

# Constant Regions in Models of Arithmetic

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## Abstract

This paper introduces a new theory of *constant regions*, which generalizes that of interstices, in nonstandard models of arithmetic. In particular, we show that two homogeneity notions introduced by Richard Kaye and the author, namely *constantness* and *pregenericity*, are equivalent. This led to some new characterizations of generic cuts in terms of existential closedness.

## 1 Introduction

Kaye [4] introduced the notions of *constant* and *small* intervals for the definition of generic cuts. Kaye and the author [6] simplified this definition to one that omits smallness but uses a two-variable version of constantness. This two-variable version of constantness was called *pregenericity* in our paper.

**Definition.** Let  $Y$  be an indicator on a model  $M \models \text{PA}$ . A  $Y$ -interval  $\llbracket a, b \rrbracket$  is *constant* over  $c$  if

$$\forall x \in \llbracket a, b \rrbracket \forall \llbracket u, v \rrbracket \subseteq \llbracket a, b \rrbracket \exists x' \in \llbracket u, v \rrbracket (M, x, c) \equiv (M, x', c).$$

A  $Y$ -interval  $\llbracket a, b \rrbracket$  is *pregeneric* over  $c$  if

$$\forall x, y \in \llbracket a, b \rrbracket \forall \llbracket u, v \rrbracket \subseteq \llbracket a, b \rrbracket \exists x', y' \in \llbracket u, v \rrbracket (M, x, y, c) \equiv (M, x', y', c).$$

We show in this paper that constantness and pregenericity actually coincide. In particular, the notion of smallness is redundant in Kaye's original paper [4]. It turns out that looking at *intervals* with end points in the model obscures some features of the notions. So instead, we consider general convex sets with the analogous homogeneity property—we call these *constant regions*.

After setting up some notation in Section 2, we develop a theory of constant regions in Sections 3–7 parallel to that of interstices. The relationship between the two theories is briefly studied in Section 9. In between, it is shown in Section 8 that essentially all cuts inside a constant region are closed under the same definable functions. This reveals a new sense of homogeneity for constant

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regions. In Section 10, we prove the key theorem of this paper, that all constant intervals are pregeneric.

This supplies useful sufficient conditions for a cut to be generic. In addition to a simplified definition of generic cuts which we present in Section 11, several new characterizations of generic cuts in terms of special cuts and existential closedness are obtained. *Specialness* is the opposite of genericity that we propose in Section 12. A few elementary facts about *existentially closed cuts* are included in Section 13. After setting up some lemmas in Section 14 about limits of coded  $\omega$ -sequences, we prove the remaining characterizations in Section 15.

## 2 Preliminaries

For background information about nonstandard models of arithmetic, we refer the reader to Kaye [3] and Kossak–Schmerl [9]. Our language  $\mathcal{L}_A$  contains the symbols  $0, 1, +, \times, <$ . Throughout this paper, we work in a fixed nonstandard model  $M \models \text{PA}$ . For  $c \in M$ , let  $\text{cl}(c)$  denote the Skolem closure of  $c$  in  $M$ . Unless otherwise stated, ‘definable’ means ‘definable with parameters’. We sometimes write  $\mathbb{Q}$  for ‘there are cofinally many’. A *cut* of  $M$  is a nonempty initial segment of  $M$  with no maximum element. Informally, we say a convex subset  $\Omega \subseteq M$  *contains* a cut  $I$  if  $\inf \Omega \subseteq I \subseteq \sup \Omega$ . For  $c \in M$ , denote by  $\text{Aut}(M, c)$  the stabilizer of  $c$  under the automorphism group of  $M$ .

We assume the reader is familiar with the general theory of *indicators* from Kaye [4] and Kaye–Wong [6, 7]. Nevertheless, we repeat a few more important definitions and facts here.

On our model  $M$  we have a monotone indicator  $Y$  which will be fixed throughout. For each  $k \in \mathbb{N}$ , set

$$Y_k(x) = (\mu y)(Y(x, y) \geq k).$$

By definition, the following hold.

- (1)  $Y$  is a function  $M^2 \rightarrow M$  whose graph is a *class*. In other words, the set

$$\{\langle x, y, Y(x, y) \rangle : x, y < b\}$$

is definable in  $M$  for every  $b \in M$ .

- (2) For all  $x, y, x', y' \in M$ , if  $(M, x, y) \equiv (M, x', y')$ , then

$$Y(x, y) > \mathbb{N} \iff Y(x', y') > \mathbb{N}.$$

- (3) For all  $x, y, z \in M$ , if  $Y(x, y) > \mathbb{N}$ , then either  $Y(x, z) > \mathbb{N}$  or  $Y(z, y) > \mathbb{N}$ .

- (4)  $M \models \forall x, y, u, v (x \leq u \wedge v \leq y \rightarrow Y(u, v) \leq Y(x, y))$ .

- (5)  $M \models \forall x, y (x \geq y \rightarrow Y(x, y) = 0)$ .

Additionally, we require that

- (6)  $Y_k$  is parameter-free definable for each  $k \in \mathbb{N}$ .

This will be needed in Lemma 5.2 and in Section 13, for example. All naturally occurring indicators  $Y$  known to the author satisfy this additional requirement.

Condition (4) above guarantees the  $Y_k$ 's are monotone in all arguments. Condition (3) ensures the family of functions  $\{Y_k : k \in \mathbb{N}\}$  is closed under composition, in the sense that

$$\forall b \in M \ \forall \ell \in \mathbb{N} \ \exists k \in \mathbb{N} \ \forall x < b \ Y_\ell^2(x) < Y_k(x).$$

To see why this holds, notice for all nonstandard  $\nu \in M$  and all  $x < b$ , we have  $Y(x, Y_\ell^2(x)) \in \mathbb{N} < Y(x, Y_\nu(x))$  by (3), and so  $Y_\ell^2(x) < Y_\nu(x)$ . An application of underspill then gives the  $k$  we want. In many cases, for example, when  $Y$  is definable, one can actually find one  $k \in \mathbb{N}$  that works for all  $b \in M$ .

Recall double brackets  $\llbracket \cdot, \cdot \rrbracket$  are reserved for  $Y$ -intervals, that is, intervals  $[a, b]$  which satisfy  $Y(a, b) > \mathbb{N}$ . We write  $a \ll b$  if  $[a, b]$  is a  $Y$ -interval. A subset of  $M$  is  $Y$ -large if it includes some  $Y$ -interval. A  $Y$ -cut is a cut that is closed under  $Y_k$  for every  $k \in \mathbb{N}$ . For  $d \in M$ , define

$$\begin{aligned} M(d) &= \{y \in M : Y(d, y) \in \mathbb{N}\}, \text{ and} \\ M[d] &= \{x \in M : Y(x, d) > \mathbb{N}\}. \end{aligned}$$

The monotonicity of  $Y$  guarantees that both of these are  $Y$ -cuts.

It would be helpful, although not necessary, for the reader to be acquainted with the theory of *interstices* from Bamber–Kotlarski [1].

### 3 Constant Regions

We hope the word *region* gives a more positive image than *gap* and *interstice* do.

**Definition.** Let  $c \in M$ . A  $Y$ -large convex subset  $\Omega$  of  $M$  is *constant* over  $c$  (with respect to  $Y$ ) if

$$\forall x \in \Omega \ \forall \llbracket u, v \rrbracket \subseteq \Omega \ \exists x' \in \llbracket u, v \rrbracket \ (M, x, c) \equiv (M, x', c).$$

A *constant region* over  $c$  is a maximal  $Y$ -large convex subset of  $M$  that is constant over  $c$ .

Recall such  $x'$  can be made to satisfy  $u \ll x' \ll v$  using the properties of indicators.

With this definition, the model  $M$  is required to be recursively saturated by many proofs in this paper. For example, recursive saturation makes sure the union of two intersecting constant convex sets is again constant. Nevertheless, as the reader can verify, most of these still go through if one defines a constant region over  $c$  to be a maximal  $Y$ -large convex set  $\Omega$  such that for every  $\varphi(x, z) \in \mathcal{L}_A$ , there exists  $k \in \mathbb{N}$  which satisfies the property

$$\exists x \in \Omega \ \varphi(x, c) \quad \Rightarrow \quad \forall \llbracket u, v \rrbracket \subseteq \Omega \ (Y(u, v) > k \rightarrow \exists x \in \llbracket u, v \rrbracket \ \varphi(x, c)).$$

Over recursive saturation, the two definitions are equivalent—see Lemma 5.2. We avoid this cumbersome definition because we are mainly interested in recursively saturated models of PA.

Kaye [4] showed that constant regions exist in abundance in arithmetically saturated models of PA.

**Theorem 3.1.** Suppose  $M$  is an arithmetically saturated model of PA, and let  $c \in M$ . Then every  $Y$ -interval in  $M$  contains a  $Y$ -large subinterval that is constant over  $c$ .  $\square$

An alternative proof demonstrating the existence of constant regions can be found in an earlier paper by Schmerl [10, Proposition 1.3] using *interstices*. In fact, we will develop a theory of constant regions mimicking that of interstices. For example, the following lemma has a counterpart for interstices [2]. The difference is that the lemma, as stated below, no longer needs  $\mathbb{N}$  to be strong. The use of arithmetic saturation in Theorem 3.1 above is necessary though—see Corollary 8.4.

**Lemma 3.2.** Assume the model  $M$  is recursively saturated. Let  $\Omega$  be a constant region over some element  $c \in M$ . Then for every  $d \in \Omega$ ,

$$\inf \Omega < M[d] < M(d) < \sup \Omega.$$

In particular, both  $\inf \Omega$  and  $\sup \Omega$  are  $Y$ -cuts.

*Proof.* Pick any  $d \in \Omega$ . We show  $M[d] > \inf \Omega$ . The argument for  $\sup \Omega$  is symmetric. Recall that  $\Omega$  is  $Y$ -large. Take  $\llbracket r, s \rrbracket \subseteq \Omega$ . Using constantness, pick  $d^* \in \llbracket r, s \rrbracket$  such that  $(M, d, c) \equiv (M, d^*, c)$  and  $r \ll d^* \ll s$ . Then find  $r', s' \in M$  which make  $(M, d, c, r', s') \equiv (M, d^*, c, r, s)$  and thus  $r' \ll d \ll s'$ . Such  $r', s'$  can be found using recursive saturation. Notice  $\Omega \cup \llbracket r', s' \rrbracket$  has the constantness property over  $c$  and  $d \in \Omega \cap \llbracket r', s' \rrbracket$ . So by the maximality of  $\Omega$ , we must have  $\llbracket r', s' \rrbracket \subseteq \Omega$ . It follows that  $M[d] > r' > \inf \Omega$ , as required.  $\square$

This is already sufficient to give us a nice corollary, which in a sense, says we can overspill and underspill across a constant region.

**Corollary 3.3.** Suppose  $M$  is recursively saturated. Let  $\Omega$  be a bounded constant region over a parameter  $c \in M$ . Then for every  $\mathcal{L}_A$ -formula  $\theta(x, z)$ , the following are equivalent.

- (a) There are arbitrarily large  $x < \inf \Omega$  satisfying  $\theta(x, c)$ .
- (b) There are arbitrarily small  $x > \sup \Omega$  satisfying  $\theta(x, c)$ .  $\square$

## 4 Special Points

Informally speaking, a special point is a point that can be distinguished from its neighbours by a formula.

**Definition.** Let  $c \in M$ . An element  $w \in M$  is said to be a *special point* of  $M$  over  $c$  (with respect to  $Y$ ) if there exist  $\varphi(x, z) \in \mathcal{L}_A$  and  $\llbracket r, s \rrbracket \subseteq M$  such that

- $M \models \varphi(w, c)$ ;
- $M \models \forall x \in \llbracket r, s \rrbracket \neg \varphi(x, c)$ ; and
- $w + 1 = r$  or  $s + 1 = w$ .

**Lemma 4.1.** A constant region contains no special point over the same parameter.

*Proof.* Immediate. □

The converse is also true.

**Lemma 4.2.** Fix a parameter  $c \in M$ , and suppose  $M$  is recursively saturated. If a convex subset  $\Omega \subseteq M$  contains no special point over  $c$ , then it is constant over  $c$ .

*Proof.* Suppose  $w \in \Omega$ . Let  $\llbracket r, s \rrbracket \subseteq \Omega$  in which no  $w'$  satisfies  $(M, w, c) \equiv (M, w', c)$ . Using recursive saturation, find  $\varphi(x, z) \in \mathcal{L}_A$  such that

$$M \models \varphi(w, c) \wedge \forall x \in \llbracket r, s \rrbracket \neg \varphi(x, c).$$

Then either  $(\max x < r)(\varphi(x, c))$  or  $(\min x > s)(\varphi(x, c))$  is a special point over  $c$ . Such a point is in  $\Omega$  because  $\Omega$  is convex. □

Recall that interstices are convex sets that do not contain definable points. Analogously, constant regions are convex sets that do not contain special points.

## 5 The Type of a Region

**Definition.** Let  $c \in M$ , and  $\Omega$  be a constant region over  $c$ . An  $\mathcal{L}_A(c)$ -formula  $\theta(x, c)$  is said to be *satisfied* in  $\Omega$  if there is an  $x \in \Omega$  such that  $M \models \theta(x, c)$ . Otherwise, it is said to be *excluded* from  $\Omega$ . The set of all formulas  $\theta(x, c)$  that are satisfied in  $\Omega$  is referred to as the *type* of  $\Omega$  over  $c$ , denoted by  $\text{tp}(\Omega/c)$ .

It was shown in Section 6 of Kaye–Wong [6] that if  $Y(x, y) = z$  is a single formula, then there are countably many types of constant regions over any given  $c$ .

**Lemma 5.1.** If  $M$  is recursively saturated, then  $\text{tp}(\Omega/c)$  is coded for all  $c \in M$  and all constant regions  $\Omega$  over  $c$ .

*Proof.* For any  $\llbracket r, s \rrbracket \subseteq \Omega$ , we have

$$\text{tp}(\Omega/c) = \{\theta(x, c) : M \models \exists x \in \llbracket r, s \rrbracket \theta(x, c)\}. \quad \square$$

Clearly, a formula satisfied in a constant region is densely satisfied with respect to the indicator. Using recursive saturation, we can get a uniform bound on this density.

**Lemma 5.2.** Suppose  $M$  is recursively saturated, and let  $c \in M$ . If  $\theta(x, c)$  is an  $\mathcal{L}_A(c)$ -formula satisfied in a constant region  $\Omega$  over  $c$ , then there is  $k \in \mathbb{N}$  such that

$$\forall [u, v] \subseteq \Omega (Y(u, v) > k \rightarrow \exists x \in [u, v] \theta(x, c)).$$

*Proof.* Pick any  $\llbracket r, s \rrbracket \subseteq \Omega$ . We know  $\theta(x, c)$  is satisfied in  $\llbracket r, s \rrbracket$  by constantness. Hence recursive saturation gives  $k \in \mathbb{N}$  such that

$$\forall u \in \llbracket r, s \rrbracket \exists x \in [u, Y_k(u)] \theta(x, c).$$

This  $k$  is what we want because both

$$\begin{aligned} &(\max u < r)(\forall x \in [u, Y_k(u)] \neg \theta(x, c)) \text{ and} \\ &(\min u > s)(\forall x \in [u, Y_k(u)] \neg \theta(x, c)) \end{aligned}$$

are special points over  $c$ , and thus have to lie outside  $\Omega$ , if they exist. □

In countable recursively saturated models  $M \models \text{PA}$ , the type of a constant region determines its orbit under the action of the automorphism group of  $M$ .

**Proposition 5.3.** Suppose  $M$  is a countable recursively saturated model of PA. Let  $c \in M$ . If  $\Omega_1$  and  $\Omega_2$  are constant regions with the same type over  $c$ , then there exists  $g \in \text{Aut}(M, c)$  such that  $g(\Omega_1) = \Omega_2$ .

*Proof.* If  $x \in \Omega_1$  and  $\llbracket r, s \rrbracket \subseteq \Omega_2$ , then recursive saturation says there is  $x' \in \llbracket r, s \rrbracket$  which has the same type as  $x$  over  $c$ .  $\square$

## 6 Maxima and Minima

By underspill, if a formula  $\theta(x, c)$  excluded from a constant region  $\Omega$  is satisfied below  $\Omega$ , then  $(\max x < \Omega)(\theta(x, c))$  exists. Similarly, by overspill, if  $\theta(x, c)$  is excluded from  $\Omega$  but is satisfied above  $\Omega$ , then  $(\min x > \Omega)(\theta(x, c))$  exists. These maxima and minima are clearly special points over  $c$ .

**Lemma 6.1.** Suppose  $M$  is recursively saturated. Let  $c \in M$  and  $\Omega$  be a constant region over  $c$ . Then

$$\inf \Omega = \sup\{(\max x < \Omega)(\theta(x, c)) : \theta \text{ is satisfied below but not in } \Omega\}.$$

If, in addition, the region  $\Omega$  is bounded, then we also have

$$\sup \Omega = \inf\{(\min x > \Omega)(\theta(x, c)) : \theta \text{ is satisfied above but not in } \Omega\}.$$

*Proof.* We concentrate on  $\inf \Omega$ —the proof for  $\sup \Omega$  is symmetric. Pick  $w \in \inf \Omega$ . By Lemma 4.2, we can assume  $w$  is a special point over  $c$  without loss of generality. Let  $\theta(x, c)$  witness the speciality of  $w$ . There are two possibilities.

If  $\theta(x, c)$  is excluded from  $\Omega$ , then

$$(\max x < \Omega)(\theta(x, c)) \geq w.$$

So we are done in this case.

Suppose  $\theta(x, c)$  is satisfied in  $\Omega$ . Using Lemma 5.2, find  $k \in \mathbb{N}$  such that

$$\forall [u, v] \subseteq \Omega \ (Y(u, v) > k \rightarrow \exists x \in [u, v] \ \theta(x, c)).$$

Depending on the behaviour of  $\theta(x, c)$  around  $w$ ,

$$\begin{aligned} & \text{either } (\max u < \Omega)(\exists v \ (Y(u, v) > k \wedge \forall x \in [u, v] \ \neg \theta(x, c))) \geq w \\ & \text{or } (\max v < \Omega)(\exists u \ (Y(u, v) > k \wedge \forall x \in [u, v] \ \neg \theta(x, c))) + 1 \geq w. \quad \square \end{aligned}$$

This lemma implies there are coded monotone sequences of special points that converge to a bounded constant region from either side. An interstitial counterpart of this can be found in Bamber–Kotlarski [1].

**Corollary 6.2.** Assume  $M$  is recursively saturated. Let  $c \in M$  and  $\Omega$  be a constant region over  $c$ . Then there is  $\alpha \in M$  such that

- $(\alpha)_n$  is a special point over  $c$  for every  $n \in \mathbb{N}$ ;
- $(\alpha)_n \leq (\alpha)_{n+1}$  for every  $n \in \mathbb{N}$ ; and

- $\sup\{(\alpha)_n : n \in \mathbb{N}\} = \inf \Omega$ .

If in addition, the region  $\Omega$  is bounded, then there is  $\beta \in M$  such that

- $(\beta)_n$  is a special point over  $c$  for every  $n \in \mathbb{N}$ ;
- $(\beta)_n \geq (\beta)_{n+1}$  for every  $n \in \mathbb{N}$ ; and
- $\inf\{(\beta)_n : n \in \mathbb{N}\} = \sup \Omega$ .

*Proof.* Fix a recursive enumeration  $(\theta_n(x, z))_{n \in \mathbb{N}}$  of  $\mathcal{L}_A$ -formulas, and pick any  $\llbracket r, s \rrbracket \subseteq \Omega$ . Using recursive saturation, find  $\alpha \in M$  such that for all  $n \in \mathbb{N}$ ,

$$(\alpha)_n = \begin{cases} (\max x < r)(\theta_n(x, c)), & \text{if } \theta_n(x, c) \text{ is satisfied below } r \\ & \text{but not in } \llbracket r, s \rrbracket; \\ 0, & \text{otherwise.} \end{cases}$$

If  $\Omega$  is bounded above, say, by the special point  $b$ , then we may take  $\beta$  to be an element of  $M$  which satisfies

$$(\beta)_n = \begin{cases} (\min x > s)(\theta_n(x, c)), & \text{if } \theta_n(x, c) \text{ is satisfied above } s \\ & \text{but not in } \llbracket r, s \rrbracket; \\ b, & \text{otherwise,} \end{cases}$$

for all  $n \in \mathbb{N}$ . Such  $\beta$  exists by recursive saturation.  $\square$

## 7 Infimum and Supremum

Given a cut  $I$ , it is useful to know how the definable functions under which  $I$  is closed relate to points of the forms

$$(\max x \in I)(\theta(x, c)) \quad \text{and} \quad (\min x > I)(\theta(x, c)),$$

where  $\theta(x, z) \in \mathcal{L}_A$ . To turn a definable function into a formula, one iterates the function. This is what we do in the following proof. More applications of this trick can be found in the proofs of Theorem 8.1 and Proposition 9.5.

**Lemma 7.1.** Let  $I$  be a cut. Denote by  $\mathfrak{E}_I$  the set of all increasing functions  $F: M \rightarrow M$  such that  $F$  is definable over  $c$  and  $I$  is *not* closed under  $F$ . Suppose  $\mathfrak{E}_I$  is nonempty.

- (a) For every  $F \in \mathfrak{E}_I$ , there is a formula  $\theta(x, z) \in \mathcal{L}_A$  such that

$$F((\max x \in I)(\theta(x, c))) = (\min x > I)(\theta(x, c)).$$

- (b) If  $u = (\max x \in I)(\theta(x, c))$  for some  $\theta(x, z) \in \mathcal{L}_A$ , then there is  $F \in \mathfrak{E}_I$  such that  $F(u) = u$ .
- (c) If  $v = (\min x > I)(\theta(x, c))$  for some  $\theta(x, z) \in \mathcal{L}_A$ , then there is  $F \in \mathfrak{E}_I$  such that  $F(I) \leq v$ .

*Proof.* First consider (a). Define  $G(0) = 0$  and

$$G(u+1) = \begin{cases} F(G(u)), & \text{if } F(G(u)) > G(u); \\ G(u) + 1, & \text{otherwise.} \end{cases}$$

If  $\text{Im}(G) \cap I \subseteq_{\text{cf}} I$ , then  $I$  is closed under  $F$ , which is not the case. So both  $(\max v \in I)(v \in \text{Im}(G))$  and  $(\min v > I)(v \in \text{Im}(G))$  exist, and

$$F((\max v \in I)(v \in \text{Im}(G))) = (\min v > I)(v \in \text{Im}(G)),$$

as required.

For (b) and (c), use the function  $F$  defined by

$$F(x) = \begin{cases} (\mu x' \geq x)(\theta(x', c)), & \text{if } \exists x' \geq x \theta(x', c); \\ G(x), & \text{otherwise,} \end{cases}$$

where  $G$  is some fixed element of  $\mathfrak{E}_I$ . □

The reader is invited to unravel the following corollary to see what picture the lemma above gives.

**Corollary 7.2.** Let  $I$  be a cut. Denote by  $\bar{\mathfrak{E}}_I$  the set of all definable functions over  $c$  under which  $I$  is *not* closed. If  $\bar{\mathfrak{E}}_I$  is nonempty, then

$$\begin{aligned} & \bigcup \{ \sup(F^{-1}(I) \cap I) : F \in \bar{\mathfrak{E}}_I \} \\ &= \sup \{ (\max x \in I)(\theta(x, c)) : \theta \in \mathcal{L}_A \text{ for which the max exists} \} \end{aligned}$$

and

$$\begin{aligned} & \bigcap \{ \sup F(I) : F \in \bar{\mathfrak{E}}_I \} \\ &= \inf \{ (\min x > I)(\theta(x, c)) : \theta \in \mathcal{L}_A \text{ for which the min exists} \}. \quad \square \end{aligned}$$

Lemma 7.1(a) tells us we can transfer closure conditions between the infimum and the supremum of a constant region. This is similar to the situation for interstices.

**Proposition 7.3.** Suppose  $M$  is a recursively saturated model of PA. Let  $\Omega$  be a constant region over a parameter  $c \in M$ . Then the following are equivalent for a function  $F: M \rightarrow M$  definable over  $c$ .

- (a) The cut  $\inf \Omega$  is not closed under  $F$ .
- (b) The cut  $\sup \Omega$  is not closed under  $F$ .
- (c) There is a special point  $w$  over  $c$  in  $\inf \Omega$  such that  $F(w) > \sup \Omega$ .

*Proof.* Without loss of generality, suppose  $F$  is nondecreasing. It is clear that (c)  $\Rightarrow$  (a) and (c)  $\Rightarrow$  (b). To prove the converses, notice if  $I$  is a  $Y$ -cut, then all points of the forms

$$(\max x \in I)(\theta(x, c)) \quad \text{and} \quad (\min x > I)(\theta(x, c)),$$

where  $\theta(x, z)$  is an  $\mathcal{L}_A$ -formula, are special over  $c$ . Therefore, we are done by Lemmas 4.1 and 7.1(a). □

**Corollary 7.4.** Assume  $M$  is recursively saturated. Let  $c \in M$  and  $\Omega$  be a constant region over  $c$ . Then  $\inf \Omega$  and  $\sup \Omega$  are closed under the same  $\mathcal{L}_A$ -definable functions over  $c$ .  $\square$

Consequently, if  $\Omega$  is unbounded, then  $\inf \Omega \prec \sup \Omega$ . The converse, however, is not true. The most obvious example is when  $Y$  is the indicator for elementary cuts, in which case Lemma 3.2 makes both  $\inf \Omega$  and  $\sup \Omega$  elementary. Section 6 of Bigorajska–Kotlarski–Schmerl [2] contains examples where  $\sup \Omega$  is not elementary in  $M$ . See Section 9 for some explanations of why that paper is relevant.

## 8 Closedness under Functions

We investigate, in this section, under which functions the cuts  $\inf \Omega$  and  $\sup \Omega$  are closed for a constant region  $\Omega$ . It turns out our result works for all the  $Y$ -cuts in between too.

**Theorem 8.1.** Suppose  $M$  is a recursively saturated model of PA. Let  $\Omega$  be a constant region over some element  $c \in M$ . Then for all functions  $F: M \rightarrow M$  definable over  $c$ , the following are equivalent.

- (a) There exists a  $Y$ -cut  $I$  in  $\Omega$  that is closed under  $F$ .
- (b) All  $Y$ -cuts  $I$  in  $\Omega$  are closed under  $F$ .
- (c) There exists  $k \in \mathbb{N}$  such that for every  $x \in \Omega$ , we have  $F(x) \leq Y_k(x)$ .

*Proof.* Clearly (c)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (a). So it remains to prove (a)  $\Rightarrow$  (c).

Let  $I$  be a  $Y$ -cut in  $\Omega$  that is closed under  $F$ . By Proposition 7.3, if  $I = \sup \Omega$ , then  $\inf \Omega$  is closed under  $F$  too. So it is safe to assume  $I \neq \sup \Omega$ , and hence  $I$  is a proper cut of  $M$ . We may also assume  $x < F(x) \leq F(x+1)$  for all  $x \in M$ .

Define  $G(u) = F^u(0)$ . Since  $I$  is closed under  $F$ , we know  $\text{Im}(G) \cap I \subseteq_{\text{cf}} I$ . If  $I = \inf \Omega$ , then this overflows. Otherwise, one directly sees  $\text{Im}(G) \cap \Omega \neq \emptyset$ . Using Lemma 5.2, find  $\ell \in \mathbb{N}$  such that

$$\text{Im}(G) \cap [d+1, Y_\ell(d)] \neq \emptyset$$

for all  $d \in \Omega$ . We claim  $F(x) \leq Y_\ell^2(x)$  for every  $x \in \Omega$ . Since  $I \neq M$  and the  $Y_k$ 's are closed under composition, this will imply the existence of  $k \in \mathbb{N}$  that makes  $F(x) \leq Y_k(x)$  for all small enough  $x \in \Omega$ . An application of constantness together with Lemma 3.2 will then finish the proof.

Pick any  $x \in \Omega$ . Let  $d = (\max w \in \text{Im } G)(w \leq x)$ . Then

$$\begin{aligned} F(x) &\leq F(F(d)) && \text{since } x < F(d) \text{ by maximality,} \\ &\leq Y_\ell(F(d)) && \text{since } F(d) \in \text{Im } G, \\ &\leq Y_\ell^2(d) && \text{since } d \in \text{Im } G \text{ and so } F(d) \leq Y_\ell(d), \\ &\leq Y_\ell^2(x) && \text{since } d \leq x, \end{aligned}$$

as required.  $\square$

This theorem can be restated in terms of domination of functions.

**Definition.** Let  $I \subseteq_e M$  and  $F, G: M \rightarrow M$ . We say that  $F$  dominates  $G$  on  $I$  if there is an element of  $I$  above which all  $x \in I$  satisfy  $F(x) \geq G(x)$ .

**Corollary 8.2.** Assume our model  $M$  is recursively saturated. Fix a parameter  $c \in M$  and a constant region  $\Omega$  over  $c$ . Let  $I$  be a  $Y$ -cut in  $\Omega$  and  $F: M \rightarrow M$  that is  $\mathcal{L}_A$ -definable over  $c$ . Then  $I$  is closed under  $F$  if and only if  $F$  is dominated by  $Y_k$  on  $I$  for some  $k \in \mathbb{N}$ .  $\square$

It is important that the function  $F$  above is definable over the parameter  $c$  alone. For instance, with extra parameters, we can find faster growing functions under which  $\inf \Omega$  is closed—see the proof of Lemma 14.2 and Corollary 6.2. Actually, the proof of Lemma 14.2 shows  $\inf \Omega \neq M(d)$  for any  $d \in \text{cl}(c)$  over which  $Y$  is definable. However, it is not hard to find an indicator  $Y$  for which  $M(0)$  is the infimum of a constant region, especially when  $M(0) = \mathbb{N}$ .

Theorem 8.1 puts a tight restriction on unbounded constant intervals. For example, it tells us if a constant region  $\Omega$  is unbounded, then all  $Y$ -cuts in  $\Omega$  must be elementary. So not many constant regions can be unbounded. The next corollary further supports this claim. Some resemblance with the proof of Proposition 6.5 in our previous paper [6] may be observed.

**Corollary 8.3.** Suppose  $M$  is recursively saturated, and the indicator  $Y$  is definable over a parameter  $c \in M$ . If  $\Omega$  is a constant region over the same parameter  $c$ , then  $\Omega$  must be bounded.

*Proof.* Suppose not. Then by Corollary 8.2, every definable function over  $c$  is dominated by  $Y_k$  on  $M$  for some  $k \in \mathbb{N}$ . This is impossible. Consider, for example, the function  $F: x \mapsto Y_x(x) + 1$ . It is definable over  $c$ , and for every  $x > k$ , we have

$$F(x) = Y_x(x) + 1 > Y_x(x) \geq Y_k(x).$$

Thus, no  $Y_k$  can dominate  $F$  on  $M$ .  $\square$

Another piece of information that Theorem 8.1 provides is the necessity of arithmetic saturation in Theorem 3.1. This is similar to Proposition 5.13 in Kaye–Wong [6].

**Corollary 8.4.** Assume recursive saturation of our model  $M$ . Suppose there is a  $Y$ -interval  $\llbracket a, b \rrbracket$  in which we can find for each  $c \in M$  a subinterval that is constant over  $c$ . Then  $\mathbb{N}$  is strong in  $M$ .

*Proof.* Fix a code  $Y$  for the graph of the indicator below  $b$ . Let  $c \in M$  be a code for a function  $f: \mathbb{N} \rightarrow M$ . Using the hypothesis, find  $\llbracket r, s \rrbracket \subseteq \llbracket a, b \rrbracket$  that is constant over  $\langle c, Y \rangle$ . Overspill then gives a nonstandard  $\nu \in M$  which satisfies

$$M \models \forall x < b \exists y Y(x, y) > \nu.$$

We claim that for all  $m \in \mathbb{N}$ ,

$$f(m) > \mathbb{N} \iff f(m) > \min(Y(r, s), \nu).$$

Let  $m \in \mathbb{N}$  for which  $f(m) > \mathbb{N}$ . If  $f(m) > \nu$ , then we are done. So suppose  $f(m) \leq \nu$ . Consider the function  $F$  defined by

$$F(x) = (\max y)(Y(x, y) < f(m)).$$

By the choice of  $\nu$ , we know  $F(x)$  exists for every  $x < b$ . Notice  $F$  is monotone wherever it is defined.

Suppose we have  $k \in \mathbb{N}$  and  $x < b$  such that  $F(x) \leq Y_k(x)$ . Then

$$F(x) \leq Y_k(x) = (\mu y)(Y(x, y) \geq k) \in M(x),$$

and so  $F(x) + 1 \in M(x)$  too. However, by the maximality of  $F(x)$ , we must have  $Y(x, F(x) + 1) \geq f(m) > \mathbb{N}$ . This contradicts the definition of  $M(x)$ . Therefore  $F(x) > Y_k(x)$  for all  $k \in \mathbb{N}$  and all  $x < b$ .

Let  $\Omega$  be the constant region over  $\langle c, Y \rangle$  around  $\llbracket r, s \rrbracket$ . Corollary 8.2 says  $\inf \Omega$  is not closed under  $F$ . Using Proposition 7.3, find  $w \in \inf \Omega$  such that  $F(w) > \sup \Omega$ . Notice  $[r, s] \subseteq \Omega \subseteq [w, F(w)]$  as a result. Therefore,

$$Y(r, s) \leq Y(w, F(w)) < f(m),$$

as required. □

The role of the strength of  $\mathbb{N}$  in the context of interstices was studied in Bamber–Kotlarski [1] and Bigorajska–Kotlarski–Schmerl [2].

## 9 Interstices

The word *interstice* was introduced by Bamber–Kotlarski [1], although the notion already appeared in Kaye–Kossak–Kotlarski [5].

**Definition.** An *interstice* over  $c$  is a maximal nonempty convex subset of  $M$  that contains no point from  $\text{cl}(c)$ . Denote by  $\mathfrak{F}_\Lambda$  the set of all functions definable over  $c$  under which  $\inf \Lambda$  is closed. The *interstitial gap* around an element  $d \in \Lambda$  (with respect to  $\Lambda$ ) is defined to be

$$\{x \in M : F(x) > d \text{ and } G(d) > x \text{ for some } F, G \in \mathfrak{F}_\Lambda\}.$$

Each interstice can, in a sense, be considered as a constant region for some local indicator. This is summarized in the next two propositions. The first of these was proved by Kaye–Kossak–Kotlarski [5, Lemma 6.4], while the second appeared in Schmerl [10, Proposition 1.3].

**Proposition 9.1.** Let  $M$  be an arithmetically saturated model of PA. Then for all  $c \in M$  and all interstices  $\Lambda$  over  $c$ , the set  $\mathfrak{F}_\Lambda$  is coded in  $M$  as a set of Gödel numbers. □

**Proposition 9.2.** Suppose  $M$  is a recursively saturated model of PA. Let  $\Lambda$  be an interstice over an element  $c \in M$ . If  $x \in \Lambda$  and  $[u, v] \subseteq \Lambda$  such that  $F(u) < v$  for all  $F \in \mathfrak{F}_\Lambda$ , then there exists  $x' \in [u, v]$  making  $(M, x, c) \equiv (M, x', c)$ . □

More explicitly, if the model  $M$  is arithmetically saturated, then given an interstice  $\Lambda$  over a parameter  $c \in M$ , we can make  $\Lambda$  into a constant region over  $c$  by setting  $Y_k$  to be the  $k$ th element of  $\mathfrak{F}_\Lambda$  in the coded sequence that Proposition 9.1 provides. Notice, however, our definability assumption on the indicator mostly fails for  $Y_k$ 's obtained in this way.

Clearly, constant regions have to be subsets of interstices. If a constant region  $\Omega$  is included in an interstice  $\Lambda$ , then  $Y_k \in \mathfrak{F}_\Lambda$  for every  $k \in \mathbb{N}$ . So by

Theorem 3.1, every arithmetically saturated model of PA contains an interstice  $\Lambda$  for which  $Y_k \in \mathfrak{F}_\Lambda$  for all  $k \in \mathbb{N}$ . Conversely, if an interstice  $\Lambda$  makes  $Y_k \in \mathfrak{F}_\Lambda$  for every  $k \in \mathbb{N}$ , then, assuming the model  $M$  is arithmetically saturated, it includes a constant region for  $Y$  by Theorem 3.1. Furthermore, we know exactly when the inclusion between a constant region and an interstice is proper.

**Proposition 9.3.** Suppose  $M$  is recursively saturated. Let  $\Omega$  be a constant region over an element  $c \in M$ , and let  $\Lambda$  be the interstice over  $c$  containing  $\Omega$ . The following are equivalent.

- (a)  $\Omega = \Lambda$ .
- (b)  $\inf \Omega = \inf \Lambda$ .
- (c)  $\inf \Omega$  is closed under all functions in  $\mathfrak{F}_\Lambda$ .
- (d)  $\forall F \in \mathfrak{F}_\Lambda \exists k \in \mathbb{N} \forall x \in \Lambda Y_k(x) \geq F(x)$ .

*Proof.* The implication (a)  $\Rightarrow$  (d) is part of Theorem 8.1, and (d)  $\Rightarrow$  (c) is an easy exercise. We will prove (c)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (a).

Suppose (b) is false. Using Lemma 6.1, find an  $\mathcal{L}_A$ -formula  $\theta(x, z)$  such that  $\theta(x, c)$  is satisfied below but not in  $\Omega$ , and

$$(\max x < \Omega)(\theta(x, c)) \in \Lambda \cap \inf \Omega.$$

Call this maximum  $w$ . Define  $F(x) = (\mu x' > x)(\theta(x', c))$ . Then  $F \in \mathfrak{F}_\Lambda$  and  $F(w) > \Omega$ . So (c) is false.

Now, suppose (b) holds. We show that  $\Lambda$  has the constantness property over  $c$ , and so (a) holds because of the maximality of  $\Omega$ . Pick  $x \in \Lambda$  and  $\llbracket u, v \rrbracket \subseteq \Lambda$ . We claim that  $F(u) < v$  for every  $F \in \mathfrak{F}_\Lambda$ . Fix  $F \in \mathfrak{F}_\Lambda$ . Let  $k \in \mathbb{N}$  such that  $Y_k$  dominates  $F$  on  $\inf \Omega$ . Notice there is a point in  $\text{cl}(c) \cap \inf \Omega$  above which all  $x \in \inf \Omega$  satisfy  $Y_k(x) \geq F(x)$ . Thus actually, we have  $Y_k(x) \geq F(x)$  for all  $x \in \Lambda$ , because  $\Lambda$  does not contain any definable point over  $c$ . In particular,  $F(u) \leq Y_k(u) < v$ . Since the choice of  $F \in \mathfrak{F}_\Lambda$  is arbitrary, Proposition 9.2 tells us there exists  $x' \in \llbracket u, v \rrbracket$  such that  $(M, x, c) \equiv (M, x', c)$ , as required.  $\square$

Consequently, provided a constant region is not an interstice, it is always a proper subset of an interstitial gap.

**Corollary 9.4.** Fix  $c \in M$ , and suppose  $M$  is recursively saturated. Let  $\Omega$  be a constant region over  $c$  that is strictly included in an interstice  $\Lambda$  over  $c$ . Then there is an interstitial gap  $\lambda \subseteq \Lambda$  which satisfies

$$\inf \lambda < \inf \Omega < \sup \Omega < \sup \lambda.$$

*Proof.* Combine Propositions 9.3(c) and 7.3.  $\square$

It is clear that constant regions in different interstices cannot have the same type. The next proposition says, if the model is recursively saturated, then distinct constant regions of the same type must live in distinct interstitial gaps. The proof we give resembles that of Ehrenfeucht's Lemma [9, Theorem 1.7.2].

**Proposition 9.5.** Assume  $M$  is recursively saturated. Fix an element  $c \in M$  and an interstice  $\Lambda$  over  $c$ . Let  $\Omega$  and  $\Omega'$  be constant regions over  $c$  inside  $\Lambda$  such that  $\text{tp}(\Omega/c) = \text{tp}(\Omega'/c)$  and  $\Omega < \Omega'$ . Then  $F(\Omega) < \Omega'$  for all  $F \in \mathfrak{F}_\Lambda$ .

*Proof.* Let  $d \in \Omega$  and  $d' \in \Omega'$ . Suppose we can find an increasing  $F \in \mathfrak{F}_\Lambda$  such that  $F(d) \geq d'$ . Define  $G(u) = F^u(0)$ . Since  $F \in \mathfrak{F}_\Lambda$ , we know  $\text{Im}(G) \cap \text{inf } \Lambda \subseteq_{\text{cf}} \text{inf } \Lambda$ . So by overspill, there are images of  $G$  inside  $\Lambda$ . However,  $\text{inf } \Omega$  cannot be closed under  $F$  by Corollary 7.4. Thus  $\text{Im}(G) \cap \text{inf } \Omega \not\subseteq_{\text{cf}} \text{inf } \Omega$ . Let  $u \in M$  be maximum such that  $G(u) < \Omega$ . Notice  $\text{Im}(G) \cap \Omega = \emptyset$  by overspill. So  $G(u+1) > \sup \Omega > d$ . This implies

$$G(u+2) = F(G(u+1)) \geq F(d) \geq d'.$$

Since  $\text{tp}(\Omega/c) = \text{tp}(\Omega'/c)$ , we know  $\text{Im}(G) \cap \Omega' = \emptyset$ . Hence  $G(u+2) > \Omega'$ . Putting everything together, we obtain

$$\begin{aligned} u+1 &= (\min x)(G(x) > \Omega) \text{ and} \\ u+2 &= (\min x)(G(x) > \Omega'). \end{aligned}$$

This contradicts the hypothesis that  $\text{tp}(\Omega/c) = \text{tp}(\Omega'/c)$ .  $\square$

## 10 Pregeneric Intervals

We are ready for the key theorem of this paper, that all constant intervals are pregeneric over the same parameter.

**Theorem 10.1.** Suppose  $M$  is a countable recursively saturated model of PA. Fix  $c \in M$  and a constant region  $\Omega$  over  $c$ . If  $x, y \in \Omega$  and  $\llbracket r, s \rrbracket \subseteq \Omega$ , then there are  $x', y' \in \llbracket r, s \rrbracket$  such that  $(M, x, y, c) \equiv (M, x', y', c)$ .

*Proof.* By recursive saturation, it suffices to show that the type

$$p(u, v) = \{\varphi(u, v, c) \leftrightarrow \varphi(x, y, c) : \varphi \in \mathcal{L}_A\} + \{u, v \in \llbracket r, s \rrbracket\}$$

is finitely satisfied in  $M$ . Pick  $\varphi(u, v, w) \in \mathcal{L}_A$ , and suppose  $M \models \varphi(x, y, c)$ . Define

$$F(u) = (\mu v > u)(\exists u', v' \in [u, v] \varphi(u', v', c)).$$

Notice  $F$  is monotone. If  $\text{inf } \Omega$  is not closed under  $F$ , then by Proposition 7.3, there is  $w \in \text{inf } \Omega$  such that  $F(w) > \sup \Omega$ . This contradicts the existence of  $x$  and  $y$ . So  $\text{inf } \Omega$  must be closed under  $F$ . Theorem 8.1 then implies  $F(r) \leq s$ . Therefore,  $M \models \exists u, v \in \llbracket r, s \rrbracket \varphi(u, v, c)$ , as required.  $\square$

The issues for small intervals are more delicate. It is known that small intervals are not closed under taking subintervals [11, Proposition 5.1]. Therefore, not all pregeneric intervals are small. Since the proof is short and has not appeared elsewhere in the literature, we include it here.

**Definition.** An interval  $\llbracket a, b \rrbracket$  is *small* over  $c$  if  $Y(a, b) < x$  for each nonstandard  $x \in \text{cl}(a, c)$ .

**Proposition 10.2** (Kaye). Let  $c \in M$ . Then every  $Y$ -interval  $\llbracket a, b \rrbracket$  contains a  $Y$ -large subinterval that is not small over  $c$ .

*Proof.* Without loss of generality, assume  $a$  is nonstandard. Let  $d \in \llbracket a, b \rrbracket$  which satisfies  $a \ll d \ll b$ . Choose a nonstandard  $m \in M$  such that

$$m < \min\{\lfloor \log_2(a) \rfloor, \lfloor \log_2(d-a) \rfloor, Y(d, b)\}.$$

Notice  $2^m < 2^{\lfloor \log_2(a) \rfloor} \leq a$ . So there is an odd  $w \in M$  such that  $2^m w < a$ . Let

$$x = (\max w)(\exists y (w = 2y + 1) \wedge 2^m w < a).$$

Then  $2^m(x+2) \geq a$  by the maximality of  $x$ . Also,

$$2^m(x+2) = 2^m x + 2^{m+1} < a + 2^{\lfloor \log_2(d-a) \rfloor} \leq a + d - a = d.$$

Hence  $2^m(x+2) \in \llbracket a, d \rrbracket$ . Let  $a' = 2^m(x+2)$ . Since  $a' \leq d \ll b$ , we know  $\llbracket a', b \rrbracket$  is a  $Y$ -interval. However, by the monotonicity of  $Y$ ,

$$Y(a', b) \geq Y(d, b) > m = (\max u)(2^u \mid a') \in \text{cl}(a', c) \setminus \mathbb{N}.$$

Therefore  $\llbracket a', b \rrbracket$  is not small over  $c$ . □

## 11 Generic Cuts

We can thus redefine generic cuts as follows.

**Definition.** A  $Y$ -cut  $I$  is *generic* if for every  $c \in M$ , there exists a constant region over  $c$  that contains  $I$ .

**Corollary 11.1.** Suppose  $M$  is recursively saturated, and let  $I$  be a  $Y$ -cut. Then  $I$  is generic if and only if over no parameter in  $M$  is  $I$  a limit of special points. □

Kaye [4] showed if  $M$  is a countable arithmetically saturated model of PA, then the class of generic cuts is the smallest comeagre set in the space of  $Y$ -cuts that is invariant under the action of  $\text{Aut}(M)$ . This is a simple consequence of Theorem 3.1. In particular, the generic cuts form a dense subset of the space under such hypotheses.

Recall two cuts  $I, J$  are said to be *conjugate* over an element  $c \in M$  exactly when  $(M, I, c) \cong (M, J, c)$ . On the one hand, Kaye–Wong [6] showed using a back-and-forth argument that—provided  $M$  is countable—generic cuts in the same constant region are all conjugate. On the other hand, a constant region is the maximal convex subset of  $M$  in which all generic cuts are conjugate. Compare the next proposition with the results in Section 6 of Kaye–Wong [6].

**Proposition 11.2.** Assume  $M$  is recursively saturated. Fix  $c \in M$ . Let  $I$  be a generic cut, and  $\Omega$  be the constant region over  $c$  containing  $I$ . Then between  $\Omega$  and any generic cut  $J$  outside  $\Omega$ , there is a  $Y$ -interval in which *no* generic cut  $I'$  satisfies  $(M, I, c) \equiv (M, I', c)$ .

*Proof.* Without loss of generality, suppose  $(M, I, c) \equiv (M, J, c)$ . It follows that  $I$  and  $J$  are in the same interstice over  $c$ , and so  $\Omega$  is not an interstice over  $c$ . Using Corollary 9.4, take a  $Y$ -interval  $\llbracket r, s \rrbracket$  that is disjoint from  $\Omega$  but is in the same interstitial gap as  $\Omega$ . Let  $I'$  be a generic cut in  $\llbracket r, s \rrbracket$ , and  $\Omega'$  be the constant region over  $c$  containing  $I'$ . Proposition 9.5 says  $\text{tp}(\Omega/c) \neq \text{tp}(\Omega'/c)$ . If  $\theta(x, c) \in \text{tp}(\Omega/c) \setminus \text{tp}(\Omega'/c)$ , then points  $x \in M$  satisfying  $\theta(x, c)$  are cofinal in  $I$  but not in  $I'$ . Therefore  $(M, I, c) \not\equiv (M, I', c)$ . □

## 12 Special Cuts

Imitating the definition of special points, we devise the notion of *special cuts*. It seems an additional countability assumption on our model  $M$  is needed for most of this section, because apparently we cannot avoid dealing with automorphisms.

**Definition.** A cut  $I$  is *special* over an element  $c \in M$  if there exists an interval around  $I$  in which the only cut conjugate to  $I$  over  $c$  is  $I$  itself.

Special points and special cuts are naturally rather close to each other.

**Lemma 12.1.** Let  $c \in M$  and  $w$  be a special point over  $c$ . Then either  $M(w)$  or  $M[w]$  is a special cut over  $c$ .

*Proof.* Let  $\theta(x, z) \in \mathcal{L}_A$  that witnesses the specialness of  $w$ . Without loss of generality, suppose we have  $s \gg w$  satisfying

$$M \models \theta(w, c) \wedge \forall x \in [w+1, s] \neg \theta(x, c).$$

We claim that no  $g \in \text{Aut}(M, c)$  maps  $M(w)$  into  $\llbracket w+1, s \rrbracket$  without fixing  $M(w)$ . To see this, take any  $g \in \text{Aut}(M, c)$  making  $g(M(w)) \in \llbracket w+1, s \rrbracket$ . If there is  $k \in \mathbb{N}$  such that  $g(M(w)) < Y_k(w) \in M(w)$ , then

$$\begin{aligned} g(Y_k(w)) &= g(Y_k((\max x \in M(w))(\theta(x, c)))) \\ &= Y_k((\max x \in g(M(w)))(\theta(x, c))) \\ &= Y_k(w) > g(M(w)), \end{aligned}$$

which is not possible. Similarly, one cannot have  $M(w) < g(M(w))$ . Therefore  $g(M(w)) = M(w)$ , as required.  $\square$

A consequence of this lemma is a characterization of constant regions in terms of special cuts.

**Proposition 12.2.** Suppose  $M$  is recursively saturated. Fix a parameter  $c \in M$ , and let  $\Omega$  be a convex subset of  $M$  which satisfies

$$\inf \Omega < M[d] < M(d) < \sup \Omega$$

for all  $d \in \Omega$ . Then  $\Omega$  is constant over  $c$  if and only if it contains no  $Y$ -cut that is special over  $c$ .

*Proof.* Lemmas 4.2 and 12.1 imply that if  $\Omega$  is not constant over  $c$ , then it contains a special  $Y$ -cut over  $c$ . This proves one direction of the proposition.

Conversely, suppose  $\Omega$  is constant over  $c$ . Let  $I$  be a  $Y$ -cut in  $\Omega$ , and  $\llbracket r, s \rrbracket$  be an interval around  $I$ . Without loss, assume  $\llbracket r, s \rrbracket \subseteq \Omega$ . Using the axioms of an indicator, we can find a subinterval  $\llbracket u, v \rrbracket \subseteq \llbracket r, s \rrbracket$  which does not contain  $I$ , and so by Theorem 10.1, there is  $g \in \text{Aut}(M, c)$  such that  $g(I) \in \llbracket u, v \rrbracket$ . Since the choice of  $\llbracket r, s \rrbracket$  is arbitrary, the cut  $I$  cannot be special over  $c$ .  $\square$

It is conceivable that specialness is the opposite of genericity. The next theorem describes the precise relationship between these two notions.

**Theorem 12.3.** Let  $M$  be a countable recursively saturated model of PA. Then a  $Y$ -cut  $I$  is generic if and only if over no element of  $M$  is  $I$  a limit of special  $Y$ -cuts.

*Proof.* Suppose  $I$  is a generic cut. If  $c \in M$  and  $\Omega$  is a constant region over  $c$  containing  $I$ , then Proposition 12.2 says  $\Omega$  cannot contain any special  $Y$ -cut over  $c$ , and so  $I$  is not a limit of such cuts.

Conversely, suppose  $I$  is not generic. Let  $c \in M$  over which no constant interval can contain  $I$ . Then in view of Lemma 4.2, there must be special points over  $c$  arbitrarily close to  $I$ . Notice we may assume  $I$  is neither  $M(d)$  nor  $M[d]$  for any  $d \in M$ , because otherwise  $I$  would itself be a special cut over  $d$ . So we are done by Lemma 12.1.  $\square$

We also have a theorem that loosely connects specialness with genericity.

**Theorem 12.4.** Let  $M$  be a countable recursively saturated model of PA. Then a  $Y$ -cut  $I$  is generic if and only if for every  $c \in M$ , there exists an interval containing  $I$  in which the orbit of  $I$  under the action of  $\text{Aut}(M, c)$  is  $Y$ -dense.

*Proof.* The ‘only if’ part is an easy consequence of Theorem 10.1. The argument we use for the ‘if’ part is similar to that from Theorem 3.10 in Kaye–Wong [6]. Let  $c \in M$ . Pick an interval  $\llbracket r, s \rrbracket$  containing  $I$  in which the orbit of  $I$  under the action of  $\text{Aut}(M, c)$  is dense. Using Theorem 3.1, find a subinterval  $\llbracket a, b \rrbracket \subseteq \llbracket r, s \rrbracket$  that is constant over  $c$ . By hypothesis, there is  $g \in \text{Aut}(M, c)$  mapping  $I$  into  $\llbracket a, b \rrbracket$ , and so it makes  $g^{-1}(\llbracket a, b \rrbracket)$  a constant interval over  $c$  containing  $I$ .  $\square$

### 13 Existentially Closed Cuts

Let us recapitulate Kaye–Wong [7]. We work with the language  $\mathcal{L}_{\text{cut}}$ , which, in addition to the symbols in  $\mathcal{L}_A$ , contains a new function symbol for each  $\mathcal{L}_A$ -Skolem function, and a new unary predicate symbol  $\mathbb{I}$  intended to interpret a cut. Recall from Section 2 that each  $Y_k$ , where  $k \in \mathbb{N}$ , is assumed to be parameter-free definable. So they make sense across a class  $\mathbf{K}$  of models of PA containing  $M$ . This class  $\mathbf{K}$  will be additionally required to be *cofinal* in the class of all models of PA under elementary embeddings. In other words, we want every model of PA to have an elementary extension in  $\mathbf{K}$ . The theory  $\text{PA}_Y$  consists of the axioms of PA, the definitions of the  $\mathcal{L}_A$ -Skolem functions, an axiom saying  $\mathbb{I}$  is a proper cut, and a scheme saying  $\mathbb{I}$  is closed under  $Y_k$  for all  $k \in \mathbb{N}$ .

Given a cut  $I$ , the *existential type* of an element  $c \in M$ , denoted by  $\text{etp}_{(M, I)}(c)$ , is the set of all  $\exists_1$ -formulas satisfied by  $c$  in  $(M, I)$ . Kaye and the author [7] observed that over  $\text{PA}_Y$ , every  $\exists_1$ -formula  $\theta(z) \in \mathcal{L}_{\text{cut}}$  says

$$\mathbb{I} \text{ is not closed under the function } x \mapsto F(x, z)$$

for some  $\mathcal{L}_A$ -Skolem function  $F$ . So, for example, if  $M$  is recursively saturated and  $\Omega$  is a constant region over  $c$ , then Corollary 7.4 says  $\text{etp}_{(M, \inf \Omega)}(c) = \text{etp}_{(M, \sup \Omega)}(c)$ . An *existentially closed model* of  $\text{PA}_Y$  is a model  $(M, I) \models \text{PA}_Y$  such that whenever  $(K, J)$  is an extension of  $(M, I)$  satisfying  $\text{PA}_Y$ , where  $K \in \mathbf{K}$ , we have

$$(K, J) \models \theta(c) \quad \Rightarrow \quad (M, I) \models \theta(c)$$

for all  $\exists_1$ -formulas  $\theta(z) \in \mathcal{L}_{\text{cut}}$  and all  $c \in M$ . Abusing the notation slightly, I will sometimes write ‘the  $Y$ -cut  $I$  is existentially closed’ when I mean ‘ $(M, I)$  is an existentially closed model of  $\text{PA}_Y$ ’. General model-theoretic facts tell us existentially closed models of  $\text{PA}_Y$  exist, because  $\text{PA}_Y$  is  $\forall_2$ .

Thanks to the following proposition, we can work on existentially closed models of  $\text{PA}_Y$  without mentioning extensions and the class  $\mathbf{K}$  in the later sections.

**Proposition 13.1.** Let  $I$  be a proper  $Y$ -cut. Then  $(M, I)$  is an existentially closed model of  $\text{PA}_Y$  if and only if for every  $\mathcal{L}_A$ -definable function  $F$  under which  $I$  is closed, there is  $k \in \mathbb{N}$  such that  $Y_k$  dominates  $F$  on  $I$ .

*Proof.* Let  $F$  be a definable function on  $M$  under which  $I$  is closed. Without loss of generality, suppose  $F$  is increasing. Assume  $F$  is not dominated by  $Y_k$  on  $I$  for any  $k \in \mathbb{N}$ . Using this assumption and the Compactness Theorem, find an elementary extension  $K \succ M$  that contains two new elements  $r, s$  satisfying

$$x < r \ll s \leq F(r) < y$$

for all  $x \in I$  and all  $y \in M \setminus I$ . By passing to an elementary extension if necessary, we may suppose  $K \in \mathbf{K}$ . Let  $J$  be any  $Y$ -cut in  $K$  between  $r$  and  $s$ . Then  $(K, J) \supseteq (M, I)$  but  $J$  is not closed under  $F$ . So  $(M, I)$  is not existentially closed.

Conversely, fix some  $(K, J) \supseteq (M, I)$ , where  $K \in \mathbf{K}$  and  $(K, J) \models \text{PA}_Y$ . Let  $F$  be a definable function on  $M$  under which  $I$  is closed. Without loss, we assume  $F$  is increasing again. Using the hypothesis, find  $k \in \mathbb{N}$  such that  $Y_k$  dominates  $F$  on  $I$ . Then there is  $\llbracket a, b \rrbracket \subseteq M$  containing  $I$  which makes

$$M \models \forall x \in \llbracket a, b \rrbracket Y_k(x) \geq F(x).$$

This transfers to  $K$  by elementarity. So  $Y_k$  dominates  $F$  on  $J$  too. In particular,  $J$  is closed under  $F$  because  $J$  is a  $Y$ -cut.  $\square$

There are numerous corollaries to this proposition, the most straightforward one being the following. Recall from Kirby [8] that the  $Y$ -index of a cut  $I$  is defined by

$$\text{index}_Y(I) = \{n \in M : \forall x \in I \forall y > I Y(x, y) \geq n\}.$$

The  $Y$ -index of a  $Y$ -cut is always bigger than  $\mathbb{N}$ .

**Corollary 13.2.** Suppose we have an existentially closed model  $(M, I) \models \text{PA}_Y$ . Then  $\text{index}_Y(I) = \mathbb{N}$ . In particular,  $(M, I)$  is not  $\forall_1$ -recursively saturated.  $\square$

The following corollary was first proved in Kaye–Wong [7, Theorem 5.2].

**Corollary 13.3.** If  $M$  is recursively saturated and  $I$  is a generic cut, then  $(M, I)$  is an existentially closed model of  $\text{PA}_Y$ .

*Proof.* Combine Theorem 8.1 and Proposition 13.1.  $\square$

Taking away the arithmetic saturation part from Kaye’s proof of Theorem 3.1, we obtain the next proposition.

**Proposition 13.4.** Let  $M$  be a countable nonstandard model of PA. Then for every  $\llbracket a, b \rrbracket \subseteq M$ , there is a cut  $I$  of  $M$  contained in  $\llbracket a, b \rrbracket$  such that  $(M, I)$  is an existentially closed model of  $\text{PA}_Y$ .

*Proof.* Enumerate all parametrically definable functions  $M \rightarrow M$  in the sequence  $(F_n)_{n \in \mathbb{N}}$ . We will construct, recursively, an  $\omega$ -sequence of intervals

$$\llbracket a, b \rrbracket = \llbracket a_0, b_0 \rrbracket \supseteq \llbracket a_1, b_1 \rrbracket \supseteq \llbracket a_2, b_2 \rrbracket \supseteq \dots$$

such that  $(M, \sup\{a_n : n \in \mathbb{N}\})$  is an existentially closed model of  $\text{PA}_Y$ . Suppose we already have  $\llbracket a_n, b_n \rrbracket$ . Consider  $F_n$ . There are two possibilities.

- (i) There is  $\llbracket r, s \rrbracket \subseteq \llbracket a_n, b_n \rrbracket$  such that  $F_n(r) = s$ .
- (ii) For every  $r \in \llbracket a_n, b_n \rrbracket$ , we have  $Y(r, F_n(r)) \in \mathbb{N}$ .

In case (i), no  $Y$ -cut in  $\llbracket r, s \rrbracket$  can be closed under  $F_n$ . So we can safely let  $\llbracket a_{n+1}, b_{n+1} \rrbracket$  be any such  $\llbracket r, s \rrbracket$ . In case (ii), there must be  $k \in \mathbb{N}$  such that

$$M \models \forall r \in \llbracket a_n, b_n \rrbracket Y(r, F_n(r)) \leq k. \quad (*)$$

Thus, any  $Y$ -cut  $I$  in  $\llbracket a_n, b_n \rrbracket$  is closed under  $F_n$ . Furthermore, if  $(M, I) \subseteq (K, J) \models \text{PA}_Y$  where  $I \in \llbracket a_n, b_n \rrbracket$  and  $K \in \mathbf{K}$ , then  $J$  must be closed under  $F_n$  too, because  $(*)$  transfers to  $K$ . So we can set  $\llbracket a_{n+1}, b_{n+1} \rrbracket$  to  $\llbracket a_n, b_n \rrbracket$ . In either case, the function  $F_n$  cannot be a counterexample to the existential closedness of the outcome. Since every definable function is considered in this construction, we conclude that  $(M, \sup\{a_n : n \in \mathbb{N}\})$  is existentially closed.  $\square$

Apparently, it is not impossible that for every cut  $I$ , one can find a suitable indicator  $Y$  to make  $(M, I)$  an existentially closed model of  $\text{PA}_Y$ . In view of Proposition 13.1, this is related to the following question.

**Question 13.5.** Is there a cut  $I$  for which no cofinal subclass of  $\mathfrak{F}_I$  is coded in  $M$ ? Here  $\mathfrak{F}_I$  denotes the class of all parameter-free definable functions  $M \rightarrow M$  under which  $I$  is closed, ordered under domination on  $I$ .

## 14 $\omega$ -limits

Probably the simplest cuts are those of the form

$$\sup\{(a)_n : n \in \mathbb{N}\} \quad \text{or} \quad \inf\{(b)_n : n \in \mathbb{N}\},$$

where  $a, b \in M$ . None of these cuts turns out to be existentially closed. We start with a slightly more general proposition which goes in the direction of Question 13.5.

**Proposition 14.1.** For a  $Y$ -cut  $I$  and a monotonically increasing definable function  $f: M \rightarrow M$ , the following are equivalent.

- (a)  $I$  is closed under  $x \mapsto Y_{f(x)}(x)$ .
- (b)  $f(I) \subseteq \text{index}_Y(I)$ .

*Proof.* If  $f(I) \subseteq \text{index}_Y(I)$ , then clearly  $I$  is closed under  $x \mapsto Y_{f(x)}(x)$ . Conversely, suppose  $I$  is closed under  $x \mapsto Y_{f(x)}(x)$ . Let  $a \in I$ . Then for each  $x \in I$  above  $a$  and each  $y > I$ ,

$$\begin{aligned} Y(x, y) &\geq Y(x, Y_{f(x)}(x)) && \text{since } Y_{f(x)}(x) \in I < y, \\ &\geq Y(x, Y_{f(a)}(x)) && \text{since } a < x \text{ and } f, Y \text{ are monotone,} \\ &= f(a). \end{aligned}$$

Therefore  $f(a) \in \text{index}_Y(I)$ .  $\square$

**Lemma 14.2.** If  $I$  is a  $Y$ -cut of the form

$$\sup\{(a)_n : n \in \mathbb{N}\},$$

where  $a \in M$ , then  $(M, I)$  is not an existentially closed model of  $\text{PA}_Y$ .

*Proof.* Without loss of generality, suppose  $(a)_n < (a)_{n+1}$  for all  $n \in \mathbb{N}$ . We may also assume  $\text{index}_Y(I) = \mathbb{N}$  because of Corollary 13.2. If  $f$  is a monotonically increasing definable function  $M \rightarrow M$  such that  $f(I) = \mathbb{N}$ , then  $x \mapsto Y_{f(x)}(x)$  dominates  $Y_k$  on  $I$  for every  $k \in \mathbb{N}$ . Therefore, by Propositions 13.1 and 14.1, it suffices to find a monotonically increasing definable function  $f: M \rightarrow M$  such that  $f(I) = \mathbb{N}$ . One example is given by  $f(x) = (\mu n)((a)_n \geq x)$ .  $\square$

**Lemma 14.3.** If  $I$  is a  $Y$ -cut of the form

$$\inf\{(b)_n : n \in \mathbb{N}\},$$

where  $b \in M$ , then  $(M, I)$  is not an existentially closed model of  $\text{PA}_Y$ .

Proposition 14.1 does not help in proving this lemma, because there is no definable increasing function  $f$  mapping a downward  $\omega$ -limit to  $\mathbb{N}$ . Moreover, we do not have an analogue of Proposition 14.1 for monotonically decreasing function  $f$  satisfying  $f(I) \subseteq_{\text{dof}} M \setminus \text{index}_Y(I)$ . To see this, consider, for example,  $I = M[b]$  for some  $b \in M$ , and  $f(x) = (\mu k)(Y_k(x) \geq b)$ . Since  $f(M \setminus I) \subseteq_{\text{cf}} \mathbb{N}$ , we have  $f(I) \subseteq_{\text{dof}} M \setminus \mathbb{N}$  by underspill. However, for all  $x \in I$ ,

$$Y_{f(x)}(x) = Y_{(\mu k)(Y_k(x) \geq b)}(x) \geq b > I.$$

Fortunately, the trick used in Proposition 2.4 of Kaye [4] works.

*Proof.* Without loss of generality, assume  $(b)_{n+1} < (b)_n$  for all  $n \in \mathbb{N}$ . We may also suppose  $Y((b)_{n+1}, (b)_n) \in \mathbb{N}$  for all  $n \in \mathbb{N}$ . This can be seen using Proposition 13.1 with the function

$$x \mapsto (\mu y)(\exists n (y = (b)_n \wedge (b)_{n+1} \geq x)),$$

or be shown directly from the definitions. Define

$$F(x) = (\mu z)(Y(x, z) \geq Y(z, (b)_0)).$$

One can verify using condition (3) on page 2 that whenever  $x \in I$ , both  $[x, F(x)]$  and  $[F(x), (b)_0]$  are  $Y$ -large. Hence  $I$  is closed under  $F$ , but  $F$  dominates  $Y_k$  on  $I$  for every  $k \in \mathbb{N}$ . Proposition 13.1 then says  $(M, I)$  is not an existentially closed model of  $\text{PA}_Y$ .  $\square$

In Corollary 8.2, we saw that the infima of constant regions share some common properties with existentially closed  $Y$ -cuts. However, Corollary 6.2 and Lemma 14.2 tell us none of these cuts are existentially closed. In view of Lemma 14.3, the same is true for the suprema of constant regions.

## 15 Genericity and Existential Closedness

We conclude this paper with a few new characterizations of generic cuts in terms of existential closedness. The following lemma will be handy.

**Lemma 15.1.** Suppose  $M$  is recursively saturated. Fix an element  $c \in M$  and an existentially closed model  $(M, I) \models \text{PA}_Y$ . Let  $\llbracket r, s \rrbracket$  be an interval around  $I$  which avoids all points of the form

$$(\max x \in I)(\theta(x, c)) \quad \text{or} \quad (\min x > I)(\theta(x, c)),$$

where  $\theta(x, z)$  is some  $\mathcal{L}_A$ -formula. Then  $\llbracket r, s \rrbracket$  is constant over  $c$ .

*Proof.* We claim that if  $\varphi(x, c)$  is an  $\mathcal{L}_A$ -formula satisfied by some element of  $\llbracket r, s \rrbracket$ , then it has to be satisfied in every  $Y$ -large subinterval of  $\llbracket r, s \rrbracket$ . By recursive saturation, this is sufficient to finish the proof. There are three possibilities:

- (i)  $(M, I) \models \forall x \in \mathbb{I} \neg \varphi(x, c)$ ;
- (ii)  $(M, I) \models \exists w (w = (\max x \in \mathbb{I})(\varphi(x, c)))$ ;
- (iii)  $(M, I) \models \mathbf{Q}x \in \mathbb{I} \varphi(x, c)$ .

In case (i), we must have  $M \models \forall x \in \llbracket r, s \rrbracket \neg \varphi(x, c)$  because of our choice of  $\llbracket r, s \rrbracket$ . For a similar reason, if (ii) holds, then  $(\max x \in \mathbb{I})(\varphi(x, c)) < r$  and so no  $x \in \llbracket r, s \rrbracket$  can satisfy  $\varphi(x, c)$ . Consider case (iii). By Proposition 13.1, there is  $k \in \mathbb{N}$  such that  $Y_k$  dominates the function

$$x \mapsto (\mu x' > x)(\varphi(x', c))$$

on  $I$ . By our hypothesis on  $\llbracket r, s \rrbracket$ , this domination must start below  $r$ . Therefore, we have  $M \models \forall x \in \llbracket r, s \rrbracket \exists x' \in [x, Y_k(x)] \varphi(x', c)$  for some  $k \in \mathbb{N}$ , as required.  $\square$

**Theorem 15.2.** Let  $(M, I)$  be an existentially closed model of  $\text{PA}_Y$  in which  $M$  is recursively saturated. The following are equivalent.

- (a)  $I$  is a generic cut.
- (b)  $(M, I)$  is  $\exists_1$ -recursively saturated and  $\text{etp}_{(M, I)}(c)$  is coded for all  $c \in M$ .
- (c) There is a lower bound for

$$\{(\min x > I)(\theta(x, c)) : \theta \in \mathcal{L}_A \text{ for which the min exists}\}$$

above  $I$  for every  $c \in M$ .

- (d) There is an upper bound for

$$\{(\max x \in I)(\theta(x, c)) : \theta \in \mathcal{L}_A \text{ for which the max exists}\}$$

in  $I$  for every  $c \in M$ .

- (e) For every  $c \in M$ , there exists  $b > I$  such that for all functions  $F: M \rightarrow M$  definable over  $c$ , we have

$$\forall x \in I F(x) \leq b \quad \Rightarrow \quad \forall x \in I F(x) \in I.$$

- (f) For every  $c \in M$ , there exists  $a \in I$  such that for all functions  $F: M \rightarrow M$  definable over  $c$ , we have

$$\forall x < a \ F(x) \in I \quad \Rightarrow \quad \forall x \in I \ F(x) \in I.$$

Clauses (b)–(f) are true whenever  $(M, I)$  is an  $\forall_1$ -recursively saturated model of  $\text{PA}_Y$ . However, by Corollary 13.2, this amount of saturation is not available in existentially closed cuts.

*Proof.* Corollary 7.2 says (c)  $\Leftrightarrow$  (e) and (d)  $\Leftrightarrow$  (f). So we can ignore (e) and (f).

We first show (c)  $\Rightarrow$  (d). Pick any  $c \in M$ . Using (c), take  $s > I$  that is above every  $(\min x > I)(\theta(x, c))$  where  $\theta \in \mathcal{L}_A$  for which the minimum exists. A further application of (c) to the parameter  $s$  tells us  $I \neq M[s]$ . Let  $r' > I$  which satisfies  $r' \ll s$ . We claim that  $(\max x \in I)(\theta(x, c))$  exists if and only if

$$M \models \forall x \in \llbracket r', s \rrbracket \neg \theta(x, c) \wedge \exists x < s \ \theta(x, c),$$

for every  $\mathcal{L}_A$ -formula  $\theta(x, z)$ . For the ‘only if’ part, suppose  $(\max x \in I)(\theta(x, c))$  exists. Then clearly  $M \models \exists x < s \ \theta(x, c)$ . By overspill, we also know

$$(M, I) \models \exists b > I \ \forall x > I \ (x < b \rightarrow \neg \theta(x, c)).$$

If  $(M, I) \models \forall x > I \ \neg \theta(x, c)$ , then we are done. So it remains to consider the case when  $(\min x > I)(\theta(x, c))$  exists. If this minimum exists, then it is bigger than  $s$  by the choice of  $s$ , and so there cannot be any  $x \in \llbracket r', s \rrbracket$  that satisfies  $\theta(x, c)$ . This is what we want. For the ‘if’ part, suppose  $M \models \forall x \in \llbracket r', s \rrbracket \neg \theta(x, c) \wedge \exists x < s \ \theta(x, c)$ . Notice that since  $s$  is smaller than all the minima, there must be some  $x \in I$  which satisfies  $\theta(x, c)$ . We apply the same trick as in the previous proof: define

$$F(x) = (\mu x' > x)(\theta(x', c)).$$

If  $I$  is not closed under  $F$ , then we are done. So suppose not. Using Proposition 13.1, find  $k \in \mathbb{N}$  such that  $Y_k$  dominates  $F$  on  $I$ . By our choice of  $s$ , we must have  $F(x) \leq Y_k(x)$  for all  $x > I$  below  $s$ . In particular,  $M \models \exists x \in \llbracket r', s \rrbracket \theta(x, c)$ . This is contradictory to our hypothesis. Thus  $I$  cannot be closed under  $F$  after all. This finishes the proof of our claim. It thus follows from recursive saturation that there is  $a \in M$  making

$$\{(a)_n : n \in \mathbb{N}\} = \{(\max x \in I)(\theta(x, c)) : \theta \in \mathcal{L}_A \text{ for which the max exists}\}.$$

Lemma 14.2 says these maxima cannot be cofinal in  $I$ , showing (d).

Similarly, one shows (d)  $\Rightarrow$  (c). Therefore (c)  $\Rightarrow$  (a) and (d)  $\Rightarrow$  (a) by Lemma 15.1. We will prove (a)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (c).

The fact that (a)  $\Rightarrow$  (b) was first proved in Kaye–Wong [7]. We include an alternative proof here using the machinery developed in this paper. Suppose  $I$  is a generic cut. The codedness part is already taken care of by Lemma 5.1. For the recursive saturation part, consider the recursive  $\exists_1$ -type

$$p(w) = \{\exists \llbracket x, y \rrbracket \exists \mathbb{I} \ \varphi_i(w, x, y, c) : i \in \mathbb{N}\},$$

where  $c$  is some element of  $M$ . Take any constant interval  $\llbracket a, b \rrbracket$  over  $c$  around  $I$ . Since  $p$  is finitely satisfied in  $(M, I)$ , we have

$$M \models \exists \llbracket u, v \rrbracket \subseteq \llbracket a, b \rrbracket \left( Y(u, v) > k \wedge \exists w \exists \llbracket x, y \rrbracket \supseteq \llbracket u, v \rrbracket \prod_{i < k} \varphi_i(w, x, y, c) \right),$$

for every  $k \in \mathbb{N}$ . Using overspill, one can find  $\llbracket u, v \rrbracket \subseteq \llbracket a, b \rrbracket$  such that

$$M \models \exists w \exists [x, y] \supseteq \llbracket u, v \rrbracket \bigwedge_{i \in \mathbb{N}} \varphi_i(w, x, y, c).$$

So  $p(w)$  is realized in  $(M, I)$  by Theorem 10.1.

Finally, consider the implication (b)  $\Rightarrow$  (c). Fix  $c \in M$ . It suffices to realize

$$p(u, v) = \{ \exists x \in \mathbb{I} \exists y > \mathbb{I} \varphi(x, y, c) \rightarrow \exists x < u \exists y > v \varphi(x, y, c) : \varphi \in \mathcal{L}_A \} \\ \cup \{ u \in \mathbb{I} < v \}.$$

Since  $\text{etp}_{(M, I)}(c)$  is coded, this  $p(u, v)$  is equivalent to a recursive  $\exists_1$ -type. So we are done by  $\exists_1$ -recursive saturation.  $\square$

Recall from Corollary 8.4 that the existence of generic cuts implies the strength of  $\mathbb{N}$ . Thus if  $M$  is a countable model of PA in which  $\mathbb{N}$  is not strong, and if  $I$  is an existentially closed  $Y$ -cut given by Proposition 13.4, then all of (a)–(f) in Theorem 15.2 must fail for  $I$ .

## Note Added in Proof

The argument on page 154 of J. Hirschfeld and W. H. Wheeler's *Forcing, Arithmetic, Division Rings* (vol. 454 of *Lecture Notes in Mathematics*, Springer, Berlin, 1975) shows that the saturation condition in Theorem 15.2(b) is redundant.

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