{\in}\mathbb{N} \ \ \Gamma \vdash \theta(\underline{k}) \quad \Rightarrow \quad \exists d{\in}M \setminus \mathbb{N} \ \ M \models \theta(d).$$

Proposition 3.2. Fix a language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences which includes Q. If M is nonstandard and Γ-full, then it satisfies Γ-provable $\Pi_1(\bar{m})$-overspill.

Proof. Let $\eta(u,v)$ be a $\Delta_0(\bar{m})$ formula such that $\Gamma \vdash \forall v\ \eta(\underline{k}, v)$ for all $k \in \mathbb{N}$. Then, since the realizations of the terms $\underline{0}, \underline{1}, \underline{2}, \ldots$ form an initial segment of every model of Q,

$$\Gamma \vdash \forall b\ \exists u{\leqslant}b\ \forall v \bigwedge_{i=0}^{n} \big(\exists u', v'{\leqslant}b\ \neg\eta(u',v') \to \eta(u,v) \wedge u \geqslant \underline{i}\big)$$

for every $n \in \mathbb{N}$. An application of Γ-fullness finishes the proof. $\square$

The next proposition is the saturation-free version of Proposition 2.4.

Proposition 3.3 (folklore; see Adamowicz [4, page 8])**.** Fix a language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences which includes $\mathrm{I}\Delta_0$. If M is nonstandard and has a proper end extension K that expands to a model of Γ, then M satisfies Γ-provable $\Pi_1(\bar{m})$-overspill.

Proof. Let $\eta(u,v)$ be a $\Delta_0(\bar{m})$ formula such that $\Gamma \vdash \forall v\ \eta(\underline{k}, v)$ for all $k \in \mathbb{N}$. Pick any $b \in K \backslash M$. Then $K \models \forall v{\leqslant}b\ \eta(k,v)$ for every $k \in \mathbb{N}$. Apply Δ_0-overspill in K to find $d \in M \setminus \mathbb{N}$ such that $K \models \forall v{\leqslant}b\ \eta(d,v)$. Then $M \models \forall v\ \eta(d,v)$ because $b > M$. $\square$

Remark 3.4. A slight modification of the proof above shows that, under the hypotheses of Proposition 3.3, the model M actually satisfies Γ-provable $\Pi_1^{\mathrm{G}}(\bar{m})$-overspill. In fact, as the reader can verify, given such a set Γ and any model of $\mathrm{B}\Sigma_1$, if the model satisfies Γ-provable $\Pi_1(\bar{m})$-overspill, then it satisfies Γ-provable $\Pi_1^{\mathrm{G}}(\bar{m})$-overspill.

As alluded to before, with a definable satisfaction predicate, provable overspill implies fullness. More specifically, the following partial converse to Proposition 3.2 holds.

Theorem 3.5 (essentially Cornaros–Dimitracopoulos [14, Theorem 13(i)])**.** Fix a language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{B}\Sigma_1$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences extending $\mathrm{I}\Delta_0$. Suppose we have a $\Pi_1(\bar{m})$ formula $\mathrm{Sat}(\varphi, \varepsilon)$ and a $\Sigma_1(\bar{m})$ formula $\delta(\varphi,\varepsilon)$ such that (Sat, δ) defines a partial satisfaction predicate for Δ_0 formulas over Γ with a nonstandard domain in M. If M satisfies Γ-provable $\Pi_1(\bar{m})$-overspill, then M is Γ-full.

Proof sketch. As suggested by Cornaros and Dimitracopoulos, one follows the second part of proof of Theorem 5(2) in Wilkie–Paris [34].

To check that our axioms for definable satisfaction predicates suffice for their proof, suppose $\mathrm{Sat}(\varphi,\varepsilon) = \forall\beta\ \mathrm{Sat}_0(\varphi,\varepsilon,\beta)$ and $\delta(\varphi,\varepsilon) = \exists\beta\ \delta_0(\varphi,\varepsilon,\beta)$, where $\mathrm{Sat}_0, \delta_0 \in \Delta_0(\bar{m})$. Without loss of generality, assume

$$\begin{aligned} \mathrm{PA}^- \vdash \forall\beta,\beta'\ &\big(\delta_0(\varphi,\varepsilon,\beta) \wedge \beta \leqslant \beta' \to \delta_0(\varphi,\varepsilon,\beta')\big) \\ &\wedge \forall\beta,\beta'\ \big(\mathrm{Sat}_0(\varphi,\varepsilon,\beta') \wedge \beta \leqslant \beta' \to \mathrm{Sat}_0(\varphi,\varepsilon,\beta)\big). \end{aligned}$$

The usual bound $\xi(\varphi,\varepsilon)$ for the satisfaction predicate is then given by

$$\min\big\{\beta : \delta_0(\varphi,\varepsilon,\beta) \wedge \delta_0(\neg\varphi,\varepsilon,\beta) \wedge \big(\neg\mathrm{Sat}_0(\varphi,\varepsilon,\beta) \vee \neg\mathrm{Sat}_0(\neg\varphi,\varepsilon,\beta)\big)\big\}.$$

The compositional axioms for satisfaction transfer from Γ to M via provable overspill on the $\Pi_1(\bar{m})$ formulas

- $\forall\varphi,\varepsilon,\beta_1,\beta_2 \begin{pmatrix} \varphi \leqslant u \wedge \exists\beta\ (\beta \leqslant \beta_1 \wedge \beta \leqslant \beta_2 \wedge \xi(\varphi,\varepsilon) = \beta) \\ \to \big(\mathrm{Sat}_0(\varphi,\varepsilon,\beta_1) \leftrightarrow \mathrm{Sat}_0(\varphi,\varepsilon,\beta_2)\big) \end{pmatrix}$;
- $\forall\varphi,\varepsilon,\beta \begin{pmatrix} \varphi \leqslant u \wedge \exists\beta'{\leqslant}\beta\ \xi(\varphi,\varepsilon) = \beta' \wedge \exists\beta'{\leqslant}\beta\ \xi(\neg\varphi,\varepsilon) = \beta' \\ \to \big(\mathrm{Sat}_0(\neg\varphi,\varepsilon,\beta) \leftrightarrow \neg\mathrm{Sat}_0(\varphi,\varepsilon,\beta)\big) \end{pmatrix}$; etc.

Remark 2.1 then allows us to unravel the relevant nonstandard formula in M. Notice that some monotonicity property of this nonstandard formula is required in the unravelling process. This monotonicity can also be transferred from Γ to M via provable overspill. □

Remark 3.6. In view of Lemma 2.6, if the set of sentences Γ in Theorem 3.5 is recursive, then one sees from the proof that a single application of Γ-provable $\Pi_1(\bar{m})$-overspill is sufficient to entail Γ-fullness. This observation was first recorded by Adamowicz [1, 3].

The following corollary summarizes what we proved in this section so far, and extends the list of equivalent conditions in Corollary 2.8 under the hypothesis that a suitable definable satisfaction predicate is available. It also answers a question in the Wilkie–Paris paper [34, Remark on page 158]; cf. Adamowicz [3, page 597] and Cornaros–Dimitracopoulos [14, Corollary 16]. Conceivably, this answer was previously known, but we are not able to find a reference for it in the literature.

Corollary 3.7. Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and elements $\bar{m}$ in a countable nonstandard model $M \models \mathrm{B}\Sigma_1$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$. Let Γ be a recursive set of $\mathcal{L}_{\mathrm{A}}^*(\bar{m})$ sentences extending $\mathrm{I}\Delta_0$. If we have a $\Pi_1(\bar{m})$ formula $\mathrm{Sat}(\varphi,\varepsilon)$ and a $\Sigma_1(\bar{m})$ formula $\delta(\varphi,\varepsilon)$ such that (Sat,δ) defines a partial satisfaction predicate for Δ_0 formulas over Γ with a nonstandard domain in M, then the following are equivalent.

(i) M is Γ-full.

(ii) M has a proper end extension that expands to a model of Γ.

(iii) M satisfies Γ-provable $\Pi_1(\bar{m})$-overspill.

Proof. The implication (i) ⇒ (ii) follows from Corollary 2.8. Proposition 3.3 shows (ii) ⇒ (iii). The implication (iii) ⇒ (i) is Theorem 3.5. One can also deduce (i) ⇒ (iii) from Proposition 3.2. □

In an unpublished piece of work, the author showed that certain satisfaction classes at the Π_1 level guarantee the existence of proper end extensions. However, the Π_1-definable satisfaction predicates defined in this paper, in general, do not.

Corollary 3.8. Fix constant symbols $\bar{m}$ new to $\mathcal{L}_{\mathrm{A}}$. Let Γ be a consistent recursive set of $\Pi_2(\bar{m})$ sentences extending $\mathrm{I}\Delta_0$. If there is a $\Pi_1(\bar{m})$ formula that defines a compositional satisfaction predicate Sat for Δ_0 formulas over Γ, then some countable model of $\mathrm{B}\Sigma_1 + \Gamma$ cannot be properly end extended to a model of Γ.

Proof. This follows directly from Theorem 2.12 and Corollary 3.7. □

Remark 3.9. As the reader can verify, the condition on the $\Pi_1(\bar{m})$ formula Sat in Corollary 3.8 can be weakened. More precisely, it suffices to require that

- $(\mathrm{Sat}, 0 = 0)$ defines a partial satisfaction predicate for Δ_0 formulas over Γ; and
- there is a Γ-provably total Δ_0-definable function F such that for every standard Δ_0 formula φ,

$$\Gamma \vdash \forall\varepsilon\ \forall\beta{\geqslant}F(\varphi,\varepsilon)\ \big(\neg\mathrm{Sat}_0(\varphi,\varepsilon,\beta) \vee \neg\mathrm{Sat}_0(\neg\varphi,\varepsilon,\beta)\big),$$

where Sat_0 is as chosen in the proof of Theorem 3.5. The same remark applies to Corollary 4.1.

One finds the following question in Adamowicz [4, page 5].

Question 3.10. Is there a Π_1 sentence σ consistent with $\mathrm{B}\Sigma_1$ such that every countable model of $\mathrm{B}\Sigma_1 + \sigma$ has a proper end extension satisfying $\mathrm{I}\Delta_0$?

If some Π_1 formula defines a compositional satisfaction predicate for Δ_0 formulas over $\mathrm{I}\Delta_0$, then it is not hard to find a set of Π_1 sentences S consistent with $\mathrm{B}\Sigma_1$ such that every countable model of $\mathrm{B}\Sigma_1{+}S$ has a proper end extension satisfying $\mathrm{I}\Delta_0$. For example, take

$$S = \{\forall u\ \theta(u) : \theta \in \Pi_1 \text{ and } \mathrm{I}\Delta_0 \vdash \theta(\underline{k}) \text{ for every } k \in \mathbb{N}\}.$$

By construction, every countable nonstandard model of $\mathrm{B}\Sigma_1 + S$ satisfies $\mathrm{I}\Delta_0$-provable Π_1-overspill. However, this set S is very strong because it is deductively equivalent to Π_1-$\mathrm{Th}(\mathbb{N})$. In addition, it computes Π_1-$\mathrm{Th}(\mathbb{N})$ and so it is not recursive. Via the following lemma, one can find a recursive set of Π_1 sentences that is much weaker and does the same job. Given a recursive set of sentences Γ in a recursive language, we denote by $\mathrm{TCon}(\Gamma)$ a canonically chosen Π_1 sentence which expresses the consistency of Γ with respect to the tableau deduction system over $\mathrm{I}\Delta_0$; see Wilkie–Paris [33, paragraph 8.9] for a precise definition.

Lemma 3.11. Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$ and $\bar{m} \in M \models \mathrm{I}\Delta_0$ such that $\mathcal{L}_{\mathrm{A}}(M) \cap \mathcal{L}_{\mathrm{A}}^* = \mathcal{L}_{\mathrm{A}}$ and M is nonstandard. Let $\Gamma(\bar{w})$ be a recursive set of $\mathcal{L}_{\mathrm{A}}^*$ formulas with free variables $\bar{w}$. For every $\mathcal{L}_{\mathrm{A}}$ formula $\theta(u,\bar{w})$, if

(1) $\Gamma(\bar{m}) \vdash \theta(\underline{k},\bar{m})$ for every $k \in \mathbb{N}$; and

(2) $M \models \forall u\ \big(\neg\theta(u,\bar{m}) \rightarrow \mathrm{TCon}(\Gamma(\underline{\bar{m}}) + \neg\theta(\underline{u},\underline{\bar{m}}))\big)$,

then $M \models \theta(d, \bar{m})$ for some nonstandard $d \in M$.

Proof. Assume $M \not\models \theta(u, \bar{m})$ for all nonstandard $u \in M$. Pick any $\nu \in M \setminus \mathbb{N}$. By (1),

$$M \models \exists \pi \leqslant \nu \ \mathrm{ClTab}\big(\pi,\, \Gamma(\underline{\bar{m}}) + \neg\theta(\underline{k}, \underline{\bar{m}})\big)$$

for every $k \in \mathbb{N}$, where $\mathrm{ClTab}\big(\pi,\, \Gamma(\underline{\bar{m}}) + \neg\theta(\underline{u}, \underline{\bar{m}})\big)$ is some fixed Δ_0 formula which expresses 'π is a tableau proof of contradiction from $\Gamma(\underline{\bar{m}}) + \neg\theta(\underline{u}, \underline{\bar{m}})$' over $\mathrm{I}\Delta_0$. Use Δ_0-overspill to find a nonstandard $a \in M$ such that

$$M \models \exists \pi \leqslant \nu \ \mathrm{ClTab}\big(\pi,\, \Gamma(\underline{\bar{m}}) + \neg\theta(\underline{a}, \underline{\bar{m}})\big).$$

Then $M \models \neg\mathrm{TCon}\big(\Gamma(\underline{\bar{m}}) + \neg\theta(\underline{a}, \underline{\bar{m}})\big)$, but $M \models \neg\theta(a, \bar{m})$ by our assumption at the beginning. So (2) fails. $\square$

Remark 3.12. As the reader can readily see from the proof, condition (2) in Lemma 3.11 can be weakened to

(2′) for some nonstandard $\nu \in M$,

$$M \models \forall u \leqslant \nu \ \Big(\neg\theta(u, \bar{m}) \to \forall \pi \leqslant \nu \ \neg\mathrm{ClTab}\big(\pi,\, \Gamma(\underline{\bar{m}}) + \neg\theta(\underline{u}, \underline{\bar{m}})\big)\Big).$$

This is the main idea behind the proofs of Lemma 6 and Lemma 7 in Wilkie–Paris [34].

It follows from Lemma 3.11 that if a nonstandard model of $\mathrm{I}\Delta_0$ satisfies the tableau version of the uniform Π_1 reflection scheme for $\mathrm{I}\Delta_0$, then it satisfies $\mathrm{I}\Delta_0$-provable Π_1-overspill. We employ the tableau version of the reflection scheme here because this version is provable in $\mathrm{I}\Delta_0 + \exp$, but the usual version is not [33, Lemma 8.10 and Corollary 8.14]; see also Visser [31, Theorem 7.2.3].

If some Π_1 formula defines a compositional satisfaction predicate for Δ_0 formulas over a set of sentences Γ, then the uniform Π_1 reflection scheme can be axiomatized by a single Π_1 sentence over Γ. It follows that Question 3.10 has a positive answer in this case. We give a slightly different proof below.

Theorem 3.13 (implicit in Wilkie–Paris [34, Theorem 5(2)])**.** Fix a recursive language $\mathcal{L}_{\mathrm{A}}^* \supseteq \mathcal{L}_{\mathrm{A}}$, constant symbols $\bar{m}$ new to $\mathcal{L}_{\mathrm{A}}^*$, and a recursive set of $\mathcal{L}_{\mathrm{A}}^*$ formulas $\Gamma(\bar{w}) \supseteq \mathrm{I}\Delta_0$. Let $\mathrm{Sat}(\varphi, \varepsilon)$ be a $\Pi_1(\bar{m})$ formula and $\delta(\varphi, \varepsilon)$ be a $\Sigma_1(\bar{m})$ formula such that (Sat, δ) defines a partial satisfaction predicate for Δ_0 formulas over $\Gamma(\bar{m})$. Then there exists a $\Pi_1(\bar{m})$ sentence σ with the following property:

> given any countable model $(M, \bar{m}) \models \mathrm{B}\Sigma_1 + \sigma$ in which (Sat, δ) defines a partial satisfaction predicate for Δ_0 formulas with a nonstandard domain, one can find a recursively saturated model of $\Gamma(\bar{m})$ whose $\mathcal{L}_{\mathrm{A}}$ reduct is a proper end extension of M.

Proof. According to Remark 3.6, the proof of Theorem 3.5 gives us a Π_1 formula $\theta(u, \bar{w})$ such that

- $\Gamma(\bar{m}) \vdash \theta(\underline{k}, \bar{m})$ for every $k \in \mathbb{N}$; and
- given any $(M, \bar{m}) \models \mathrm{B}\Sigma_1$ in which (Sat, δ) defines a partial satisfaction predicate for Δ_0 formulas with a nonstandard domain, if $M \models \theta(d, \bar{m})$ for some nonstandard $d \in M$, then M is $\Gamma(\bar{m})$-full.

In view of Lemma 3.11, Corollary 3.7, and Theorem 2.7, the $\Pi_1(\bar{m})$ sentence

$$\sigma \quad = \quad \forall u \ \big(\neg\theta(u,\bar{m}) \to \mathrm{TCon}(\Gamma(\underline{\bar{m}}) + \neg\theta(\underline{u},\underline{\bar{m}}))\big)$$

satisfies our requirement. Notice that if the model M we are given is standard, then one needs a separate argument, but such an argument is much easier to find. $\square$

4 Discussions

It is clear from the Π_1 conservativity of $\mathrm{B}\Sigma_1$ over $\mathrm{I}\Delta_0$ that a model of $\mathrm{I}\Delta_0$ satisfies $\mathrm{I}\Delta_0$-provable Π_1-overspill if and only if it satisfies $\mathrm{B}\Sigma_1$-provable Π_1-overspill. Since $\mathrm{B}\Sigma_1$ is Π_1^{G}-axiomatizable [8, Lemma 1], one cannot replace Π_1 by Π_1^{G} here. Wilkie and Paris [34] left open the question whether every countable $\mathrm{I}\Delta_0$-full model of $\mathrm{B}\Sigma_1$ is $\mathrm{B}\Sigma_1$-full. The 'obvious argument' to answer this question in the positive does not work essentially because $\mathrm{B}\Sigma_1$ cannot deal with infinitary formulas. The next corollary provides an answer using a less obvious argument. It also summarizes what we know about $\mathrm{I}\Delta_0$-fullness, and shows how close $\mathrm{I}\Delta_0$ and $\mathrm{B}\Sigma_1$ are.

Corollary 4.1. The following are equivalent for a countable $M \models \mathrm{B}\Sigma_1$.

(i) M has a short Π_1-recursively saturated end extension $K \models \mathrm{B}\Sigma_1$.

(ii) M has a recursively saturated proper end extension $K \models \mathrm{B}\Sigma_1$.

(iii) M is $\mathrm{B}\Sigma_1$-full.

(iv) M is $\mathrm{I}\Delta_0$-full.

(v) M has a recursively saturated proper end extension $K \models \mathrm{I}\Delta_0$.

(vi) M has a Σ_1-recursively saturated proper end extension $K \models \mathrm{I}\Delta_0$.

Furthermore, if $M \not\cong \mathbb{N}$ and some Π_1 formula defines a compositional satisfaction predicate for Δ_0 formulas over $\mathrm{I}\Delta_0$, then the conditions above are equivalent to each of the following.

(vii) M satisfies $\mathrm{I}\Delta_0$-provable Π_1-overspill.

(viii) M satisfies $\mathrm{B}\Sigma_1$-provable Π_1-overspill.

(ix) M has a proper end extension $K \models \mathrm{I}\Delta_0$.

(x) M has a proper end extension $K \models \mathrm{B}\Sigma_1$

Proof. The implication (i) $\Rightarrow$ (ii) is a consequence of Proposition 2.11 and Theorem 2.7. The equivalences (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v) are special cases of Corollary 2.8. The implication (iii) $\Rightarrow$ (iv) follows directly from the definition of fullness. The implications (ii) $\Rightarrow$ (v) $\wedge$ (i) and (v) $\Rightarrow$ (vi) are obvious. One can prove the 'furthermore' part using Corollary 3.7.

To show (vi) $\Rightarrow$ (ii), suppose $K \supseteq_{\mathrm{e}} M$ as in (vi). Take any $a \in K \setminus M$. With Σ_1 recursive saturation, we know $a^{\mathbb{N}} \neq K$. Pick $b \in K \setminus a^{\mathbb{N}}$. Theorem 5.10 in Enayat–Wong [19] then gives us a recursively saturated $K' \subseteq_{\mathrm{e}} K$ such that $a \in K' < b$. $\square$

In view of the results in Section 2, one can easily replace the hypothesis in Corollary 3.8 by a seemingly weaker one.

Corollary 4.2. Suppose, for every countable $M \models \mathrm{B}\Sigma_1$, if M has a proper end extension satisfying $\mathrm{I}\Delta_0$, then it has a Σ_1 recursively saturated such extension. Then some countable model of $\mathrm{B}\Sigma_1$ cannot be properly end-extended to a model of $\mathrm{I}\Delta_0$.

Proof. This follows from Theorem 2.12 and Corollary 4.1. □

This suggests that one way to attempt to answer the End-Extension Question is to investigate whether the existence of a proper end extension implies the existence of any special such extension. At the moment, the following proposition is the only piece of information on this line known to the author. This proposition may have been independently discovered by other people before.

Proposition 4.3. If a model $M \models \mathrm{B}\Sigma_1$ has a proper end extension satisfying $\mathrm{I}\Delta_0$, then it has one that is not Σ_1-elementary.

Proof. Suppose M has a proper end extension $K \models \mathrm{I}\Delta_0$. Without loss of generality, assume $M \preccurlyeq_1 K$. Then $M \models \mathrm{B}\Sigma_2$ by Theorem B in Paris–Kirby [28]. Thus Theorem 5.1 in Beklemishev [9] tells us $M \models \mathrm{Con}(\mathrm{B}\Sigma_1 + \exp)$. The formalized version of the Gödel–Rosser Incompleteness Theorem then implies $M \models \mathrm{Con}(\mathrm{B}\Sigma_1 + \exp + \neg\mathrm{Con}(\mathrm{B}\Sigma_1 + \exp))$. Using the Arithmetized Completeness Theorem [20, Theorem I.4.27(1)], we get $K' \supsetneq_{\mathrm{e}} M$ satisfying $\mathrm{B}\Sigma_1 + \exp + \neg\mathrm{Con}(\mathrm{B}\Sigma_1 + \exp)$. Since the sentence $\neg\mathrm{Con}(\mathrm{B}\Sigma_1 + \exp)$ is Σ_1, this extension K' is not Σ_1-elementary over M. □

If $\mathrm{I}\Delta_0$ proves topped recursive saturation, then the following question has a positive answer by Proposition 2.10.

Question 4.4. Is it true that if a countable model of $\mathrm{B}\Sigma_1$ has a proper end extension satisfying $\mathrm{I}\Delta_0$, then it has one satisfying $\mathrm{B}\Sigma_1$?

In view of Fact 1.1, the use of $\mathrm{B}\Sigma_1$ in Corollary 2.8 is necessary. We can squeeze a little more information out of our results using Fact 1.1.

Corollary 4.5. (1) If a countable $M \models \mathrm{I}\Delta_0$ is $\mathrm{I}\Delta_0$-full but $M \not\models \mathrm{B}\Sigma_1$, then M is recursively saturated.

(2) If σ is an $\mathcal{L}_{\mathrm{A}}$ sentence such that $\mathrm{I}\Delta_0 + \neg\sigma$ is Π_1^{G}-conservative over $\mathrm{I}\Delta_0$, then $\mathrm{I}\Delta_0 + \sigma \vdash \mathrm{B}\Sigma_1$.

Proof. (1) Apply Theorem 2.7 to find a recursively saturated model $K \models \mathrm{I}\Delta_0$ which end extends M. Since $M \not\models \mathrm{B}\Sigma_1$, we know $K = M$ by Fact 1.1. It follows that M itself is recursively saturated.

(2) Let M be a countable recursively saturated model of $\mathrm{I}\Delta_0 + \sigma$. Then M is $\mathrm{I}\Delta_0$-full by Proposition 2.11. It follows from the Π_1^{G} conservativity of $\mathrm{I}\Delta_0 + \neg\sigma$ over $\mathrm{I}\Delta_0$ that M must be $(\mathrm{I}\Delta_0 + \neg\sigma)$-full. Apply Theorem 2.7 to get $K \models \mathrm{I}\Delta_0 + \neg\sigma$ end extending M. This extension is proper because $M \models \sigma$ and $K \models \neg\sigma$. Hence $M \models \mathrm{B}\Sigma_1$ by Fact 1.1. □

It is well known [12, Proposition 4.3] that no consistent set of Σ_3 sentences containing exp can prove $\mathrm{B}\Sigma_1$. Thus if τ is a Σ_3 sentence consistent with $\mathrm{I}\Delta_0 + \exp$, then $\mathrm{I}\Delta_0 + \neg(\tau \wedge \exp)$ is not Π_1^{G}-conservative over $\mathrm{I}\Delta_0$ by Corollary 4.5(2). This tells us that $\mathrm{I}\Delta_0 + \neg\mathrm{Con}(\mathrm{ZFC})$, for example, is not Π_1^{G}-conservative over $\mathrm{I}\Delta_0$. It is not clear to the author whether $\mathrm{I}\Delta_0 + \neg\mathrm{Con}(\mathrm{ZFC})$ is Π_1-conservative over $\mathrm{I}\Delta_0$; cf. Visser [32].

It is worth noting that Δ_0 induction is not used in the results of Section 2 at all (except in Theorem 2.12 where, in a sense, weakening $\mathrm{I}\Delta_0$ weakens the theorem). This shows the study of fullness is not specific to arithmetic, but really it is part of general model theory. What makes the End-Extension Question especially interesting for arithmetic is its connections with definable truth predicates as discussed in Section 3. This point was not easy to see before.

If there is a countable nonstandard model $M \models \mathrm{B}\Sigma_1$ that does not satisfy $\mathrm{I}\Delta_0$-provable Π_1-overspill, then the first formulation of the End-Extension Question has a negative answer by Proposition 3.3. In such M, the standard cut $\mathbb{N}$ must be parameter-free Π_1-definable, and exp must fail [34, Lemma 6].

Question 4.6. Is there a countable nonstandard $M \models \mathrm{B}\Sigma_1 + \neg\exp$ in which $\mathbb{N}$ is Π_1-definable?

All tools known to the author for building nonstandard models of $\mathrm{B}\Sigma_1$ in which $\mathbb{N}$ is Π_1-definable are mentioned at the end of Section 2. The theories of the models constructed come only from a rather restricted class.

Question 4.7. Let $M \models \mathrm{B}\Sigma_1 + \exp$. Can one always find a nonstandard $M' \models \mathrm{B}\Sigma_1$ in which $\mathbb{N}$ is Π_1-definable and $\Pi_2\text{-Th}(M') = \Pi_2\text{-Th}(M)$?

We know one other way of building models of arithmetic in which $\mathbb{N}$ is Π_1- but not Σ_1-definable, namely, using Σ_1-closed models. For instance, using Corollary 3.6 in Adamowicz–Cordón-Franco–Lara-Martín [6], one sees that Question 4.7 has a positive answer if $\mathrm{B}\Sigma_1$ is weakened to $\mathrm{I}\Delta_0$ and $\Pi_2\text{-Th}(M') = \Pi_2\text{-Th}(M)$ is weakened to $\Pi_2\text{-Th}(M') \supseteq \Pi_2\text{-Th}(M)$. However, as shown by Paris [13], no Σ_1-closed model of $\mathrm{I}\Delta_0 + \exp$ can satisfy $\mathrm{B}\Sigma_1$. The question whether some Σ_1-closed model of $\mathrm{I}\Delta_0$ can satisfy $\mathrm{B}\Sigma_1$ is open. If such a model exists, then it cannot have a proper end extension to a model of $\mathrm{I}\Delta_0$, and hence the first formulation of the End-Extension Question has a negative answer; see Adamowicz [2, proof of Theorem 1] and Cordón-Franco–Fernández-Margarit–Lara-Martín [13, Proposition 3].

In their proof that $\mathbb{N}$ is parameter-free Π_1-definable in all Σ_1-closed models of $\mathrm{I}\Delta_0 + \exp$, Adamowicz, Cordón-Franco and Lara-Martín [6, Corollary 3.5] invoke exp only to guarantee the existence of a suitably definable satisfaction predicate. Thus one is naturally led to the following question.

Question 4.8. Is there no short Π_1-recursively saturated Σ_1-closed model of $\mathrm{I}\Delta_0$?

As shown by Dimitracopoulos and Paschalis [18], the end-extension results in the Wilkie–Paris paper [34, Theorem 5] admit alternative proofs based on some versions of the Arithmetized Completeness Theorem.

Most results in this paper generalize to all levels of the arithmetic hierarchy; we refer the reader to Cornaros–Dimitracopoulos [14, pages 11ff.] to see how this can be done.

Acknowledgements

I am especially grateful to Zofia Adamowicz for her interest, guidance, and numerous suggestions. Many of the new ideas and observations presented in this paper originated from the thought-provoking questions she posed at our weekly meetings. I thank Ali Enayat for recommending Simpson's method of constructing recursively saturated models [30, proof of Lemma IX.4.2] to me, which led to the current proof of Theorem 2.7. I thank the anonymous referee for spotting numerous mathematical and typographical errors, and for the suggestions which helped make this paper more reader-friendly.

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