Model-theoretic characterizations of ATR_0 and Π^1_1-CA_0

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Abstract

We characterize countable models of ATR_0 and Π^1_1-CA_0 in terms of their conservative end extensions.

1 Introduction and preliminaries

A number of model-theoretic characterizations of WKL_0 and ACA_0 in terms of *conservative end extensions* are known. These are summarized in Theorem 1.2 and Theorem 1.3 below. This paper presents similar characterizations of other subsystems of second-order arithmetic, especially ATR_0 and Π^1_1-CA_0.

As in Simpson [20, §I.2], let L_1 be the language of first-order arithmetic, and let L_2 be the language of second-order arithmetic with equality only for the numerical sort. Thus $X = Y$ is not an atomic formula in L_2 but rather an abbreviation for the Π^0_1 formula $\forall m\ (m \in X \leftrightarrow m \in Y)$. We follow standard notation and write $\mathrm{I}\Sigma_k, \mathrm{I}\Delta_k, \mathrm{I}\Sigma^i_k, \ldots$ for Robinson arithmetic Q extended with the schemes of induction restricted to formulas in $\Sigma_k, \Delta_k, \Sigma^i_k, \ldots$ respectively. We consider only L_2 structures of the form $(M, \mathscr{X})$ where M is an L_1 structure satisfying $\mathrm{I}\Delta_0$ and $\mathscr{X}$ is a nonempty collection of subsets of M.

Definition. Let $(M, \mathscr{X}) \models \mathrm{I}\Sigma^0_0$. An *extension* of $(M, \mathscr{X})$ consists of another L_2 structure $(K, \mathscr{Y})$ together with an embedding $\varepsilon\colon \mathscr{X} \to \mathscr{Y}$ such that M is an L_1 substructure of K and $X = \varepsilon(X) \cap M$ for all $X \in \mathscr{X}$. The extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is said to be *proper* if $M \neq K$. It is an *end extension* if $\forall y{\in}K \setminus M\ \forall x{\in}M\ x \leqslant y$. It is *conservative* if

$$\mathscr{X} = \{Y \cap M : Y \in \mathscr{Y}\}.$$

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Remark 1.1. Mimicking the usual argument for models of Peano arithmetic [8, page 101], one can show that all conservative extensions of a model of $\mathrm{I}\Sigma^0_0$ to a model of Σ^0_0-CA are end extensions.

A combination of results from Tanaka [21], Ressayre [15, 16] and Enayat [3, Theorem 3.6] gives the following. Recall [20, Section X.4] that RCA^*_0 is obtained from RCA_0 by replacing $\mathrm{I}\Sigma^0_1$ with $\mathrm{I}\Sigma^0_0 + \exp$.

Definition. Fix $(M,\mathscr{X}) \models \mathrm{I}\Sigma^0_0$ and an end extension $(K,\mathscr{Y})$ of $(M,\mathscr{X})$ with embedding ε. Let $b \in M$ and $S \in \mathscr{X}$.

- The extension from $(M,\mathscr{X})$ to $(K,\mathscr{Y})$ is Σ^0_1-*elementary over S below b* if for all Σ^0_1 formulas φ and all $\bar{a} < b$,
$$(K,\mathscr{Y}) \models \varphi(\bar{a},\varepsilon(S)) \quad\Leftrightarrow\quad (M,\mathscr{X}) \models \varphi(\bar{a},S).$$
- $(K,\mathscr{Y})$ and $(M,\mathscr{X})$ are 1-*equivalent over S below b* if for every $\bar{u} \in K$, there is $\bar{v} \in M$ such that for all Σ^0_0 formulas φ and all $\bar{a} < b$,
$$(K,\mathscr{Y}) \models \varphi(\bar{a},\bar{u},\varepsilon(S)) \quad\Leftrightarrow\quad (M,\mathscr{X}) \models \varphi(\bar{a},\bar{v},S).$$
- $(K,\mathscr{Y})$ and $(M,\mathscr{X})$ are *isomorphic over S below b* if there is an isomorphism $f\colon (K,\mathscr{Y}) \to (M,\mathscr{X})$ such that $f(a) = a$ for every $a < b$, and $f(\varepsilon(S)) = S$.

Theorem 1.2. Let $(M,\mathscr{X})$ be a countable model of RCA^*_0 in which $M \ncong \mathbb{N}$. Then the following are equivalent.

(i) $(M,\mathscr{X}) \models \mathrm{WKL}_0$.

(ii) $(M,\mathscr{X}) \models \mathrm{I}\Sigma^0_1$ and it has a proper conservative extension that satisfies RCA^*_0.

(iii) For every $b \in M$ and every $S \in \mathscr{X}$, there exists a (recursively saturated) proper conservative extension of $(M,\mathscr{X})$ that satisfies RCA_0 (or equivalently WKL_0) and is Σ^0_1-elementary over S below b.

(iv) For every $b \in M$ and every $S \in \mathscr{X}$, there exists a proper conservative extension of $(M,\mathscr{X})$ that satisfies RCA^*_0 and is 1-equivalent to $(M,\mathscr{X})$ over S below b.

(v) For every $b \in M$ and every $S \in \mathscr{X}$, there exists a proper conservative extension of $(M,\mathscr{X})$ that is isomorphic to $(M,\mathscr{X})$ over S below b.

A similar characterization for ACA_0 can be obtained by combining results from Mac Dowell–Specker [11], Gaifman [6], Phillips [14], Paris–Kirby [12], Kirby [9], and Yokoyama [23].

Definition. Let Γ be a class of L_2 formulas and $(M,\mathscr{X})$ be an L_2 structure. An extension $(K,\mathscr{Y})$ of $(M,\mathscr{X})$ with embedding ε is Γ-*elementary* if for every $\bar{a} \in M$, every $\bar{S} \in \mathscr{X}$, and every $\varphi \in \Gamma$,
$$(M,\mathscr{X}) \models \varphi(\bar{a},\bar{S}) \quad\Leftrightarrow\quad (K,\mathscr{Y}) \models \varphi(\bar{a},\varepsilon(\bar{S})).$$

Theorem 1.3. For a countable $(M,\mathscr{X}) \models \mathrm{RCA}^*_0$, the following are equivalent.

(i) $(M, \mathscr{X}) \models \mathrm{ACA}_0$.

(ii) $(M, \mathscr{X})$ has a proper Σ^0_1-elementary conservative extension that satisfies Σ^0_0-CA.

(iii) $(M, \mathscr{X})$ has a proper Σ^1_1-elementary conservative extension that satisfies ACA_0.

Starting from Schmerl's notion of *exclusive extensions* [17] and a proof of Theorem 1.3, we obtain in Section 2 similar characterizations of countable models of ATR_0 and Π^1_1-CA_0 in terms of conservative extensions; see Corollary 2.5 and Theorem 2.8. As is usually the case [6, 8, 9, 11, 14, ...], these conservative extensions are generated by a new *number* whose properties are carefully chosen to preserve the qualities of the ground model. In Section 4, we explore the potential of generating conservative extensions using a new *set* instead. There we prove more exotic characterizations of countable models of ATR_0 and Π^1_1-CA_0 in terms of what we call *Ramsey extensions*; see Theorem 4.5 and Theorem 4.6. The construction of Ramsey extensions involves a sequential version of the Arithmetical Ramsey Theorem, the reverse-mathematical status of which is determined in Section 3. We hope that our results will lead to better understanding of the model theory of subsystems of second-order arithmetic.

For background in first- and second-order arithmetic, we refer the reader to Kaye [8] and Simpson [20]. We write $\forall^{\mathrm{cf}} X\ \varphi(X)$ for

$$\forall X\ \big(\forall y\ \exists x{\geqslant}y\ (x \in X) \rightarrow \varphi(X)\big).$$

The abbreviation $\exists^{\mathrm{cf}} X$ is defined similarly. We write $X \subseteq_{\mathrm{cf}} Y$ for

$$X \subseteq Y \quad \text{and} \quad \forall y{\in}Y \setminus X\ \exists x{\in}X\ x \geqslant y.$$

If $(M, \mathscr{X})$ is an L_2 structure, then

$$\mathscr{X}^* = \{X \in \mathscr{X} : X \subseteq_{\mathrm{cf}} M\}.$$

Set $H_{>i} = \{x \in H : x > i\}$ and $(S)_i = \{x : \langle i, x\rangle \in S\}$.

Definition. Let $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$.

- Denote by $\mathrm{L}_2(M, \mathscr{X})$ the language obtained from L_2 by adding a new constant symbol of the numerical sort for every element of M, and a new constant symbol of the set sort for every element of $\mathscr{X}$.
- Let Γ be a class of L_2 formulas. A $\Gamma(M, \mathscr{X})$ *formula* is defined to be an $\mathrm{L}_2(M, \mathscr{X})$ formula that can be obtained from a formula in Γ by replacing some (or all) free variables with new constant symbols of the appropriate sorts in $\mathrm{L}_2(M, \mathscr{X})$.

If $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$ and $(K, \mathscr{Y})$ is an extension of $(M, \mathscr{X})$, then we view $(K, \mathscr{Y})$ as an $\mathrm{L}_2(M, \mathscr{X})$ structure in the natural way via the associated embedding $\varepsilon\colon \mathscr{X} \rightarrow \mathscr{Y}$.

2 Exclusive extensions

The notion of *exclusive extensions* was first identified by Schmerl [17, Section 2].

Definition. Let $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ and $(K, \mathscr{Y})$ be an extension of $(M, \mathscr{X})$ with embedding $\varepsilon\colon \mathscr{X} \to \mathscr{Y}$. We say the extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is *exclusive* if

$$\forall Y{\in}\mathscr{Y}\ \exists X{\in}\mathscr{X}\ \exists i{\in}K\ Y = (\varepsilon(X))_i.$$

Ali Enayat pointed out that, as Schmerl implicitly mentioned [17, proof of Corollary 2.9], the proof of the implication (i) $\Rightarrow$ (iii) in Theorem 1.3 by Gaifman [6] and Phillips [14] actually produces an exclusive extension. Here we present a reformulation of the construction which is more natural in the context of second-order arithmetic.

Theorem 2.1 (Schmerl)**.** Every countable $(M, \mathscr{X}) \models \mathrm{ACA}_0$ has a proper Σ^1_1-elementary conservative exclusive extension that satisfies ACA_0.

Proof. The construction below is essentially the same as the one in Kirby [9] except for the second-order part, which his extension lacks. The fact that Kirby's extension affords a natural second-order structure and is Σ^1_1-elementary was pointed out to the authors by Keita Yokoyama. The proof of exclusiveness arose in a discussion with Yokoyama.

The extension $(K, \mathscr{Y})$ will be an ultrapower of $(M, \mathscr{X})$. Take a sufficiently generic filter $\mathscr{U}$ in the partially ordered set $(\mathscr{X}^*, \subseteq)$. This genericity ensures that $\mathscr{U}$ is an ultrafilter in the Boolean algebra $(\mathscr{X}, \subseteq)$. The elements of K are equivalence classes of the form

$$[A] = \{A' \in \mathscr{X} : A' \text{ codes } M \to M \text{ and } \{i \in M : A(i) = A'(i)\} \in \mathscr{U}\},$$

where $A \in \mathscr{X}$ that codes $M \to M$. The elements of $\mathscr{Y}$ are equivalence classes of the form

$$\langle X \rangle = \{X' \in \mathscr{X} : \{i \in M : (X)_i = (X')_i\} \in \mathscr{U}\},$$

where $X \in \mathscr{X}$. To turn $\mathscr{Y}$ into a collection of subsets of K, we may identify each $\langle X \rangle \in \mathscr{Y}$ with

$$\{[A] \in K : \{i \in M : A(i) \in (X)_i\} \in \mathscr{U}\}.$$

Claim 2.1.1 (Łoś(Σ^0_∞))**.** Whenever $\theta(\bar{x}, \bar{Y})$ is an arithmetical formula, $[\bar{A}] \in K$ and $\langle \bar{X} \rangle \in \mathscr{Y}$,

$$(K, \mathscr{Y}) \models \theta([\bar{A}], \langle \bar{X} \rangle) \quad \Leftrightarrow \quad \{i \in M : (M, \mathscr{X}) \models \theta(\bar{A}(i), (\bar{X})_i)\} \in \mathscr{U}.$$

Proof of Claim 2.1.1. The usual induction proof works. The only part that requires slightly more attention is the right-to-left direction for the existential quantifier. So let us see this in more detail. Let $\theta(\bar{x}, \bar{Y}, z)$ be an arithmetical formula such that $U = \{i \in M : (M, \mathscr{X}) \models \exists z\ \theta(\bar{A}(i), (\bar{X})_i, z)\} \in \mathscr{U}$. Define $F\colon M \to M$ by setting, for each $i \in M$,

$$F(i) = \begin{cases} \min\{z \in M : (M, \mathscr{X}) \models \theta(\bar{A}(i), (\bar{X})_i, z)\}, & \text{if } i \in U; \\ 0, & \text{otherwise.} \end{cases}$$

This function is well-defined and is in $\mathscr{X}$ by arithmetical comprehension in $(M, \mathscr{X})$. Since $\{i \in M : (M, \mathscr{X}) \models \theta(\bar{A}(i), (\bar{X})_i, F(i))\} \supseteq U \in \mathscr{U}$, the induction hypothesis implies that $(K, \mathscr{Y}) \models \theta([\bar{A}], \langle \bar{X} \rangle, [F])$. □ Claim 2.1.1

Via the usual canonical embeddings, we can identify $(M, \mathscr{X})$ with a substructure of $(K, \mathscr{Y})$. In particular, the embedding $\varepsilon\colon \mathscr{X} \to \mathscr{Y}$ sends each $X \in \mathscr{X}$ to $\langle \tilde{X} \rangle \in \mathscr{Y}$ where $\tilde{X} = M \times X = \{\langle i, x\rangle : i \in M \text{ and } x \in X\}$.

Claim 2.1.2. The extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is Σ^1_1-elementary.

Proof of Claim 2.1.2. Fix an arithmetical formula $\theta(x, Y, Z)$. Let $a \in M$ and $X \in \mathscr{X}$ such that $(M, \mathscr{X}) \models \forall Z\ \theta(a, X, Z)$. Our aim is to show $(K, \mathscr{Y}) \models \forall Z\ \theta([A], \langle \tilde{X} \rangle, Z)$, where $A \in \mathscr{X}$ which codes the function $M \to M$ with constant value a. So pick any $\langle S \rangle \in \mathscr{Y}$. Since $(M, \mathscr{X}) \models \forall Z\ \theta(a, X, Z)$,

$$\begin{aligned} &\{i \in M : (M, \mathscr{X}) \models \theta(A(i), (\tilde{X})_i, (S)_i)\} \\ &= \{i \in M : (M, \mathscr{X}) \models \theta(a, X, (S)_i)\} = M \in \mathscr{U}. \end{aligned}$$

Hence $(K, \mathscr{Y}) \models \theta([A], \langle \tilde{X} \rangle, \langle S \rangle)$ by Łoś(Σ^0_∞), as required. □ Claim 2.1.2

Let D denote the identity function $M \to M$. Notice $[D] > a$ for every $a \in M$. So $K \neq M$. Observe that if $[A] \in K$, then

$$\{i \in M : A(i) = (\tilde{A})_i(D(i))\} = \{i \in M : A(i) = A(i)\} = M \in \mathscr{U},$$

and hence $[A] = \langle \tilde{A} \rangle([D])$ by Łoś(Σ^0_∞). Therefore,

$$K = \{\varepsilon(A)([D]) : A \in \mathscr{X} \text{ that codes } M \to M\}.$$

Similarly, if $\langle X \rangle \in \mathscr{Y}$, then

$$\{i \in M : (X)_i = ((\tilde{X})_i)_{D(i)}\} = \{i \in M : (X)_i = (X)_i\} = M \in \mathscr{U}.$$

By Łoś(Σ^0_∞), this implies $\langle X \rangle = (\langle \tilde{X} \rangle)_{[D]}$ for every $X \in \mathscr{X}$. Thus

$$\mathscr{Y} = \{(\varepsilon(\tilde{X}))_{[D]} : X \in \mathscr{X}\}.$$

In particular, the extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is exclusive.

Claim 2.1.3. $(K, \mathscr{Y}) \models \mathrm{ACA}_0$.

Proof of Claim 2.1.3. In view of the paragraph above, it suffices to consider sets of the form

$$\{x \in K : (K, \mathscr{Y}) \models \theta([D], x)\},$$

where $\theta \in \Sigma^0_\infty(M, \mathscr{X})$. Using arithmetical comprehension in $(M, \mathscr{X})$, let

$$X = \{\langle i, x\rangle \in M : (M, \mathscr{X}) \models \theta(i, x)\} \in \mathscr{X}.$$

Now $\big\{i \in M : (M, \mathscr{X}) \models \forall x\ \big(x \in (X)_i \leftrightarrow \theta(D(i), x)\big)\big\} = M \in \mathscr{U}$. So Łoś$(\Sigma^0_\infty)$ tells us that $(K, \mathscr{Y}) \models \forall x\ \big(x \in \langle X \rangle \leftrightarrow \theta([D], x)\big)$. □ Claim 2.1.3

Claim 2.1.4. The extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is conservative.

Proof of Claim 2.1.4. Take any $\langle X\rangle \in \mathscr{Y}$. Notice $(M, \mathscr{X}) \models \mathrm{COH}$, i.e.,

$$\begin{aligned}(M, \mathscr{X}) \models \forall R\ \forall^{\mathrm{cf}} S\ \exists^{\mathrm{cf}} H{\subseteq}S\ \forall j\ \exists b \\ \big(\forall i{>}b\ (i \in H \to i \in (R)_j) \vee \forall i{>}b\ (i \in H \to i \notin (R)_j)\big).\end{aligned}$$

In particular, we can apply COH to $R = X^{\mathrm{T}} = \{\langle j, i\rangle : \langle i, j\rangle \in X\}$. Since $\mathscr{U}$ is sufficiently generic, this gives us $H \in \mathscr{U}$ such that

$$(M, \mathscr{X}) \models \forall j\ \exists b\ \big(\forall i{>}b\ (i \in H \to j \in (X)_i) \vee \forall i{>}b\ (i \in H \to j \notin (X)_i)\big).$$

Recall $\mathscr{U} \subseteq \mathscr{X}^*$. So by Łoś$(\Sigma^0_\infty)$,

$$\begin{aligned}\langle X\rangle \cap M &= \{j \in M : \{i \in M : j \in (X)_i\} \in \mathscr{U}\} \\ &= \{j \in M : (M, \mathscr{X}) \models \exists b\ \forall i{>}b\ (i \in H \to j \in (X)_i)\}.\end{aligned}$$

Arithmetical comprehension then implies $\langle X\rangle \cap M \in \mathscr{X}$. □ Claim 2.1.4

This concludes the proof of Theorem 2.1. □

Many properties are preserved under exclusive extensions if the ground model satisfies some form of choice. Schmerl [17] called the choice axioms he isolated for this purpose *collection axioms*, although they are *not* the same as the usual collection axioms in first-order arithmetic [8, Chapter 7].

Definition. Let Γ be a class of L_2 formulas. The formula $\theta(x, Y)$ below may contain undisplayed number or set variables different from y and S, in which case the formulas displayed should be interpreted as their universal closures.

- Γ-AC denotes the scheme consisting of all sentences of the form
$$\forall x\ \exists Y\ \theta(x, Y) \to \exists S\ \forall x\ \theta(x, (S)_x),$$
where $\theta \in \Gamma$.
- Γ-Coll denotes the scheme consisting of all sentences of the form
$$\forall x\ \exists Y\ \theta(x, Y) \to \exists S\ \forall x\ \exists y\ \theta(x, (S)_y),$$
where $\theta \in \Gamma$.
- $\mathrm{AC} = \Sigma^1_\infty$-AC and $\mathrm{CA} = \Pi^1_\infty$-$\mathrm{CA}_0$.

There is a level-by-level correspondence between Schmerl's collection axioms and the usual choice axioms; cf. Lemma 1.1 in Schmerl [17]. Keita Yokoyama informed the authors that this correspondence was independently discovered by Takeshi Yamazaki [unpublished].

Lemma 2.2. Let $k \in \mathbb{N}$.

(1) Both Σ^1_{k+1} and Π^1_{k+1} are closed under number quantification over $\mathrm{RCA}^*_0 + \Sigma^1_{k+1}$-Coll.

(2) $\mathrm{ACA}_0 + \Sigma^1_{k+1}\text{-Coll} \vdash \Delta^1_{k+1}\text{-CA}_0$.

(3) Σ^1_{k+1}-Coll and Σ^1_{k+1}-AC are equivalent over ACA_0.

Proof. (1) Proceed by induction on k as usual.

(2) We proceed by strong induction on k. Take $(M, \mathscr{X}) \models \mathrm{ACA}_0 + \Sigma^1_{k+1}$-Coll. Let $\theta, \eta \in \Pi^1_k(M, \mathscr{X})$ such that $(M, \mathscr{X}) \models \forall x\ \big(\exists Y\ \theta(x, Y) \leftrightarrow \forall Z\ \neg\eta(x, Z)\big)$. Then
$$(M, \mathscr{X}) \models \forall x\ \exists Y\ \big(\forall Z\ \neg\eta(x, Z) \to \theta(x, Y)\big).$$
Apply Σ^1_{k+1}-Coll to find $S \in \mathscr{X}$ such that
$$(M, \mathscr{X}) \models \forall x\ \exists y\ \big(\forall Z\ \neg\eta(x, Z) \to \theta(x, (S)_y)\big).$$
Notice
$$\{x \in M : (M, \mathscr{X}) \models \exists Y\ \theta(x, Y)\} = \{x \in M : (M, \mathscr{X}) \models \exists y\ \theta(x, (S)_y)\}.$$
If $k = 0$, then this set is in $\mathscr{X}$ by ACA_0. By part (1) and the induction hypothesis, the same holds if $k \geqslant 1$.

(3) It is clear that $\mathrm{RCA}^*_0 + \Sigma^1_{k+1}$-AC $\vdash \Sigma^1_{k+1}$-Coll. Conversely, suppose $(M, \mathscr{X}) \models \mathrm{ACA}_0 + \Sigma^1_{k+1}$-Coll. Take $\theta(x, Y) \in \Pi^1_k(M, \mathscr{X})$ and assume $(M, \mathscr{X}) \models \forall x\ \exists Y\ \theta(x, Y)$. Apply Π^1_k-Coll to find $S \in \mathscr{X}$ such that $(M, \mathscr{X}) \models \forall x\ \exists y\ \theta(x, (S)_y)$. Define
$$F = \{\langle x, y\rangle \in M : (M, \mathscr{X}) \models \theta(x, (S)_y) \wedge \forall y'{<}y\ \neg\theta(x, (S)_{y'})\}.$$
Then $F \in \mathscr{X}$ by the last two parts. Notice that (2) implies $(M, \mathscr{X}) \models \mathrm{I}\Sigma^1_k$. So F actually codes a function $M \to M$. It follows that $(M, \mathscr{X}) \models \forall x\ \theta(x, (S')_x)$ where $S' = \{\langle x, v\rangle : v \in (S)_{F(x)}\}$. □

Schmerl originally formulated the following theorem in terms of collection axioms. In view of the lemma above, we can rewrite it using choice axioms.

Theorem 2.3 (Schmerl)**.** Let $k \in \mathbb{N}$ and $(M, \mathscr{X}) \models \mathrm{RCA}^*_0 + \Sigma^1_{k+1}$-AC. Whenever $(K, \mathscr{Y})$ is a Σ^0_∞-elementary exclusive extension of $(M, \mathscr{X})$ that satisfies RCA^*_0,

(1) the extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is actually Σ^1_{k+2}-elementary; and

(2) if $(M, \mathscr{X}) \models \Pi^1_{k+1}$-$\mathrm{CA}_0$, then $(K, \mathscr{Y}) \models \Pi^1_{k+1}$-$\mathrm{CA}_0 + \Sigma^1_{k+1}$-AC.

Proof. See Theorems 2.5–2.7 in Schmerl [17]. □

The theorem above readily gives a characterization of countable models of Π^1_{k+1}-$\mathrm{CA}_0 + \Sigma^1_{k+1}$-AC.

Theorem 2.4. For all $k \in \mathbb{N}$ and all countable $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$, the following are equivalent.

(i) $(M, \mathscr{X}) \models \Pi^1_{k+1}$-$\mathrm{CA}_0 + \Sigma^1_{k+1}$-AC.

(ii) $(M, \mathscr{X})$ has a proper Σ^1_{k+1}-elementary conservative exclusive extension that satisfies $\mathrm{RCA}^*_0 + \mathrm{I}\Sigma^1_{k+1}$.

(iii) $(M, \mathscr{X})$ has a proper Σ^1_{k+2}-elementary conservative exclusive extension that satisfies Π^1_{k+1}-$\mathrm{CA}_0 + \Sigma^1_{k+1}$-AC.

Proof. The implication (iii) $\Rightarrow$ (ii) is clear.

Let us show (ii) $\Rightarrow$ (i). Suppose we are given an extension $(K,\mathscr{Y})$ of $(M,\mathscr{X})$ with the properties listed in (ii). We first prove $(M,\mathscr{X}) \models \Pi^1_{k+1}\text{-CA}_0$. Let $\theta(x) \in \Pi^1_{k+1}(M,\mathscr{X})$ and $b \in K \setminus M$. Set

$$Y = \{x \leqslant b : (K,\mathscr{Y}) \models \theta(x)\}.$$

Then $Y \in \mathscr{Y}$ because $\text{RCA}^*_0 + \text{I}\Sigma^1_{k+1}$ proves bounded Π^1_{k+1} comprehension; cf. Lemma 2.1 of Simpson [19]. By the Π^1_{k+1} elementarity and the conservativity of the extension $(K,\mathscr{Y})$,

$$\{x \in M : (M,\mathscr{X}) \models \theta(x)\} = \{x \in M : (K,\mathscr{Y}) \models \theta(x)\} = Y \cap M \in \mathscr{X}.$$

Notice exclusiveness has not been invoked so far.

Next, we prove $(M,\mathscr{X}) \models \Sigma^1_{k+1}\text{-AC}$. In view of Lemma 2.2, it suffices to show $(M,\mathscr{X}) \models \Sigma^1_{k+1}\text{-Coll}$. Let $\theta(x,Y) \in \Pi^1_k(M,\mathscr{X})$ such that $(M,\mathscr{X}) \models \forall x\ \exists Y\ \theta(x,Y)$. Then for every $b \in M$,

$$(K,\mathscr{Y}) \models \forall x{\leqslant}b\ \exists Y\ \theta(x,Y)$$

by Π^1_k elementarity. This implies, for all $b \in M$,

$$(K,\mathscr{Y}) \models \exists S\ \forall x{\leqslant}b\ \theta(x,(S)_x)$$

because $\text{RCA}^*_0{+}\text{I}\Sigma^1_{k+1}$ proves bounded Σ^1_{k+1} choice; cf. Lemma 2.1 of Simpson [19]. Apply Σ^1_{k+1} overspill to find $b \in K \setminus M$ and $S \in \mathscr{Y}$ such that $(K,\mathscr{Y}) \models \forall x{\leqslant}b\ \theta(x,(S)_x)$. Recall $(K,\mathscr{Y})$ is an exclusive extension of $(M,\mathscr{X})$. Let $S_0 \in \mathscr{X}$ such that $S = (\varepsilon(S_0))_y$ for some $y \in K$. Then

$$(K,\mathscr{Y}) \models \forall x{\leqslant}b\ \exists y\ \theta\big(x,((\varepsilon(S_0))_y)_x\big).$$

Notice $b > a$ for all $a \in M$ by conservativity. Thus Σ^1_{k+1} elementarity implies $(M,\mathscr{X}) \models \forall x\ \exists y\ \theta\big(x,((S_0)_y)_x\big)$. It follows that $(M,\mathscr{X}) \models \forall x\ \exists w\ \theta(x,(S'_0)_w)$ where $S'_0 = \{\langle\langle x,y\rangle,z\rangle : \langle x,\langle y,z\rangle\rangle \in S_0\}$.

Finally, consider (i) $\Rightarrow$ (iii). Assume $(M,\mathscr{X}) \models \Pi^1_{k+1}\text{-CA}_0 + \Sigma^1_{k+1}\text{-AC}$. Apply Theorem 2.1 to find a proper Σ^1_1-elementary conservative exclusive extension of $(M,\mathscr{X})$ that satisfies ACA_0. Then use Theorem 2.3. ☐

We have a slightly cleaner statement at the bottom levels because if $k < 2$, then $\Pi^1_{k+1}\text{-CA}_0 \vdash \Sigma^1_{k+1}\text{-AC}$ [20, Theorem V.8.3 and Theorem VII.6.9].

Corollary 2.5. For all $k < 2$ and all countable $(M,\mathscr{X}) \models \text{RCA}^*_0$, the following are equivalent.

(i) $(M,\mathscr{X}) \models \Pi^1_{k+1}\text{-CA}_0$.

(ii) $(M,\mathscr{X})$ has a proper Σ^1_{k+1}-elementary conservative extension that satisfies $\text{RCA}^*_0 + \text{I}\Sigma^1_{k+1}$.

(iii) $(M,\mathscr{X})$ has a proper Σ^1_{k+2}-elementary conservative exclusive extension that satisfies $\Pi^1_{k+1}\text{-CA}_0$. ☐

For the sake of completeness, let us also include the very top level. The proof is the same. It is a simple exercise to show $\text{RCA}^*_0 + \text{AC} \vdash \text{CA}$ directly [20, Lemma VII.6.6(1)].

Theorem 2.6. For a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$, the following are equivalent.

(i) $(M, \mathscr{X}) \models \mathrm{AC}$.

(ii) $(M, \mathscr{X})$ has a proper elementary conservative exclusive extension that satisfies $\mathrm{I}\Sigma^1_\infty$. □

In a similar way, one obtains an analogous characterization for ATR_0. The key is the following easy lemma. It would be interesting to see what other natural subsystems of second-order arithmetic are preserved in such exclusive extensions.

Lemma 2.7. Let $(M, \mathscr{X}) \models \mathrm{ATR}_0$. If $(K, \mathscr{Y})$ is an exclusive extension of $(M, \mathscr{X})$ that is Σ^0_∞-elementary and satisfies RCA_0^*, then it is actually Σ^1_2-elementary and satisfies ATR_0.

Proof. Recall [20, Theorem V.8.3] that $\mathrm{ATR}_0 \vdash \mathrm{RCA}_0^* + \Sigma^1_1\text{-AC}$. So $(K, \mathscr{Y})$ is a Σ^1_2-elementary extension of $(M, \mathscr{X})$ by Theorem 2.3. Since ATR_0 is Π^1_2-axiomatizable over RCA_0^*, this implies that $(K, \mathscr{Y}) \models \mathrm{ATR}_0$. □

One may argue that Theorem 2.8 does not qualify as a characterization of countable models of ATR_0 because both (ii) and (iii) below mention ATR_0 in their statements. Nevertheless, the theorem does provide useful information. In addition, with a bit of extra effort, one can actually weaken the ATR_0 in (ii) and (iii) to a bounded version of Σ^1_1 separation; cf. Simpson [20, Theorem V.5.1].

Theorem 2.8. For a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$, the following are equivalent.

(i) $(M, \mathscr{X}) \models \mathrm{ATR}_0$.

(ii) $(M, \mathscr{X})$ has a Σ^1_1-elementary extension that satisfies ATR_0.

(iii) $(M, \mathscr{X})$ has a proper Σ^1_2-elementary conservative exclusive extension that satisfies ATR_0.

Proof. It is clear that (iii) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) holds because ATR_0 is Π^1_2-axiomatizable over RCA_0^*. For (i) $\Rightarrow$ (iii), combine Theorem 2.1 and Lemma 2.7. □

3 The Arithmetical Ramsey Theorem

The combinatorial theorems that we use for our constructions in Section 4 are variants of the Galvin–Prikry Theorem [7] on colourings of infinite subsets of $\mathbb{N}$. They can be seen as generalizations of Ramsey's Theorem.

Definition. Let Γ be a class of L_2 formulas. The formulas θ, η, ξ and ξ' below may contain undisplayed number or set variables different from b and H, in which case the formulas displayed should be interpreted as their universal closures.

- Γ-RT denotes the scheme consisting of all sentences of the form

  $$\exists^{\mathrm{cf}} H\ \bigl(\forall^{\mathrm{cf}} X{\subseteq}H\ \xi(X) \vee \forall^{\mathrm{cf}} X{\subseteq}H\ \neg\xi(X)\bigr),$$

  where $\xi \in \Gamma$.

- For $k \in \mathbb{N}$, we let Δ^0_k-RT denote the scheme consisting of all sentences of the form
$$\forall^{\mathrm{cf}} X \ \big(\xi(X) \leftrightarrow \xi'(X)\big) \rightarrow \exists^{\mathrm{cf}} H \ \big(\forall^{\mathrm{cf}} X{\subseteq}H \ \xi(X) \vee \forall^{\mathrm{cf}} X{\subseteq}H \ \neg\xi(X)\big),$$
where $\xi \in \Sigma^0_k$ and $\xi' \in \Pi^0_k$.

- Γ-$\mathrm{RT}_{<\infty}$ denotes the scheme consisting of all sentences of the form
$$\forall b \ \big(\forall^{\mathrm{cf}} X \ \exists v{<}b \ \theta(X,v) \rightarrow \exists^{\mathrm{cf}} H \ \exists v{<}b \ \forall^{\mathrm{cf}} X{\subseteq}H \ \theta(X,v)\big),$$
where $\theta \in \Gamma$.

- Γ-$\widetilde{\mathrm{RT}}$ denotes the scheme consisting of all sentences of the form
$$\exists^{\mathrm{cf}} H \ \forall i \ \big(\forall^{\mathrm{cf}} X{\subseteq}H_{>i} \ \eta(i,X) \vee \forall^{\mathrm{cf}} X{\subseteq}H_{>i} \ \neg\eta(i,X)\big),$$
where $\eta \in \Gamma$.

- For $k \in \mathbb{N}$, we let Δ^0_k-$\widetilde{\mathrm{RT}}$ denote the scheme consisting of all sentences of the form
$$\begin{aligned}&\forall i \ \forall^{\mathrm{cf}} X \ \big(\eta(i,X) \leftrightarrow \eta'(i,X)\big)\\ &\rightarrow \exists^{\mathrm{cf}} H \ \forall i \ \big(\forall^{\mathrm{cf}} X{\subseteq}H_{>i} \ \eta(i,X) \vee \forall^{\mathrm{cf}} X{\subseteq}H_{>i} \ \neg\eta(i,X)\big),\end{aligned}$$
where $\eta \in \Sigma^0_k$ and $\eta' \in \Pi^0_k$.

- Define ART, $\mathrm{ART}_{<\infty}$, and $\widetilde{\mathrm{ART}}$ to be Σ^0_∞-RT, Σ^0_∞-$\mathrm{RT}_{<\infty}$, and Σ^0_∞-$\widetilde{\mathrm{RT}}$ respectively.

Remark 3.1. The RT principles defined above are usually formulated in terms of strictly increasing functions instead of cofinal sets in the literature. Over RCA_0, one can go back and forth between the two using Δ^0_1-definable transformations:

- if X is a cofinal set, then
$$\mathrm{Enum}(X) = \{\langle x,y\rangle : \text{the } x\text{th element of } X \text{ is } y\}$$
codes a strictly increasing function;

- if F is a strictly increasing function, then
$$\mathrm{Im}^*(F) = \{y : \exists x{\leqslant}y \ F(x) = y\}$$
is a cofinal set;

- the transformations Enum and Im^* are inverse to each other; and

- for all cofinal sets X_1, X_2,
$$\begin{aligned}&X_1 \subseteq X_2\\ \leftrightarrow \quad &\exists \text{ strictly increasing function } F \ \big(\mathrm{Enum}(X_1) = \mathrm{Enum}(X_2) \circ F\big).\end{aligned}$$

We choose the formulation in terms of cofinal sets because it makes the construction in Section 4 more intuitive. On the other hand, the formulation in terms of strictly increasing functions is helpful in making the homogeneous set H a subset of some prescribed cofinal set S. To achieve this, instead of applying the relevant RT principle to the original formula $\xi(X)$, one applies it to

$$\xi^S(X) \quad = \quad \xi\big(\mathrm{Im}^*(\mathrm{Enum}(S) \circ \mathrm{Enum}(X))\big).$$

Notice that, unless $\xi \in \Sigma^0_0$, the formula ξ^S is equivalent to one in the same position of the arithmetical hierarchy as ξ over $\mathrm{B}\Sigma^0_1$.

These RT principles are rather well understood reverse-mathematically.

Theorem 3.2 (Friedman–Mc Aloon–Simpson)**.** The following are pairwise equivalent over RCA_0.

(i) ATR_0.

(ii) Δ^0_1-RT.

(iii) Σ^0_1-RT.

(iv) Σ^0_1-$\mathrm{RT}_{<\infty}$.

(v) Π^0_1-$\mathrm{RT}_{<\infty}$.

(vi) Δ^0_1-$\widetilde{\mathrm{RT}}$.

Proof. The equivalence between (i), (ii) and (iii) is Theorem 3.2 in Friedman–Mc Aloon–Simpson [5]; see also Section V.9 of Simpson's book [20]. The implications from (i) to (iv), (v) and (vi) follow respectively from Theorem 3.8, Corollary 3.11, and Corollary 3.12 in the same paper. It is straightforward to deduce (ii) from (iv), (v) or (vi). □

Theorem 3.3 (Shelah, Simpson, Solovay)**.** The following are pairwise equivalent over RCA_0.

(i) Π^1_1-CA_0.

(ii) Δ^0_2-RT.

(iii) ART.

(iv) Σ^0_1-$\widetilde{\mathrm{RT}}$.

Proof. The equivalence between (i), (ii) and (iii) is proved in Section VI.6 of Simpson [20]. For the equivalence of (i) and (iv), see Theorem 3.4 in Simpson [19]. □

The rest of this section is devoted to a proof of the following theorem.

Theorem 3.4. Π^1_1-$\mathrm{CA}_0 \vdash \widetilde{\mathrm{ART}}$.

Our proof uses two facts which are both due to Friedman [4] essentially; see also Theorem VII.2.7 and Theorem VII.2.10 of Simpson [20].

Theorem 3.5. In a model of ACA_0, any coded β-model satisfies ATR_0.

Theorem 3.6. Π^1_1-CA_0 is equivalent over RCA_0 to the statement 'for every X, there is a coded β-model containing X'.

We split our proof of Theorem 3.4 into two lemmas, both of which appear implicitly in the proof of Theorem VI.6.2 in Simpson [20]. The first lemma can be seen as a refinement of Theorem 3.2(iii) in the sense that it tells us where the homogeneous set is.

Lemma 3.7. Fix $(M,\mathscr{X}) \models \mathrm{ACA}_0$. Let $(M,\mathscr{B})$ be a coded β-model in $(M,\mathscr{X})$ and $\eta(i,X) \in \Sigma^0_1(M,\mathscr{B})$. Then for every $m \in M$, there is $A \in \mathscr{B}^*$ such that

$$(M,\mathscr{X}) \models \forall i{<}m\ \big(\forall^{\mathrm{cf}} X{\subseteq}A\ \eta(i,X) \vee \forall^{\mathrm{cf}} X{\subseteq}A\ \neg\eta(i,X)\big).$$

Proof. Theorem 3.2 and Theorem 3.5 imply

$$(M,\mathscr{B}) \models \forall i\ \forall^{\mathrm{cf}} H\ \exists^{\mathrm{cf}} A{\subseteq}H\ \big(\forall^{\mathrm{cf}} X{\subseteq}A\ \eta(i,X) \vee \forall^{\mathrm{cf}} X{\subseteq}A\ \neg\eta(i,X)\big).$$

Starting with $A_0 = M \in \mathscr{B}$, we find $A_{m+1} \subseteq_{\mathrm{cf}} A_m$ in $\mathscr{B}$ satisfying

$$(M,\mathscr{B}) \models \forall^{\mathrm{cf}} X{\subseteq}A_{m+1}\ \eta(m,X) \vee \forall^{\mathrm{cf}} X{\subseteq}A_{m+1}\ \neg\eta(m,X)$$

for each $m \in M$. Given such a sequence $(A_m)_{m\in M}$, we are done, because

$$(M,\mathscr{B}) \models \forall m\ \forall i{<}m\ \big(\forall^{\mathrm{cf}} X{\subseteq}A_m\ \eta(i,X) \vee \forall^{\mathrm{cf}} X{\subseteq}A_m\ \neg\eta(i,X)\big),$$

and the homogeneity can be transferred to $(M,\mathscr{X})$ by β-ness.

The recursive construction of (A_m) is carried out in $(M,\mathscr{X})$. As $\mathscr{B}$ is coded, all set quantifiers in $(M,\mathscr{B})$ become number quantifiers in $(M,\mathscr{X})$. So, the whole recursion is an arithmetical recursion on numbers from the point of view of $(M,\mathscr{X})$. Such recursion is available in all models of ACA_0. □

The second lemma says that one can, by restricting oneself to a cofinal subset, make every Σ^0_1 formula Π^0_1.

Lemma 3.8. Let $(M,\mathscr{X}) \models \Pi^1_1$-$\mathrm{CA}_0$. Then for every $\eta(i,X) \in \Sigma^0_1(M,\mathscr{X})$, there exist $S \in \mathscr{X}^*$ and a Π^0_1 formula $\eta'(i,X)$ with further parameters from $(M,\mathscr{X})$ such that

$$(M,\mathscr{X}) \models \forall i\ \forall^{\mathrm{cf}} X{\subseteq}S\ \big(\eta(i,X) \leftrightarrow \eta'(i,X)\big).$$

Proof. Fix $\eta(i,X) \in \Sigma^0_1(M,\mathscr{X})$. Using Theorem 3.6, find a coded β-model $(M,\mathscr{B})$ in $(M,\mathscr{X})$ containing all the parameters that appear in η. We will use arithmetical recursion in $(M,\mathscr{X})$ to define an M-sequence of Mathias conditions

$$\langle s_0, A_0\rangle \geqslant \langle s_1, A_1\rangle \geqslant \langle s_2, A_2\rangle \geqslant \cdots$$

in $\mathscr{B}$. In other words, for each $n \in M$,

- s_n is an M-finite set and $A_n \in \mathscr{B}^*$ such that $\max s_n < \min A_n$;
- $s_n \subseteq s_{n+1}$ with $s_{n+1} \setminus s_n \subseteq A_n$, and $A_n \supseteq A_{n+1}$.

Set $\langle s_0, A_0\rangle = \langle \varnothing, M\rangle$. Suppose $\langle s_0, A_0\rangle \geqslant \langle s_1, A_1\rangle \geqslant \cdots \geqslant \langle s_{2m}, A_{2m}\rangle$ are already found. Set $s_{2m+1} = s_{2m} \cup \{\min A_{2m}\}$ and $A_{2m+1} = A_{2m} \setminus \{\min A_{2m}\}$. Define $s_{2m+2} = s_{2m+1}$ and find $A_{2m+2} \subseteq_{\mathrm{cf}} A_{2m+1}$ in $\mathscr{B}$ such that

$$(M, \mathscr{X}) \models \forall i{<}2m+2\ \forall t{\subseteq}s_{2m+2} \begin{pmatrix} \forall^{\mathrm{cf}} X{\subseteq}A_{2m+2}\ \eta(i, t \cup X) \\ \vee \forall^{\mathrm{cf}} X{\subseteq}A_{2m+2}\ \neg\eta(i, t \cup X) \end{pmatrix},$$

which is possible by Lemma 3.7. At the end of the recursion, set $S = \bigcup_{n \in M} s_n$.

Define, for all $i, m, t \in M$,

$$P(i, m, t) = \begin{cases} 1, & \text{if } (M, \mathscr{X}) \models \eta(i, t \cup A_{2m+2}); \\ 0, & \text{otherwise.} \end{cases}$$

Arithmetical comprehension tells us $P \in \mathscr{X}$. The following is a procedure that decides in $(M, \mathscr{X})$ whether a given $i \in M$ and a given $X \subseteq_{\mathrm{cf}} S$ in $\mathscr{X}$ satisfy $\eta(i, X)$.

1. Find $m \in M$ such that $i < 2m + 2$.
2. Set $t = s_{2m+2} \cap X$.
3. Return 'true' if $P(i, m, t) = 1$, and return 'false' otherwise.

This procedure is recursive in (s_n) and P. In particular, it is represented by a Π^0_1 formula $\eta'(i, X)$. $\square$

As with model completeness, Lemma 3.8 enables us to view every arithmetical formula as Σ^0_1. This reduces Theorem 3.4 to Theorem 3.3.

Proof of Theorem 3.4. Let $(M, \mathscr{X}) \models \Pi^1_1\text{-CA}_0$. In view of Remark 3.1, one can require the set S given by Lemma 3.8 to be a subset of any prescribed element of $\mathscr{X}^*$. A simple induction on k then tells us for all $k \in \mathbb{N}$ and all $\eta(i, X) \in \Sigma^0_{k+1}(M, \mathscr{X})$, there exist $S \in \mathscr{X}^*$ and $\eta'(i, X) \in \Sigma^0_1(M, \mathscr{X})$ such that

$$(M, \mathscr{X}) \models \forall i\ \forall^{\mathrm{cf}} X{\subseteq}S\ \big(\eta(i, X) \leftrightarrow \eta'(i, X)\big).$$

Hence $(M, \mathscr{X}) \models \widetilde{\mathrm{ART}}$ because $\Pi^1_1\text{-CA}_0 \vdash \Sigma^0_1\text{-}\widetilde{\mathrm{RT}}$ by Theorem 3.3. $\square$

Corollary 3.9. The following are pairwise equivalent over RCA_0.

(i) $\Pi^1_1\text{-CA}_0$.

(ii) $\mathrm{ART}_{<\infty}$.

(iii) $\widetilde{\mathrm{ART}}$. $\square$

4 Ramsey extensions

In this section, we will develop in detail a general method for constructing Ramsey extensions. The characterizations of ATR_0 and $\Pi^1_1\text{-CA}_0$ in terms of such extensions will only come at the end.

Definition. Fix $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ and an extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ with embedding $\varepsilon\colon \mathscr{X} \to \mathscr{Y}$. Recall that $(K, \mathscr{Y})$ can naturally be viewed as an $\mathrm{L}_2(M, \mathscr{X})$ structure in this case. Let Γ be a class of L_2 formulas. All the free variables in ξ and η below are shown.

- If $G \in \mathscr{Y}$, then $\mathrm{Filt}_{\mathscr{X}}(G) = \{S \in \mathscr{X}^* : G \subseteq \varepsilon(S)\}$.
- The extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is Γ-*Ramsey* if there is $G \in \mathscr{Y}$ such that for all $\xi(X) \in \Gamma(M, \mathscr{X})$,
$$(K, \mathscr{Y}) \models \xi(G) \quad \Leftrightarrow \quad \exists H {\in} \mathrm{Filt}_{\mathscr{X}}(G)\ (M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H\ \xi(X).$$
The set G here is said to be a *witness to* Γ-*Ramseyness.*
- The extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is *uniformly* Γ-*Ramsey* if there is $G \in \mathscr{Y}$ such that for all $\eta(i, X) \in \Gamma(M, \mathscr{X})$,
$$\begin{aligned}&\exists H {\in} \mathrm{Filt}_{\mathscr{X}}(G)\ \forall i {\in} M \\ &\quad \big(\, (K, \mathscr{Y}) \models \eta(i, G) \quad \Leftrightarrow \quad (M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H_{>i}\ \eta(i, X)\, \big).\end{aligned}$$
The set G here is said to be a *witness to uniform* Γ-*Ramseyness.*

We list a few basic facts about Ramsey and uniformly Ramsey extensions in the following lemma.

Lemma 4.1. Fix $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ and an extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ with embedding $\varepsilon\colon \mathscr{X} \to \mathscr{Y}$. Let Γ be a class of L_2 formulas.

(1) If $(K, \mathscr{Y})$ is a Σ_0^0-Ramsey extension of $(M, \mathscr{X})$ with witness G, then
   (a) $\mathrm{Filt}_{\mathscr{X}}(G) \neq \varnothing$;
   (b) $\{v \in G : v < b \text{ for some } b \in M\} = \varnothing$; and
   (c) $G \neq \varepsilon(S)$ for any nonempty $S \in \mathscr{X}$.

(2) Suppose $\Gamma \supseteq \Sigma_0^0$. If $(K, \mathscr{Y})$ is a (uniformly) Γ-Ramsey extension of $(M, \mathscr{X})$, then it is a Γ-elementary extension.

(3) If $(K, \mathscr{Y})$ is a Σ_1^0-elementary extension of $(M, \mathscr{X})$ and $G \in \mathscr{Y}$, then $\mathrm{Filt}_{\mathscr{X}}(G)$ contains M and is closed upwards with respect to inclusion.

(4) If $(K, \mathscr{Y})$ is either a Σ_1^0-Ramsey extension or a Π_1^0-Ramsey extension of $(M, \mathscr{X})$, and G is a witness to such Ramseyness, then
   (a) $G \neq \varepsilon(S)$ for any $S \in \mathscr{X}$;
   (b) $G \neq \varnothing$;
   (c) $M \not\subseteq_{\mathrm{cf}} K$;
   (d) $\mathrm{Filt}_{\mathscr{X}}(G)$ is closed under finite intersection; and
   (e) $H_{>i} \in \mathrm{Filt}_{\mathscr{X}}(G)$ whenever $H \in \mathrm{Filt}_{\mathscr{X}}(G)$ and $i \in M$.

(5) Suppose $\Gamma \supseteq \Sigma_1^0$ and is closed under negation. If $(K, \mathscr{Y})$ is a uniformly Γ-Ramsey extension of $(M, \mathscr{X})$, then it is a Γ-Ramsey extension.

(6) Let $(K, \mathscr{Y}')$ be an extension of $(K, \mathscr{Y})$ and suppose $\Gamma \subseteq \Sigma^0_\infty$. Then any witness to (uniform) Γ-Ramseyness for $(K, \mathscr{Y})$ is a witness to (uniform) Γ-Ramseyness for $(K, \mathscr{Y}')$, and vice versa if this witness is in $\mathscr{Y}$. Consequently, if $(K, \mathscr{Y})$ is a (uniformly) Γ-Ramsey extension of $(M, \mathscr{X})$, then so is $(K, \mathscr{Y}')$.

Proof. These are all elementary exercises. As a demonstration, let us show the proof of (4b). If G is a witness to Σ^0_1-Ramseyness, then, as $\mathrm{Filt}_{\mathscr{X}}(G) \neq \varnothing$ by (1a),

$$\exists H{\in}\mathrm{Filt}_{\mathscr{X}}(G)\ (M, \mathscr{X}) \models \forall^{\mathrm{cf}} X{\subseteq}H\ \exists v\ (v \in X),$$

and so $G \neq \varnothing$ by Σ^0_1-Ramseyness. If G is a witness to Π^0_1-Ramseyness, then

$$\forall H{\in}\mathrm{Filt}_{\mathscr{X}}(G)\ (M, \mathscr{X}) \models \exists^{\mathrm{cf}} X{\subseteq}H\ \exists v\ (v \in X),$$

and so $G \neq \varnothing$ by Π^0_1-Ramseyness. □

It will be convenient to formulate our construction of Ramsey extensions in forcing terminology. Let us first set up the forcing language.

Definition. Let $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$.

- Denote by $\mathrm{L}^*_2(M, \mathscr{X})$ the language obtained from $\mathrm{L}_2(M, \mathscr{X})$ by adding a new constant symbol G of the set sort. This G is always displayed when writing an $\mathrm{L}^*_2(M, \mathscr{X})$ formula.
- $\mathrm{L}^*_2(M, \mathscr{X})$ formulas made up from atomic formulas using only Boolean operations and bounded number quantification are called $\Sigma^*_0(M, \mathscr{X})$.
- Let $k \in \mathbb{N}$. Then $\Pi^*_k(M, \mathscr{X})$ formulas are those formulas of the form $\forall \bar{v}_1\ \exists \bar{v}_2 \cdots \mathrm{Q}\bar{v}_k\ \alpha(G, \dots)$, where $\mathrm{Q} \in \{\forall, \exists\}$ and $\alpha(G, \dots) \in \Sigma^*_0(M, \mathscr{X})$. A $\Sigma^*_k(M, \mathscr{X})$ formula is the negation of some $\Pi^*_k(M, \mathscr{X})$ formula.
- Denote by $\Sigma^*_\infty(M, \mathscr{X})$ the set of all $\mathrm{L}^*_2(M, \mathscr{X})$ formulas built up from atomic formulas using only Boolean operations and number quantification.

The forcing poset will be $(\mathscr{X}^*, \subseteq)$, where $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$. Given a sufficiently generic filter $\mathscr{G}$ in this poset, the generic extension will be a restricted Skolem hull of the following set of $\mathrm{L}^*_2(M, \mathscr{X})$ sentences:

$$\Xi(G) = \{\xi(G) \in \mathrm{L}^*_2(M, \mathscr{X}) : (M, \mathscr{X}) \models \forall^{\mathrm{cf}} X{\subseteq}H\ \xi(X) \text{ for some } H \in \mathscr{G}\}.$$

This restricted Skolem hull construction, which we now define, is a simple adaptation of the usual one [8, Section 10.1] to the second-order context. In what follows, we let $\infty + 1 = \infty$.

Definition. Fix $(M, \mathscr{X}) \models \mathrm{RCA}^*_0$ and $k \in \mathbb{N} \cup \{\infty\}$. The Σ^0_{k+1} *hull* of a set $\Xi(G)$ of $\mathrm{L}^*_2(M, \mathscr{X})$ sentences is the L_2 structure $(K, \mathscr{Y})$ defined as follows. A Σ^0_{k+1} *number name over* $\Xi(G)$ is a $\Sigma^*_{k+1}(M, \mathscr{X})$ formula $\theta(G, v)$, whose only free variable is v, such that $\Xi(G) \vdash \exists! v\ \theta(G, v)$. Let

$$K = \{(\iota v)(\theta(G, v)) : \theta(G, v) \text{ is a } \Sigma^0_{k+1} \text{ number name over } \Xi(G)\},$$

where $(\iota v)(\theta(G, v))$ is simply a formal symbol with index $\theta(G, v)$. (We will see in Corollary 4.3(1) that this notation makes sense in the situations we will consider.) The arithmetic operations on K are defined according to $\Xi(G)$: for all Σ^0_{k+1} number names $\theta_1(G, v), \theta_2(G, v), \theta_3(G, v)$ over $\Xi(G)$,

- $(\iota v)(\theta_1(G,v)) = (\iota v)(\theta_2(G,v))$ if and only if
$$\Xi(G) \vdash \exists v \ \big(\theta_1(G,v) \wedge \theta_2(G,v)\big);$$
- $(\iota v)(\theta_1(G,v)) \leqslant (\iota v)(\theta_2(G,v))$ if and only if
$$\Xi(G) \vdash \exists v_1, v_2 \ \big(\theta_1(G,v_1) \wedge \theta_2(G,v_2) \wedge v_1 \leqslant v_2\big);$$
- $(\iota v)(\theta_1(G,v)) + (\iota v)(\theta_2(G,v)) = (\iota v)(\theta_3(G,v))$ if and only if
$$\Xi(G) \vdash \exists v_1, v_2, v_3 \ \big(\theta_1(G,v_1) \wedge \theta_2(G,v_2) \wedge \theta_3(G,v_3) \wedge v_1 + v_2 = v_3\big);$$
- $(\iota v)(\theta_1(G,v)) \times (\iota v)(\theta_2(G,v)) = (\iota v)(\theta_3(G,v))$ if and only if
$$\Xi(G) \vdash \exists v_1, v_2, v_3 \ \big(\theta_1(G,v_1) \wedge \theta_2(G,v_2) \wedge \theta_3(G,v_3) \wedge v_1 \times v_2 = v_3\big).$$

Every $a \in M$ has a *canonical name* $\breve{a}$, which is defined to be the formula $v = a$. Let
$$\mathscr{Y} = \{\mathrm{Set}_{k+1}(S/\Xi(G)) : S \in \mathscr{X}\} \cup \{\mathrm{Set}_{k+1}(G/\Xi(G))\},$$
where, for all Σ^0_{k+1} number names $\theta(G,v)$ over $\Xi(G)$,

- $(\iota v)(\theta(G,v)) \in \mathrm{Set}_{k+1}(S/\Xi(G))$ if and only if
$$\Xi(G) \vdash \exists v \ \big(\theta(G,v) \wedge v \in S\big);$$
- $(\iota v)(\theta(G,v)) \in \mathrm{Set}_{k+1}(G/\Xi(G))$ if and only if
$$\Xi(G) \vdash \exists v \ \big(\theta(G,v) \wedge v \in G\big).$$

We can view $(K, \mathscr{Y})$ as an $\mathrm{L}_2^*(M, \mathscr{X})$ structure by interpreting

- the constant symbol a as $(\iota v)(\breve{a}(G,v))$ for each $a \in M$;
- the constant symbol S as $\mathrm{Set}_{k+1}(S/\Xi(G))$ for each $S \in \mathscr{X}$; and
- the constany symbol G as $\mathrm{Set}_{k+1}(G/\Xi(G))$.

As in the L_1 case [8, Theorem 10.1], with enough completeness and induction, Skolem hulls behave well. This can alternatively be seen as a variant of Łoś's Theorem for ultrapowers.

Theorem 4.2. Fix $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ and $k \in \mathbb{N} \cup \{\infty\}$. Let $(K, \mathscr{Y})$ be the Σ^0_{k+1} hull of a consistent set of $\mathrm{L}_2^*(M, \mathscr{X})$ sentences $\Xi(G)$ that is complete for $\Sigma^*_{k+1}(M, \mathscr{X})$ sentences and includes $\mathrm{I}\Sigma^0_k$. For a $\Sigma^*_{k+1}(M, \mathscr{X})$ or $\Pi^*_{k+1}(M, \mathscr{X})$ formula $\varphi(\bar{v}, G)$ and elements $(\iota \bar{v})(\bar{\theta}(G, \bar{v})) \in K$, the following are equivalent.

(i) $(K, \mathscr{Y}) \models \varphi\big((\iota \bar{v})(\bar{\theta}(G, \bar{v})), G\big)$.

(ii) $\Xi(G) \vdash \exists \bar{v} \ \big(\bigwedge_j \theta_j(G, v_j) \wedge \varphi(\bar{v}, G)\big)$.

(iii) $\Xi(G) \vdash \forall \bar{v} \ \big(\bigwedge_j \theta_j(G, v_j) \to \varphi(\bar{v}, G)\big)$.

Proof. We know (ii) is equivalent to (iii) in view of the definition of number names. For the equivalence of (i) with the other two, we proceed by induction on φ. The base case is true by definition. The $\wedge$ case of the induction step goes through easily.

Consider the $\neg$ case. Take $\varphi(\bar{v}, G) \in \Sigma^*_{k+1}(M, \mathscr{X})$. To show (i) $\Rightarrow$ (iii), suppose $(K, \mathscr{Y}) \models \neg\varphi\big((\iota\bar{v})(\bar{\theta}(G, \bar{v})), G\big)$. Then by the induction hypothesis,

$$\Xi(G) \nvdash \exists\bar{v}\ \Big(\bigwedge_j \theta_j(G, v_j) \wedge \varphi(\bar{v}, G)\Big).$$

Since $\Xi(G)$ is complete for $\Sigma^*_{k+1}(M, \mathscr{X})$ sentences, we deduce that

$$\Xi(G) \vdash \forall\bar{v}\ \Big(\bigwedge_j \theta_j(G, v_j) \rightarrow \neg\varphi(\bar{v}, G)\Big).$$

For (iii) $\Rightarrow$ (i), suppose $(K, \mathscr{Y}) \not\models \neg\varphi\big((\iota\bar{v})(\bar{\theta}(G, \bar{v})), G\big)$. Then the induction hypothesis implies $\Xi(G) \vdash \exists\bar{v}\ \big(\bigwedge_j \theta_j(G, v_j) \wedge \varphi(\bar{v}, G)\big)$. Since $\Xi(G)$ is consistent,

$$\Xi(G) \nvdash \forall\bar{v}\ \Big(\bigwedge_j \theta_j(G, v_j) \rightarrow \neg\varphi(\bar{v}, G)\Big).$$

Consider the existential number quantifier case. Pick any $\Pi^*_k(M, \mathscr{X})$ formula $\varphi(\bar{v}, w_1, w_2, \ldots, w_\ell, G)$. First, suppose $(K, \mathscr{Y}) \models \exists\bar{w}\ \varphi\big((\iota\bar{v})(\bar{\theta}(G, \bar{v})), \bar{w}, G\big)$. Let $(\iota\bar{w})(\bar{\theta}^\sharp(G, \bar{w})) \in K$ such that $(K, \mathscr{Y}) \models \varphi\big((\iota\bar{v})(\bar{\theta}(G, \bar{v})), (\iota\bar{w})(\bar{\theta}^\sharp(G, \bar{w})), G\big)$. By the induction hypothesis, this implies

$$\Xi(G) \vdash \exists\bar{v}, \bar{w}\ \Big(\bigwedge_j \theta_j(G, v_j) \wedge \bigwedge_i \theta^\sharp_i(G, w_i) \wedge \varphi(\bar{v}, \bar{w}, G)\Big).$$

In particular, we know $\Xi(G) \vdash \exists\bar{v}\ \big(\bigwedge_j \theta_j(G, v_j) \wedge \exists\bar{w}\ \varphi(\bar{v}, \bar{w}, G)\big)$. Conversely, suppose $\Xi(G) \vdash \exists\bar{v}\ \big(\bigwedge_j \theta_j(G, v_j) \wedge \exists\bar{w}\ \varphi(\bar{v}, \bar{w}, G)\big)$. For each $i < \ell$, find a $\Sigma^*_{k+1}(M, \mathscr{X})$ formula $\theta^\sharp_{i+1}(G, w)$ that is equivalent over $\Xi(G)$ to

$$\begin{array}{l}
\exists c, \bar{v}, w_1, w_2, \ldots, w_\ell \\
\left(\begin{array}{l}
\bigwedge_j \theta_j(G, v_j) \\
\wedge\ c = \langle w_1, w_2, \ldots, w_i, w, w_{i+2}, \ldots, w_\ell\rangle \\
\wedge\ \varphi(\bar{v}, w_1, w_2, \ldots, w_i, w, w_{i+2}, \ldots, w_\ell, G) \\
\wedge\ \forall c', w'_1, w'_2, \ldots, w'_\ell{<}c\ \big(c' = \langle\bar{w}'\rangle \rightarrow \neg\varphi(\bar{v}, \bar{w}', G)\big)
\end{array}\right).
\end{array}$$

This uses the provability of $\mathrm{B}\Sigma^0_k$ in $\Xi(G)$ if $k \geqslant 1$. As $\Xi(G) \supseteq \mathrm{I}\Sigma^0_k$, it is not hard to see that each $\theta^\sharp_{i+1}(G, v)$ is a Σ^0_{k+1} number name over $\Xi(G)$, and that

$$\Xi(G) \vdash \exists\bar{v}, \bar{w}\ \Big(\bigwedge_j \theta_j(G, v_j) \wedge \bigwedge_i \theta^\sharp_i(G, w_i) \wedge \varphi(\bar{v}, \bar{w}, G)\Big).$$

So $(K, \mathscr{Y}) \models \varphi\big((\iota\bar{v})(\bar{\theta}(G, \bar{v})), (\iota\bar{w})(\bar{\theta}^\sharp(G, \bar{w})), G\big)$ by the induction hypothesis. $\square$

It follows from Theorem 4.2 that a Skolem hull satisfies a certain part of the set of sentences from which it originates. To formulate this more precisely, we borrow a piece of terminology from Patey–Yokoyama [13].

Definition. If Γ is a class of L_2 formulas, then

$$\tilde{\Gamma} = \{\forall \bar{Z}\ \gamma(\bar{u}, \bar{W}, \bar{Z}) : \gamma(\bar{u}, \bar{W}, \bar{Z}) \in \Gamma\}.$$

Corollary 4.3. Fix $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ and $k \in \mathbb{N} \cup \{\infty\}$. Let $(K, \mathscr{Y})$ be the Σ^0_{k+1} hull of a consistent set of $\mathrm{L}_2^*(M, \mathscr{X})$ sentences $\Xi(G)$ that is complete for $\Sigma^*_{k+1}(M, \mathscr{X})$ sentences and includes $\mathrm{I}\Sigma^0_k$.

(1) $(K, \mathscr{Y}) \models \forall u\ \big(\theta(G, u) \leftrightarrow u = (\iota v)(\theta(G, v))\big)$ for every $(\iota v)(\theta(G, v)) \in K$.

(2) $(K, \mathscr{Y})$ satisfies all the $\tilde{\Pi}^0_{k+2}(M, \mathscr{X})$ sentences provable in $\Xi(G)$, including the axioms of $\mathrm{I}\Sigma^0_k$.

(3) If $\Xi(G) \vdash \forall y\ \exists x{\geqslant}y\ x \in G$, then $\mathrm{Set}_{k+1}(G/\Xi(G)) \subseteq_{\mathrm{cf}} K$.

Proof. (1) Theorem 4.2 quickly implies $(K, \mathscr{Y}) \models \theta\big(G, (\iota v)(\theta(G, v))\big)$. For the converse, let $(\iota v)(\theta'(G, v)) \in K$ such that $(K, \mathscr{Y}) \models \theta\big(G, (\iota v)(\theta'(G, v))\big)$. Then Theorem 4.2 tells us

$$\Xi(G) \vdash \exists v\ \big(\theta'(G, v) \wedge \theta(G, v)\big).$$

Hence $(\iota v)(\theta'(G, v)) = (\iota v)(\theta(G, v))$ by the definition of Σ^0_{k+1} hulls.

(2) Suppose $\Xi(G) \vdash \forall \bar{Z}\ \forall \bar{v}\ \varphi(\bar{v}, \bar{Z})$, where $\varphi \in \Sigma^0_{k+1}(M, \mathscr{X})$. Then for all $\bar{S} \in \mathscr{X}$ and all $(\iota \bar{v})(\bar{\theta}(G, \bar{v})) \in K$,

$$\Xi(G) \vdash \forall \bar{v}\ \Big(\bigwedge_j \theta_j(G, v_j) \to \varphi(\bar{v}, \bar{S})\Big),$$

and so $(K, \mathscr{Y}) \models \varphi\big((\iota \bar{v})(\bar{\theta}(G, \bar{v})), \bar{S}\big)$ by Theorem 4.2. The same argument works even when some elements of the tuple $\bar{S}$ are replaced by the constant symbol G. Thus $(K, \mathscr{Y}) \models \forall \bar{Z}\ \forall \bar{v}\ \varphi(\bar{v}, \bar{Z})$.

(3) If $\Xi(G) \vdash \forall y\ \exists x{\geqslant}y\ x \in G$ and $(\iota v)(\theta(G, v)) \in K$, then

$$\Xi(G) \vdash \exists v\ \big(\theta(G, v) \wedge \exists x{\in}G\ x > v\big),$$

and so Theorem 4.2 implies $(K, \mathscr{Y}) \models \exists x{\in}G\ \big(x > (\iota v)(\theta(G, v))\big)$. □

Next we define the forcing relation. The intended meaning for $H \Vdash \xi(G)$ is usually 'the extension associated with a sufficiently generic filter containing H must satisfy $\xi(G)$'. As Theorem 4.2 shows, in a generic construction based on Σ^0_{k+1} hulls where an appropriate amount of completeness and induction is available, this accords with the definition given below whenever $\xi(G)$ is a $\Sigma^*_{k+1}(M, \mathscr{X})$ or $\Pi^*_{k+1}(M, \mathscr{X})$ sentence.

Definition. Let $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$. For a condition $H \in \mathscr{X}^*$ and a sentence $\xi(G)$ in the forcing language $\mathrm{L}_2^*(M, \mathscr{X})$, we write $H \Vdash \xi(G)$ to mean

$$(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X{\subseteq}H\ \xi(X).$$

One may indicate in the notation the model $(M, \mathscr{X})$ in which the forcing relation is evaluated. We do not do this because there will be no risk of ambiguity in the present paper.

It is not hard to deduce from Theorem 4.2 that our method of construction naturally gives rise to Ramsey extensions. In fact, Ramseyness is related to the Truth Lemma in our forcing construction.

Corollary 4.4. Fix $k \in \mathbb{N} \cup \{\infty\}$ and $(M, \mathscr{X}) \models \mathrm{RCA}_0^* + \mathrm{I}\Sigma_k^0$. Let $\mathscr{G}$ be a filter in the poset $(\mathscr{X}^*, \subseteq)$ such that the set of $\mathrm{L}_2^*(M, \mathscr{X})$ sentences

$$\Xi(G) = \{\xi(G) \in \mathrm{L}_2^*(M, \mathscr{X}) : H \Vdash \xi(G) \text{ for some } H \in \mathscr{G}\}$$

is complete for $\Sigma_{k+1}^*(M, \mathscr{X})$ sentences, and let $(K, \mathscr{Y})$ be the Σ_{k+1}^0 hull of $\Xi(G)$.

(1) The following are equivalent for all $\Sigma_{k+1}^*(M, \mathscr{X})$ or $\Pi_{k+1}^*(M, \mathscr{X})$ formulas $\varphi(\bar{v}, G)$ and all $(\iota\bar{v})(\bar{\theta}(G, \bar{v})) \in K$.

  (i) $(K, \mathscr{Y}) \models \varphi\big((\iota\bar{v})(\bar{\theta}(G, \bar{v})), G\big)$.

  (ii) $\exists H \in \mathscr{G}\ H \Vdash \exists \bar{v}\ \big(\bigwedge_j \theta_j(G, v_j) \wedge \varphi(\bar{v}, G)\big)$.

  (iii) $\exists H \in \mathscr{G}\ H \Vdash \forall \bar{v}\ \big(\bigwedge_j \theta_j(X, v_j) \to \varphi(\bar{v}, G)\big)$.

(2) $(K, \mathscr{Y})$ is a $\tilde{\Pi}_{k+2}^0$-elementary extension of $(M, \mathscr{X})$ and $\mathrm{Set}_{k+1}(G/\Xi(G)) \subseteq_{\mathrm{cf}} K$.

(3) $\mathrm{Filt}_{\mathscr{X}}\big(\mathrm{Set}_{k+1}(G/\Xi(G))\big) = \mathscr{G}$.

(4) $(K, \mathscr{Y})$ is a $(\Sigma_{k+1}^0 \cup \Pi_{k+1}^0)$-Ramsey extension of $(M, \mathscr{X})$ with witness $\mathrm{Set}_{k+1}(G/\Xi(G))$.

Proof. (1) Since $\mathscr{G}$ is a filter, its elements are compatible with each other. So $\Xi(G)$ is consistent and includes $\mathrm{I}\Sigma_k^0$. Now Theorem 4.2 applies. An application of the Compactness Theorem finishes the proof.

(2) By Corollary 4.3(2), we can view $(K, \mathscr{Y})$ as a $\tilde{\Pi}_{k+2}^0$-elementary extension of $(M, \mathscr{X})$, since $\Xi(G)$ includes the elementary diagram of $(M, \mathscr{X})$. The rest is a straightforward application of Corollary 4.3(3).

(3) Let $H \in \mathscr{G}$. It is obvious that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H\ (X \subseteq H)$. Hence $H \in \mathrm{Filt}_{\mathscr{X}}\big(\mathrm{Set}_{k+1}(G/\Xi(G))\big)$ by (1). Conversely, let $H \in \mathrm{Filt}_{\mathscr{X}}\big(\mathrm{Set}_{k+1}(G/\Xi(G))\big)$. Then (1) gives us $H' \in \mathscr{G}$ such that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H'\ (X \subseteq H)$. This implies $H' \subseteq H$. Thus $H \in \mathscr{G}$ because $\mathscr{G}$ is a filter.

(4) This follows from the previous parts. □

At last, we are ready to prove our characterization of countable models of ATR_0 in terms of Ramsey extensions, which is Theorem 4.5 below. Clauses (ii)–(iv) there correspond respectively to Σ_1^0-RT, Σ_1^0-$\mathrm{RT}_{<\infty}$ and Π_1^0-$\mathrm{RT}_{<\infty}$ in Theorem 3.2. Let us adopt a piece of notation from Clote [1, page 43].

Definition. An extension $(K, \mathscr{Y})$ of $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$ is said to satisfy M-$\mathrm{B}\Sigma_1^0$ if for all $a \in M$ and all $\Sigma_1^0(K, \mathscr{Y})$ formulas $\varphi(u, v)$,

$$(K, \mathscr{Y}) \models \forall u{<}a\ \exists v\ \varphi(u, v) \to \exists b\ \forall u{<}a\ \exists v{<}b\ \varphi(u, v).$$

Theorem 4.5. For a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(i) $(M, \mathscr{X}) \models \mathrm{ATR}_0$.

(ii) $(M, \mathscr{X})$ has a $(\Sigma_1^0 \cup \Pi_1^0)$-Ramsey extension.

(iii) $(M, \mathscr{X})$ has a Σ^0_1-Ramsey end extension.

(iv) $(M, \mathscr{X})$ has a Π^0_1-Ramsey end extension that satisfies M-$\mathrm{B}\Sigma^0_1$.

(v) $(M, \mathscr{X})$ has a $\tilde{\Pi}^0_2$-elementary $(\Sigma^0_1 \cup \Pi^0_1)$-Ramsey conservative extension that satisfies M-$\mathrm{B}\Sigma^0_1$ and Σ^0_0-CA.

Proof. We know (v) $\Rightarrow$ (ii) $\wedge$ (iii) $\wedge$ (iv) by Remark 1.1. We will show (ii) $\vee$ (iii) $\vee$ (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (v).

For (ii) $\Rightarrow$ (i), suppose $(K, \mathscr{Y})$ is a $(\Sigma^0_1 \cup \Pi^0_1)$-Ramsey extension of $(M, \mathscr{X})$ and G is a witness to $(\Sigma^0_1 \cup \Pi^0_1)$-Ramseyness for this extension. We show $(M, \mathscr{X}) \models \Sigma^0_1$-RT, which suffices by Theorem 3.2. Let $\xi(X)$ be a $\Sigma^0_1(M, \mathscr{X})$ formula. If $(K, \mathscr{Y}) \models \xi(G)$, then by Σ^0_1-Ramseyness, we have $H \in \mathrm{Filt}_{\mathscr{X}}(G)$ such that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H\ \xi(X)$. Similarly, if $(K, \mathscr{Y}) \models \neg\xi(G)$, then we can find $H \in \mathrm{Filt}_{\mathscr{X}}(\mathscr{G})$ such that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H\ \neg\xi(X)$ by Π^0_1-Ramseyness.

For (iii) $\Rightarrow$ (i), we make use of Σ^0_1-$\mathrm{RT}_{<\infty}$ in Theorem 3.2. Suppose $(K, \mathscr{Y})$ is a Σ^0_1-Ramsey end extension of $(M, \mathscr{X})$ and G is a witness to Σ^0_1-Ramseyness for this extension. Let $\theta(X, v)$ be a $\Sigma^0_1(M, \mathscr{X})$ formula and $b \in M$ such that

$$(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X\ \exists v{<}b\ \theta(X, v).$$

Notice $M \in \mathrm{Filt}_{\mathscr{X}}(G)$ by Lemma 4.1(2) and (3). So Σ^0_1-Ramseyness implies

$$(K, \mathscr{Y}) \models \exists v{<}b\ \theta(G, v).$$

Pick $a < b$ in K such that $(K, \mathscr{Y}) \models \theta(G, a)$. As K is an end extension of M and $b \in M$, we know $a \in M$ too. By Σ^0_1-Ramseyness again, we find $H \in \mathrm{Filt}_{\mathscr{X}}(G)$ such that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H\ \theta(X, a)$, as required.

Similarly, we show (iv) $\Rightarrow$ (i) using Π^0_1-$\mathrm{RT}_{<\infty}$ in Theorem 3.2. Suppose $(M, \mathscr{X})$ has a Π^0_1-Ramsey end extension $(K, \mathscr{Y})$ satisfying M-$\mathrm{B}\Sigma^0_1$, and G is a witness to Π^0_1-Ramseyness for this extension. Let $\varphi(u, v, X)$ be a $\Sigma^0_0(M, \mathscr{X})$ formula and $b \in M$ such that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X\ \exists v{<}b\ \forall u\ \varphi(u, v, X)$. In particular,

$$(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X\ \forall a\ \exists v{<}b\ \forall u{<}a\ \varphi(u, v, X).$$

Notice $M \in \mathrm{Filt}_{\mathscr{X}}(G)$ by Lemma 4.1(2) and (3) again. So by Π^0_1-Ramseyness, we have $(K, \mathscr{Y}) \models \forall a\ \exists v{<}b\ \forall u{<}a\ \varphi(u, v, G)$. As $(K, \mathscr{Y})$ satisfies M-$\mathrm{B}\Sigma^0_1$, this implies $(K, \mathscr{Y}) \models \exists v{<}b\ \forall u\ \varphi(u, v, G)$. Pick $v < b$ in K which makes $(K, \mathscr{Y}) \models \forall u\ \varphi(u, v, G)$. Since $(K, \mathscr{Y})$ is an end extension of $(M, \mathscr{X})$ and $b \in M$, we know $v \in M$ too. Hence by Π^0_1-Ramseyness, there is $H \in \mathrm{Filt}_{\mathscr{X}}(G)$ such that $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X {\subseteq} H\ \forall u\ \varphi(u, v, X)$.

Finally, consider (i) $\Rightarrow$ (v). Suppose $(M, \mathscr{X}) \models \mathrm{ATR}_0$. As alluded to earlier in this section, we force with the poset $\mathbb{P} = (\mathscr{X}^*, \subseteq)$. Let $\mathscr{G}$ be a sufficiently generic filter in $\mathbb{P}$; for example, requiring $\mathscr{G}$ to meet all $(M, \mathscr{X})$-definable dense sets is enough. Define

$$\Xi(G) = \{\xi(G) \in \mathrm{L}^*_2(M, \mathscr{X}) : H \Vdash \xi(G) \text{ for some } H \in \mathscr{G}\}.$$

The following claim enables us to use our knowledge about Skolem hulls here.

Claim 4.5.1. The set $\Xi(G)$ is complete for $\Sigma^*_1(M, \mathscr{X})$ sentences.

Proof of Claim 4.5.1. By the genericity of $\mathscr{G}$, it suffices to show that for every $\Sigma_1^*(M, \mathscr{X})$ sentence $\xi(G)$, the set

$$\{H \in \mathscr{X}^* : H \Vdash \xi(G) \text{ or } H \Vdash \neg\xi(G)\}$$

is dense in $\mathbb{P}$. This is exactly what Σ_1^0-RT, which we get from Theorem 3.2, provides. $\square$ Claim 4.5.1

Let $(K, \mathscr{Y})$ be the Σ_1^0 hull of $\Xi(G)$. Corollary 4.4 implies that $(K, \mathscr{Y})$ is a $\tilde{\Pi}_2^0$-elementary extension of $(M, \mathscr{X})$. Moreover, we know $\mathrm{Filt}_{\mathscr{X}}\big(\mathrm{Set}_1(G/\Xi(G))\big) = \mathscr{G}$ and $\mathrm{Set}_1(G/\Xi(G))$ is a witness to the $(\Sigma_1^0 \cup \Pi_1^0)$-Ramseyness of the extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$.

Claim 4.5.2. The model $(K, \mathscr{Y})$ satisfies M-$\mathrm{B}\Sigma_1^0$.

Proof of Claim 4.5.2. Pick any $\Sigma_0^0(M, \mathscr{X})$ formula $\varphi(u, v, \bar{w}, X)$. Let $a \in M$ and $(\iota\bar{w})(\bar{\theta}(G, \bar{w})) \in K$ such that

$$(K, \mathscr{Y}) \models \forall b\ \exists u{<}a\ \forall v{<}b\ \neg\varphi\big(u, v, (\iota\bar{w})(\bar{\theta}(G, \bar{w})), G\big).$$

Apply Corollary 4.4(1) to find $S \in \mathscr{G}$ which forces

$$\forall \bar{w}\ \Big(\bigwedge_j \theta_j(G, w_j) \to \forall b\ \exists u{<}a\ \forall v{<}b\ \neg\varphi\big(u, v, \bar{w}, G\big)\Big).$$

Since $(M, \mathscr{X}) \models \mathrm{B}\Sigma_1^0$, this implies

$$S \Vdash \forall \bar{w}\ \Big(\bigwedge_j \theta_j(G, w_j) \to \exists u{<}a\ \forall v\ \neg\varphi\big(u, v, \bar{w}, G\big)\Big).$$

Replacing S by a stronger condition in $\mathscr{G}$ if necessary, we may assume that S forces each $\theta_j(G, v)$ to be a number name, i.e., that $S \Vdash \bigwedge_j \exists! w\ \theta_j(G, w)$. A bit of rewriting then reveals

$$S \Vdash \exists u{<}a\ \forall \bar{w}\ \Big(\bigwedge_j \theta_j(G, w_j) \to \forall v\ \neg\varphi\big(u, v, \bar{w}, G\big)\Big).$$

Thanks to Theorem 3.2, we know $(M, \mathscr{X}) \models \Pi_1^0\text{-RT}_{<\infty}$. Hence, in view of Remark 3.1 and the genericity of $\mathscr{G}$, one can find $u < a$ in M and $H \in \mathscr{G}$ included in S such that

$$H \Vdash \forall \bar{w}\ \Big(\bigwedge_j \theta_j(G, w_j) \to \forall v\ \neg\varphi\big(u, v, \bar{w}, G\big)\Big).$$

Thus $(K, \mathscr{Y}) \models \exists u{<}a\ \forall v\ \neg\varphi\big(u, v, (\iota\bar{w})(\bar{\theta}(G, \bar{w})), G\big)$ by Corollary 4.4(1). $\square$ Claim 4.5.2

The next claim says that the extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is, in a sense, uniformly Δ_1^0-Ramsey.

Claim 4.5.3. For every $\Sigma_1^0(M, \mathscr{X})$ formula $\eta(i, X)$ and every $\Pi_1^0(M, \mathscr{X})$ formula $\eta'(i, X)$, if $(M, \mathscr{X}) \models \forall^{\mathrm{cf}} X\ \big(\eta(i, X) \leftrightarrow \eta'(i, X)\big)$, then there is $H \in \mathscr{G}$ such that for all $i \in M$, the following are equivalent.

(i) $(K, \mathscr{Y}) \models \eta(i, G)$.

(ii) $H_{>i} \Vdash \eta(i, G)$.

(iii) $H_{>i} \nVdash \neg\eta(i, G)$.

Proof of Claim 4.5.3. Since $(M, \mathscr{X}) \models \Delta^0_1\text{-}\widetilde{\mathrm{RT}}$ by Theorem 3.2, the set

$$\big\{H \in \mathscr{X}^* : \forall i{\in}M \; \big(H_{>i} \Vdash \eta(i, G) \vee H_{>i} \Vdash \neg\eta(i, G)\big)\big\}$$

is dense in $\mathbb{P}$ in view of Remark 3.1. Any H at which $\mathscr{G}$ meets this set satisfies the requirements of the claim, because $H_{>i} \in \mathscr{G}$ for every $i \in M$ by Lemma 4.1(4e). $\square$ Claim 4.5.3

It is clear from the definition of Σ^0_1 hulls that $(K, \mathscr{Y}) \not\models \Sigma^0_0$-CA. So we need to close $(K, \mathscr{Y})$ under Σ^0_0 definability. The proof of Corollary 4.3 can easily be modified to show that $\tilde{\Pi}^0_2$ elementarity is preserved in this process. Since the first-order part does not change, Ramseyness and M-$\mathrm{B}\Sigma^0_1$ are not affected, The only condition we need to be careful about is conservativity: instead of plain conservativity, we want Σ^0_0 *conservativity*, in the sense that for every $\Sigma^*_0(M, \mathscr{X})$ formula $\eta(i, G)$,

$$\{i \in M : (K, \mathscr{Y}) \models \eta(i, G)\} \in \mathscr{X}.$$

This follows readily from the structure of $(K, \mathscr{Y})$ and the uniform Ramseyness we have.

Claim 4.5.4. The extension $(K, \mathscr{Y})$ of $(M, \mathscr{X})$ is Σ^0_0-conservative.

Proof of Claim 4.5.4. For every $\Sigma^*_0(M, \mathscr{X})$ formula $\eta(i, G)$, one can find $H \in \mathscr{G}$ using Claim 4.5.3 such that

$$\{i \in M : (K, \mathscr{Y}) \models \eta(i, G)\} = \{i \in M : (M, \mathscr{X}) \models \eta(i, H_{>i})\},$$

which we know is in $\mathscr{X}$ because $(M, \mathscr{X}) \models \Sigma^0_0$-CA. $\square$ Claim 4.5.4

The following shows that $(K, \mathscr{Y})$ is not an exclusive extension of $(M, \mathscr{X})$.

Claim 4.5.5. $(K, \mathscr{Y}) \models \forall v \; G \neq (A)_v$ for all $A \in \mathscr{X}$.

Proof of Claim 4.5.5. Fix $(\iota v)(\theta(G, v)) \in K$. In view of Corollary 4.4(1), it suffices to prove that

$$\forall H{\in}\mathscr{G} \; H \nVdash \exists v \; \big(\theta(G, v) \wedge G = (A)_v\big).$$

Pick any $H \in \mathscr{G}$. Use RCA_0 to obtain a strictly increasing sequence $(h_w)_{w \in M}$ in $\mathscr{X}$ such that $H = \{h_w : w \in M\}$. Then

$$X = \{h_{2v} : v \in M\} \cup \{h_{2v+1} : v \in M \text{ and } h_{2v+1} \notin (A)_v\}$$

is a cofinal subset of H in $\mathscr{X}$ by Σ^0_0 comprehension, and $(M, \mathscr{X}) \models \forall v \; X \neq (A)_v$ by construction. In particular, we know $(M, \mathscr{X}) \models \neg\exists v \; \big(\theta(X, v) \wedge X = (A)_v\big)$. $\square$ Claim 4.5.5

This concludes the proof of Theorem 4.5. $\square$

Apparently, one cannot make the extension in Theorem 4.5(v) satisfy Δ^0_1-CA using a trick similar to that in our construction because from $\Xi(G)$ one cannot directly read off whether a set of numbers is Δ^0_1-definable in $(K,\mathscr{Y})$. Nevertheless, by a slight refinement of Corollary 4.4, some special Δ^0_1 properties of $(M,\mathscr{X})$ do remain Δ^0_1 in $(K,\mathscr{Y})$. This enables one to add a clause to Theorem 4.5 that corresponds to Δ^0_1-RT in Theorem 3.2. We have not added such a clause because the conditions are rather technical. For the same reason, we do not include any clause corresponding to Δ^0_2-RT below, but we have clauses (ii) and (iii) there that correspond respectively to Σ^0_2-RT and Σ^0_1-$\widetilde{\mathrm{RT}}$ in Theorem 3.3.

Theorem 4.6. For a countable $(M,\mathscr{X})\models\mathrm{RCA}_0$, the following are equivalent.

(i) $(M,\mathscr{X})\models\Pi^1_1\text{-CA}_0$.

(ii) $(M,\mathscr{X})$ has a $(\Sigma^0_2\cup\Pi^0_2)$-Ramsey extension.

(iii) $(M,\mathscr{X})$ has a uniformly $(\Sigma^0_1\cup\Pi^0_1)$-Ramsey extension.

(iv) $(M,\mathscr{X})$ has a Σ^1_1-elementary uniformly Σ^0_∞-Ramsey conservative extension that satisfies ACA_0.

Proof of Theorem 4.6. It is clear from Lemma 4.1(5) that (iv) $\Rightarrow$ (ii) $\wedge$ (iii). As in the proof of (ii) $\Rightarrow$ (i) in Theorem 4.5, one can show (ii) $\Rightarrow$ (i) using Σ^0_2-RT from Theorem 3.3. To demonstrate (i) $\Rightarrow$ (iv), imitate the proof of (i) $\Rightarrow$ (v) in Theorem 4.5, but use a Σ^0_∞ hull instead of a Σ^0_1 hull. This is the place where Theorem 3.4 is used.

It remains to prove (iii) $\Rightarrow$ (i). Suppose $(K,\mathscr{Y})$ is a uniformly $(\Sigma^0_1\cup\Pi^0_1)$-Ramsey extension of $(M,\mathscr{X})$, where G is a witness to uniform $(\Sigma^0_1\cup\Pi^0_1)$-Ramseyness. We show $(M,\mathscr{X})\models\Sigma^0_1$-$\widetilde{\mathrm{RT}}$, which suffices by Theorem 3.3. Let $\eta(i,X)$ be a $\Sigma^0_1(M,\mathscr{X})$ formula. Use uniform $(\Sigma^0_1\cup\Pi^0_1)$-Ramseyness to find $S,S'\in\mathrm{Filt}_{\mathscr{X}}(G)$ such that

$$\forall i{\in}M\ \big(\,(K,\mathscr{Y})\models\eta(i,G)\ \Leftrightarrow\ (M,\mathscr{X})\models\forall^{\mathrm{cf}}X{\subseteq}S_{>i}\ \eta(i,X)\,\big)\text{ and}$$
$$\forall i{\in}M\ \big(\,(K,\mathscr{Y})\models\neg\eta(i,G)\ \Leftrightarrow\ (M,\mathscr{X})\models\forall^{\mathrm{cf}}X{\subseteq}S'_{>i}\ \neg\eta(i,X)\,\big).$$

Let $H=S\cap S'$. By Lemma 4.1(5) and (4d), we know $G\in\mathrm{Filt}_{\mathscr{X}}(G)\subseteq\mathscr{X}^*$. If $i\in M$ such that $(K,\mathscr{Y})\models\eta(i,G)$, then $(M,\mathscr{X})\models\forall^{\mathrm{cf}}X{\subseteq}H_{>i}\ \eta(i,X)$ because $H\subseteq S$. Similarly, if $i\in M$ such that $(K,\mathscr{Y})\models\neg\eta(i,G)$, then $(M,\mathscr{X})\models\forall^{\mathrm{cf}}X{\subseteq}H_{>i}\ \neg\eta(i,X)$ because $H\subseteq S'$. $\square$

The extensions we constructed for Theorem 4.5(v) and Theorem 4.6(iv) are not exclusive. In fact, one can show that $G\neq(\varepsilon(A))_v$ for any $A\in\mathscr{X}$ and $v\in K$ there. It is also worth noting that the extension in Theorem 4.5(v) is actually more elementary than it looks.

Proposition 4.7. All Σ^0_2-elementary extensions of a model of ACA_0 are Σ^0_∞-elementary.

Proof. Let $(K,\mathscr{Y})$ be a Σ^0_2-elementary extension of a model $(M,\mathscr{X})\models\mathrm{ACA}_0$. Consider the formula

$$\mathrm{Q}_k v_k\ \mathrm{Q}_{k-1}v_{k-1}\ \ldots\ \mathrm{Q}_1 v_1\ \varphi(v_1,v_2,\ldots,v_k),$$

where $\varphi \in \Sigma^0_0(M, \mathscr{X})$ and $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_k \in \{\forall, \exists\}$. Define

$$U_0 = \{\langle v_k, v_{k-1}, \ldots, v_1\rangle \in M : (M, \mathscr{X}) \models \varphi(\bar{v})\}, \text{ and}$$
$$U_{i+1} = \{\langle v_k, v_{k-1}, \ldots, v_{i+2}\rangle \in M : (M, \mathscr{X}) \models \mathrm{Q}_{i+1} v_{i+1}\ \langle v_k, v_{k-1}, \ldots, v_{i+1}\rangle \in U_i\}$$

for each $i < k$. All these U_i's are in $\mathscr{X}$ because of arithmetical comprehension in $(M, \mathscr{X})$. A straightforward induction on i using Σ^0_2 elementarity shows the equivalence of the following for all $i \leqslant k$ and all $v_k, v_{k-1}, \ldots, v_{i+1} \in M$.

(i) $(K, \mathscr{Y}) \models \mathrm{Q}_i v_i\ \mathrm{Q}_{i-1} v_{i-1}\ \cdots\ \mathrm{Q}_1 v_1\ \varphi(\bar{v})$.

(ii) $(M, \mathscr{X}) \models \mathrm{Q}_i v_i\ \mathrm{Q}_{i-1} v_{i-1}\ \cdots\ \mathrm{Q}_1 v_1\ \varphi(\bar{v})$.

(iii) $\langle v_k, v_{k-1}, \ldots, v_{i+1}\rangle \in U_i$.

This gives what we want. □

5 Further problems

From Friedman–Mc Aloon–Simpson [5, Corollary 3.12], we know $\mathrm{ATR}_0 \vdash \Pi^0_1\text{-}\widetilde{\mathrm{RT}}_{<\infty}$. Here $\Pi^0_1\text{-}\widetilde{\mathrm{RT}}_{<\infty}$ denotes the set of all universal closures of formulas of the form

$$\forall b \left(\begin{array}{l} \forall i\ \forall^{\mathrm{cf}} X\ \exists v{<}b\ \zeta(i, X, v) \\ \quad \rightarrow \exists^{\mathrm{cf}} H\ \forall i\ \exists v{<}b\ \forall^{\mathrm{cf}} X{\subseteq}H_{>i}\ \zeta(i, X, v) \end{array} \right),$$

where $\zeta \in \Pi^0_1$ which may contain undisplayed free variables.

Question 5.1. What special properties does $\Pi^0_1\text{-}\widetilde{\mathrm{RT}}_{<\infty}$ give to the extension $(K, \mathscr{Y})$ we constructed for Theorem 4.5(v)?

In addition to the variants of the Galvin–Prikry Theorem discussed in Section 3, the reverse-mathematical strengths of Δ^1_1-RT and Σ^1_1-RT are also known [20, Theorem VI.7.3 and Theorem VI.7.5]. However, our construction in Section 4 does not seem directly applicable to these stronger principles.

Question 5.2. Are there characterizations of countable models of Δ^1_1-RT and Σ^1_1-RT in terms of Ramsey extensions?

Recall a complete type $p(x)$ over a structure $\mathfrak{M}$ is said to be *definable* if for every formula $\eta(i, x)$ without parameters, the set

$$\{i \in \mathfrak{M} : \eta(i, x) \in p(x)\}$$

is parametrically definable in $\mathfrak{M}$. Definable types play a central role in stability theory, and are intimately connected to conservative extensions [6]. In our context, we may say that a complete type $p(x)$ is *definable* over an L_2 structure $(M, \mathscr{X})$ if for every *arithmetical* formula $\eta(i, x)$ without parameters,

$$\{i \in M : \eta(i, x) \in p(x)\} \in \mathscr{X}.$$

Kirby's version of Theorem 1.3 is formulated in this terminology.

Theorem 5.3 (Kirby [9, Theorem A])**.** A countable model of RCA_0 admits a non-principal definable type if and only if it satisfies ACA_0.

The Σ^0_∞ type of a witness to uniform Σ^0_∞-Ramseyness has a specific defining scheme.

Question 5.4. Can one characterize countable models of ATR_0 and Π^1_1-CA_0 in terms of some variants of definable types, without requiring the defining scheme to be in any specific form?

It is assumed throughout the paper that our models are countable. On the one hand, Enayat [2] showed that the implication (i) ⇒ (iii) in Theorem 1.3 does not generalize to uncountable models. His counterexample is an ω-model of size $\aleph_1$. In fact, Enayat pointed out in a personal communication that this model of his can be arranged to satisfy AC. As a result, the implication (i) ⇒ (ii) in Theorem 2.4, the implication (i) ⇒ (ii) in Theorem 2.8, the implication (i) ⇒ (v) in Theorem 4.5, and the implication (i) ⇒ (iv) in Theorem 4.6 all become false when the countability assumption on the model $(M, \mathscr{X})$ is dropped. On the other hand, Knight and Nadel [10] showed that the implication (i) ⇒ (ii) in Theorem 1.2 is true for all ω-models of size $\aleph_1$. Scott [18] asked whether the same holds for ω-models of all cardinalities. This question is currently one of the biggest open problems in the model theory of arithmetic.

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