Adding standardness to nonstandard arithmetic

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Abstract

We discuss structures (M, ω) where M is a nonstandard model of PA and ω is a unary predicate for the standard cut. For a given theory T of a cut in a model of PA, we give a Henkin–Orey style characterization of theories of models (M, ω), where $(M, \omega) \vDash T$. In the rest of the paper, we study sets of reals that are representable in theories of models (M, ω) and sets of reals which are definable in (M, ω). In particular, we give a generalization of the easier direction of the theorem of Kanovei characterizing countable Scott sets that are standard systems of models (M, ω).

1 Introduction

The content of this paper concerns nonstandard models of arithmetic M when an additional unary relation named N is added to represent the standard cut ω. Throughout, the larger model M will be called the *ground model* and in this paper will always be a model of first-order Peano Arithmetic, PA.

Such structures (M, ω) arise naturally in the attempt to learn more about the arithmetic of $\mathbb{N}$ (i.e. the *standard model of arithmetic* with domain ω) by nonstandard means: one initially takes an ultrapower M of $\mathbb{N}$ and studies its first-order properties. This M has a copy of $\mathbb{N}$ as an initial segment, and so M is much more than a first-order structure, and has in particular the (non-definable) predicate $\mathsf{N}(x)$ expressing the standardness or finiteness of x. More generally, the same is true when M is an arbitrary nonstandard model of weaker theories than $\mathrm{Th}(\mathbb{N})$. In this paper we consider arguably the most natural of these, PA.

Working with the structure (M, ω), one's attention is naturally drawn to the cut given by the predicate N and the additional structure on it induced from M. With a suitably general idea of 'coding', structure induced by M corresponds

to subsets and relations on N coded by elements of M. Thus one important motivation for studying models (M, ω) is that they interpret a number of ω-models of second-order arithmetic, $(\omega, \mathscr{X})$ where the sets $A \in \mathscr{X}$ are coded in M in some sense. An important example is the standard system $\mathscr{X} = \mathrm{SSy}(M)$ of the ground model. Another consists of the sets represented in its theory $\mathrm{Th}(M)$. These constructions generalise to the expanded structure (M, ω) and we can look at the set $\mathscr{X}$ of sets coded or represented in (M, ω). This will be a major topic for discussion in this paper, starting at Section 3.

Parallel to the literature on the theory of structures such as $(\omega, \mathscr{X})$ is a sequence of work starting with Paris and Kirby [13] on cuts: their programme is to consider the ways in which an initial segment such as our N embeds in its end extension M, and in particular (motivated by the idea that N is analogous to a limit ordinal λ in set theory) to consider the combinatorial and Ramsey properties of N in M, such as strength—analogous perhaps to measurability of cardinals. In the particular case of the standard cut ω or $\mathbb{N}$ surprisingly little is known: ω is regular in M and may be, but need not be, strong.

This paper reviews a number of known results on such structures (M, ω) in a way that is as self-contained as possible, and presents some additional results and observations on these structures, as well as some alternative proofs of these known theorems. This is intended to facilitate further study of models of this form, and more generally models of the form (M, I) where I is an initial segment of M. Further work in these directions has already been started by Kaye and Wong and we give some indications of forthcoming results here.

The notation and terminology we use in standard in the literature on models of PA [8, 14]. In particular, the *language of arithmetic* $\mathscr{L}_{\mathrm{A}}$ is the language with non-logical symbols $+, \cdot, <, 0, 1$. The standard model of PA with domain ω is denoted $\mathbb{N}$ and its complete theory denoted TA.

As already mentioned, we shall add a unary predicate N to $\mathscr{L}_{\mathrm{A}}$ representing an initial segment and denote the expanded language by $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$. All models of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ in this paper will implicitly satisfy PA together with

$$\mathsf{N}(1) \wedge \forall x \ (\mathsf{N}(x) \to \mathsf{N}(x+x) \wedge \mathsf{N}(x^2) \wedge \forall y < x \ \mathsf{N}(y)) \wedge \exists x \ \neg \mathsf{N}(x).$$

This base theory will be denoted $\mathrm{PA}^{\mathrm{cut}}$. In the sequel, we shall often use the predicate N in a more natural way, writing $x \in \mathsf{N}$ for $\mathsf{N}(x)$, and using the obvious quantifiers $\forall x {\in} \mathsf{N}$ and $\exists x {\in} \mathsf{N}$. When N is interpreted by the standard cut ω in a nonstandard model $M \vDash \mathrm{PA}$, we obtain the expansion $(M, \omega) \vDash \mathrm{PA}^{\mathrm{cut}}$.

2 The theory

This section contains a short investigation into the theory of models of the form (M, ω). More precisely, we attempt to determine the relevant extensions of $\mathrm{PA}^{\mathrm{cut}}$.

1. Given $M \vDash \mathrm{PA}$ what is $\mathrm{Th}(M, \omega)$?

2. What is $\mathrm{Th}(\mathrm{PA}, \omega) = \bigcap \{ \mathrm{Th}(M, \omega) : M \vDash \mathrm{PA} \}$?

3. What is $\mathrm{Th}(\mathrm{TA}, \omega) = \bigcap \{ \mathrm{Th}(M, \omega) : M \succ \mathbb{N} \}$?

Note that all such theories include the full induction scheme for all $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ formulas over the predicate N, as well as (of course) full induction for $\mathscr{L}_{\mathrm{A}}$ over the ground model.

The Completeness Theorem for ω-logic gives plenty of useful information here. This is of course a more general result than is needed for our context. We outline the main results here with sketch proofs for the set-up we are considering. For detailed expositions in more general settings, see Grzegorczyk–Mostowski–Ryll-Nardzewski [3], Keisler [10], Leblanc–Roeper–Thau–Weaver [16] and Isaacson [6].

A *theory* is to be considered as just a set of axioms, i.e. not necessarily complete nor even necessarily deductively closed. As is common, we carelessly use the $+$ operation to mean $\cup$ and identify a single axiom σ with the theory $\{\sigma\}$. We say two theories are equal when their deductive closures are equal. (So, for us, two equivalent axiomatisations of the same theory are equal.) Our logic is first-order throughout.

Given an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ theory T we define

$$\Omega_1(T) = T + \{\forall x{\in}\mathsf{N}\ \theta(x) : \theta \in \mathscr{L}_{\mathrm{A}}^{\mathrm{cut}},\ T \vdash \theta(\underline{k}) \text{ for all } k \in \omega\},$$

where $\underline{k}$ is the canonical $\mathscr{L}_{\mathrm{A}}$ notation for the natural number k. Then by recursion on ordinals we set $\Omega_0(T) = T$, $\Omega_{\alpha+1}(T) = \Omega_1(\Omega_\alpha(T))$ and $\Omega_\lambda(T) = \bigcup\{\Omega_\alpha(T) : \alpha < \lambda\}$ for limit λ. We write $\Omega(T)$ for

$$\bigcup_{\alpha \in \mathrm{On}} \Omega_\alpha(T + \mathrm{PA}^{\mathrm{cut}}),$$

noting that a full union over all ordinals is of course not needed: a union over countable ordinals or actually even less suffices [1].

By an easy induction on ordinals, the equation $\Omega_\alpha(\Omega_\beta(T)) = \Omega_{\beta+\alpha}(T)$ can be verified.

The Ω_1 operation encodes the familiar ω-rule, and the following is obvious.

Proposition 2.1. If $M \vDash \mathrm{PA}$ is nonstandard and $(M, \omega) \vDash T$ for some $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-theory T, then $(M, \omega) \vDash \Omega(T)$.

The converse to this uses the following lemma.

Lemma 2.2. Given an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-theory T and a single $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-sentence σ,

$$\Omega_\alpha(T + \sigma) = \Omega_\alpha(T) + \sigma$$

for all $\alpha \in \mathrm{On}$.

Proof. The $\supseteq$ direction is clear, and the other direction is proved by a simple induction on ordinals. $\square$

Note that the lemma fails for a set of sentences Σ replacing our σ. This is associated with the failure of compactness for ω-logic—see Proposition 2.4.

Theorem 2.3 (Henkin [5], Orey [17])**.** For an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-theory T,

$$\Omega(T) = \bigcap\{\mathrm{Th}(M, \omega) : (M, \omega) \vDash T\}.$$

Proof. One direction has already been noted. For the converse, suppose that $\Omega(T) \nvdash \sigma$. We shall show that σ is false in some $(M, \omega) \vDash T$.

From the assumption we have that $\Omega(T) + \neg\sigma$ is consistent, and hence by the lemma $\Omega(T + \neg\sigma)$ is consistent. We now find a complete theory $S = \bigcup_{i \in \omega} T_i$ (complete in first-order logic) extending $T_0 = T + \neg\sigma$ such that $\Omega(T_i)$ is consistent for each i and such that if $S \vdash \exists x{\in}\mathsf{N}\ \theta(x)$ then $S \vdash \theta(\underline{k})$ for some $k \in \omega$. Given a finite extension T_i of T_0 such that $\Omega(T_i)$ is consistent, and considering $\theta(x)$ at this stage, if at all possible we let $T_{i+1} = T_i + \theta(\underline{k})$ for some k. This is not possible when all such $\Omega(T_i + \theta(\underline{k}))$ are inconsistent, and by the lemma this holds just in case $\Omega(T_i) \vdash \neg\theta(\underline{k})$ for all k. In such a case there is then $\alpha \in \mathrm{On}$ such that $\Omega_\alpha(T_i) \vdash \neg\theta(\underline{k})$ for all k, hence $\Omega_{\alpha+1}(T_i) \vdash \forall x{\in}\mathsf{N}\ \neg\theta(x)$. In this case let $T_{i+1} = T_i + \forall x{\in}\mathsf{N}\ \neg\theta(x)$. Note too that if $\forall x{\in}\mathsf{N}\ \neg\theta(x)$ is added to the T_is then $S \nvdash \exists x{\in}\mathsf{N}\ \theta(x)$ as by construction it is consistent in first-order logic.

By applying the omitting types theorem to the complete theory S just constructed and the type $p_\omega(x) = x \in \mathsf{N} + \{x \neq \underline{k} : k \in \omega\}$ we obtain a model $(M, N) \vDash S$ omitting $p_\omega(x)$ and hence $N = \omega$. But then $(M, \omega) \vDash T + \neg\sigma$, which is as required. $\qquad\square$

Compactness does not hold for ω-logic. It is not hard to give examples of theories in $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ with an additional constant symbol. A more elaborate example in pure $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ is given in the next proposition. The third author would like to thank Yue Yang for some helpful discussions related to the proof.

Proposition 2.4. Let σ_0 be an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-sentence saying

> 'there is a subset of N coded in the ground model that is Σ_1- but not Π_1-definable in N',

and σ_{k+1} be an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-sentence saying

> 'if the kth Σ_1-definable subset of N is not Π_1-definable, then it is not coded in the ground model',

for each $k \in \omega$. Then each of the theories $T_k = \mathrm{PA}^{\mathrm{cut}} + \sigma_0 + \cdots + \sigma_k$ has an ω-model (M_k, ω) but $T_\omega = \mathrm{PA}^{\mathrm{cut}} + \{\sigma_k : k \in \omega\}$ does not.

Proof. It is easy to see that T_ω has no ω-model. So we only need to show for any particular $k \in \omega$, the theory T_k has an ω-model. We will prove this using the omitting types theorem directly. It can also be proved using Fact 3.4 and Theorem VIII.2.24 in Simpson [20].

Fix $k \in \omega$. Let $S_0, S_1, \ldots, S_{k-1}$ be the first k recursively enumerable sets. Set $S = S_0 \times S_1 \times \cdots \times S_{k-1}$. Assuming k is large enough, we may suppose S is not recursive. Pick a recursively enumerable non-recursive set X such that S is not recursive in X. (This is possible by a theorem of Muchnik from 1956. See Exercise 10-11 in Rogers [18].) Our aim is to produce a nonstandard model $M \vDash \mathrm{PA}$ in which X is coded but S is not.

Define PA_X^+ to be the $\mathscr{L}_{\mathrm{A}}(c)$-theory

$$\mathrm{PA} + \{n \in c : n \in X\} + \{n \notin c : n \in \omega \setminus X\}.$$

This is consistent and recursive in X. Consider the type

$$p(v) = \{n \in v : n \in S\} \cup \{n \notin v : n \in \omega \setminus S\},$$

which is clearly finitely consistent with PA_X^+. If it has no support over PA_X^+, then we are done by the omitting types theorem. Suppose $p(v)$ has a support in $\mathscr{L}_A(c)$, say $\theta(c, v)$, over PA_X^+. Then $n \in S$ is equivalent in $\mathbb{N}$ to the $\Sigma_1(X)$-formula

'there is a proof of $\forall v \ (\theta(c, v) \to n \in v)$ from PA_X^+'

as well as the $\Pi_1(X)$-formula

'there is no proof of $\forall v \ (\theta(c, v) \to n \notin v)$ from PA_X^+'.

Hence S is $\Delta_1(X)$. This contradicts the choice of X. □

We feel that cuts N of some $K \vDash \mathrm{PA}$ are of particular interest when they satisfy the first order properties of ω, in the language $\mathrm{PA}^{\mathrm{cut}}$.

Definition 2.5. If (K, N) is an $\mathscr{L}_A^{\mathrm{cut}}$-structure that is elementarily equivalent to some $(M, \omega) \vDash \mathrm{PA}^{\mathrm{cut}}$, then we shall say that the cut N of K is *pseudo-standard*.

Corollary 2.6. There is no $\mathscr{L}_A^{\mathrm{cut}}$-theory T such that, for all $\mathscr{L}_A^{\mathrm{cut}}$-structures $(M, N) \vDash \mathrm{PA}^{\mathrm{cut}}$, we have $(M, N) \vDash T$ if and only if N is a pseudo-standard cut of M.

Proof. If T were such a theory, then it must contradict T_ω in Proposition 2.4, and so by the Compactness Theorem, it contradicts T_k for some $k \in \omega$. This violates our choice of T. □

Henkin's theorem answers two of our questions.

Corollary 2.7. (a) $\mathrm{Th}(\mathrm{PA}, \omega) = \bigcap \{\mathrm{Th}(M, \omega) : M \vDash \mathrm{PA}\} = \Omega(\mathrm{PA}^{\mathrm{cut}})$.

(b) $\mathrm{Th}(\mathrm{TA}, \omega) = \bigcap \{\mathrm{Th}(M, \omega) : M \succ \mathbb{N}\} = \Omega(\mathrm{TA} + \mathrm{PA}^{\mathrm{cut}})$.

In contrast, we do not get a complete answer to the remaining question.

Corollary 2.8. For a nonstandard $M \vDash \mathrm{PA}$,

$$\mathrm{Th}(M, \omega) \supsetneq \bigcap \{\mathrm{Th}(K, \omega) : M \equiv K\} = \Omega(\mathrm{PA}^{\mathrm{cut}} + \mathrm{Th}(M)).$$

Proof. This proof uses the notion of *strength* of ω, something that will be discussed in more detail in the next section—see in particular Fact 3.6. For now it suffices that for any $\mathscr{L}_A$-completion T of PA there are models K_1, K_2 of T such that ω is strong in K_1 but not in K_2 and that the strength of ω is a first-order property of (K_i, ω). This shows that $\Omega(\mathrm{PA}^{\mathrm{cut}} + T)$ is not complete. □

With reference to Corollary 2.7(a), it is perhaps worth pointing out that there is still something mysterious about $\Omega(\mathrm{PA}^{\mathrm{cut}})$, in that its most visible consequence is the scheme of full induction on $\mathscr{L}_A^{\mathrm{cut}}$ formulas restricted to N, but this scheme is recursive so cannot axiomatise $\Omega(\mathrm{PA}^{\mathrm{cut}})$ which decides all $\mathscr{L}_A^{\mathrm{cut}}$ sentences where quantifiers are restricted to N.

The theories $\Omega(\mathrm{PA}^{\mathrm{cut}} + T)$ and $\Omega_1(\mathrm{PA}^{\mathrm{cut}} + T)$ where T is an $\mathscr{L}_A$-completion of PA are also of interest in their own right. A sample (easy) proposition is as follows. The notation used here is standard: for a syntactic object X, X^K represents its realisation or interpretation in a structure K.

Proposition 2.9. Suppose $(K, N) \vDash \Omega_1(\mathrm{PA}^{\mathrm{cut}} + \mathrm{Th}(M))$ for some $M \vDash \mathrm{PA}$. Then for each closed Skolem term t of $\mathscr{L}_{\mathrm{A}}$ with $t^K \in N$ we have $t^K \in \omega$.

Proof. If t is such a term and $t^K \in N$ then $\mathrm{Th}(M) \vdash t = \underline{k}$ for some $k \in \omega$. For if not, as $\mathrm{Th}(M)$ is complete $\mathrm{Th}(M) \vdash \neg t = \underline{k}$ for all k hence $\Omega_1(\mathrm{PA}^{\mathrm{cut}} + \mathrm{Th}(M)) \vdash \forall x \in \mathsf{N} \; \neg t = x$. $\square$

In particular, each nonstandard $\mathscr{L}_{\mathrm{A}}$-definable element of a model $M \vDash \mathrm{PA}$ must be greater than all cuts $N \subseteq_{\mathrm{e}} M$ such that $(M, N) \vDash \Omega_1(\mathrm{PA}^{\mathrm{cut}} + \mathrm{Th}(M))$. It follows that there are recursively saturated models of PA in which the standard cut is the only pseudo-standard cut.

We end this section with a simple characterisation of pseudo-standard cuts in terms of definable sets and definable elements. Notice the 'parameter-free' condition in (a) below is clearly necessary, but that in (b) is not.

Proposition 2.10. Let $(K, N) \vDash \mathrm{PA}^{\mathrm{cut}}$. Then N is pseudo-standard in K if and only if both of the following hold.

  (a) If $a \in N$ is a parameter-free $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-definable element of (K, N), then $a \in \omega$.

  (b) If $I \subseteq_{\mathrm{e}} N$ that is parameter-free $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-definable in (K, N), then $I = N$.

Proof. For the 'only if' direction, pick $M \vDash \mathrm{PA}$ such that $(M, \omega) \equiv (K, N)$. The proof of (a) is similar to that of Proposition 2.9. For (b), simply note that all $I \subseteq_{\mathrm{e}} \omega$ definable in (M, ω) must be ω itself. This transfers to (K, N) by elementary equivalence.

For the 'if' direction, suppose (a) and (b) hold. We show that $\Omega(\mathrm{Th}(K, N))$ is consistent, which suffices by the omitting types theorem. Suppose $\phi(x)$ is an $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-formula such that $\phi(\underline{k}) \in \mathrm{Th}(K, N)$ for each $k \in \omega$. Replacing $\phi(x)$ with $\forall x' \leqslant x \; \phi(x')$ if necessary, we may assume $\phi^{(K,N)}$ is closed downwards. For convenience, we denote this downward-closed set by X. If $X \supseteq N$, then we are done. So suppose $X \subsetneq N$. If X has a biggest element, then $\max X \in \omega$ by (a), which is a contradiction. If X has no biggest element, then $X = N$ by (b), which is also a contradiction. $\square$

3   Coded reals

A *real* is a set of natural numbers $A \subseteq \omega$. Reals are *coded* in nonstandard models of arithmetic, and the set of reals that are coded forms an important isomorphism invariant of the model under study. There are two related notions of 'coded' that are studied.

Definition 3.1. The *standard system*, $\mathrm{SSy}(M)$, of a nonstandard model $M \vDash \mathrm{PA}$ is the set of reals $A \subseteq \omega$ such that there is $a \in M$ and a formula $\theta(x, y)$ in two free variables such that

$$A = \{n \in \omega : M \vDash \theta(n, a)\}.$$

One can show, using Gödel's β-function perhaps, that the formula θ here can be taken to be the same formula for all reals, i.e. that for a careful choice of θ the only quantification needed in the definition above is over the parameter $a \in M$. Thus it is common to say 'a codes A'.

The parameter-free version of the standard system is as follows.

Definition 3.2. The *external Scott set*, $\mathrm{Rep}(M)$, of a nonstandard model $M \vDash$ PA is the set of reals $A \subseteq \omega$ such that there is a formula $\theta(x)$ in one free variable such that

$$A = \{n \in \omega : M \vDash \theta(n)\}.$$

Obviously $\mathrm{Rep}(M) \subseteq \mathrm{SSy}(M)$. Unlike $\mathrm{SSy}(M)$, the set $\mathrm{Rep}(M)$ only depends on the first-order theory of M so it may or may not be the case that these sets of reals are the same. Models M for which $\mathrm{Rep}(M) = \mathrm{SSy}(M)$ are called *thrifty* in the literature. Clearly, if P_T is the prime model of some (false) completion T of PA, then P_T is thrifty. There are many other thrifty models of PA. For instance, Hájek [4] showed that for every nonstandard $M \vDash$ PA and every $n \in \omega$, there are thrifty models $I, K \vDash$ PA such that $I \subseteq_{\mathrm{e}} M \subseteq_{\mathrm{e}} K$ and $\mathrm{Th}_{\Sigma_n}(I) = \mathrm{Th}_{\Sigma_n}(M) = \mathrm{Th}_{\Sigma_n}(K)$.

We now repeat the definitions above for a model of $\mathrm{PA}^{\mathrm{cut}}$. In the following, the cut N will usually be the standard cut ω.

Definition 3.3. The *standard system*, $\mathrm{SSy}(M, N)$, of a model $(M, N) \vDash \mathrm{PA}^{\mathrm{cut}}$ is the set of reals $A \subseteq \omega$ such that there is $a \in M$ and a formula $\theta(x, y)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ in two free variables such that

$$A = \{n \in \omega : (M, N) \vDash \theta(n, a)\}.$$

The *external Scott set*, $\mathrm{Rep}(M, N)$, of (M, N) is the set of reals $A \subseteq \omega$ such that there is a formula $\theta(x)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ in one free variable such that

$$A = \{n \in \omega : (M, N) \vDash \theta(n)\}.$$

Thus for (M, N) a model of $\mathrm{PA}^{\mathrm{cut}}$ we have that $\mathrm{Rep}(M)$ is contained in both $\mathrm{Rep}(M, N)$ and $\mathrm{SSy}(M)$, and that these two are contained in $\mathrm{SSy}(M, N)$. Examples can be given (using the tools outlined in this section and in Section 4) for which these are all different. On the other hand if the pair (M, N) is recursively saturated then $\mathrm{SSy}(M, N) = \mathrm{SSy}(M)$.

Note that $\mathrm{SSy}(M)$ can be defined using a single formula θ (such as one based on Gödel's β function) but the same is not obviously true of $\mathrm{SSy}(M, N)$, and in fact this is not true in general.[1]

Scott [19] proved that a countable set of reals is realised as $\mathrm{SSy}(M)$ or $\mathrm{Rep}(M)$ for some $M \vDash$ PA if and only if it is what is now called a *Scott set*, i.e., a boolean algebra closed under relative recursion and Weak König's Lemma. In Friedman's terminology [20], this is stated as follows.

Fact 3.4 (Scott). (a) Both $(\omega, \mathrm{SSy}(M))$ and $(\omega, \mathrm{Rep}(M))$ satisfy WKL_0 when $M \vDash$ PA is nonstandard.

(b) Let $(\omega, \mathscr{X}) \vDash \mathrm{WKL}_0$ be countable. Then for all recursive extensions $T \supseteq$ PA, there exist nonstandard models $M \vDash T$ such that $\mathrm{SSy}(M) = \mathrm{Rep}(M) = \mathscr{X}$.

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[1] In forthcoming work, Kaye shows that if there a single formula $\theta(n, x)$ of $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ such that each $A \in \mathrm{SSy}(M, \omega)$ is defined by $\theta(n, a)$ for some $a \in M$, then $(\omega, \mathrm{SSy}(M, \omega))$ satisfies the full comprehension scheme of second order arithmetic, and that this does not happen in general.

The proof of part (a) is somewhat different for Rep and for SSy: the argument that $\mathrm{Rep}(M)$ is a Scott set (Fact 3.4) is by writing down appropriate formulas, mimicking the proof of König's lemma, and the proof of (b) is by a clever construction using infinitely many diagonalisations. The same argument as in (a) applies, showing $\mathrm{Rep}(M, \omega)$ and $\mathrm{SSy}(M, \omega)$ are also Scott sets. The analogue of (b) for $\mathrm{Rep}(M, \omega)$ and $\mathrm{SSy}(M, \omega)$ appears to be much harder: we shall investigate at other possible closure conditions on $\mathrm{Rep}(M, \omega)$ and $\mathrm{SSy}(M, \omega)$ below. Since $\mathrm{Rep}(M)$ only depends on the theory T of M, we shall also write it as $\mathrm{Rep}(T)$. One may ask, in general, what the relationship between a (false) completion $T \supseteq \mathrm{PA}$ and the second-order structure $(\omega, \mathrm{Rep}(T))$ is. Scott's theorem tells us that no nontrivial recursive proper extension Γ of WKL_0 can be captured by a recursive extension S of PA in the sense that

$$(\omega, \mathrm{Rep}(T)) \vDash \Gamma \Leftrightarrow T \supseteq S$$

for all false completions T of PA.

The following simple proposition shows that the answer is probably not simple.

Proposition 3.5. Let Γ be an extension of WKL_0 in second-order arithmetic such that

$$\{\mathscr{X} \subseteq \mathscr{P}(\omega) : \mathscr{X} \text{ is countable and } (\omega, \mathscr{X}) \vDash \Gamma\}$$

is nonempty but does not exhaust all countable Scott sets.

(a) For all consistent recursive $S \supseteq \mathrm{PA}$, there are false completions T_1, T_2 of S such that $(\omega, \mathrm{Rep}(T_1)) \vDash \Gamma$ and $(\omega, \mathrm{Rep}(T_2)) \nvDash \Gamma$.

(b) No recursive $S \supseteq \mathrm{PA}$ satisfies

$$(\omega, \mathrm{Rep}(T)) \vDash \Gamma \Leftrightarrow T \supseteq S$$

for all false completions T of PA.

Proof. Since (b) follows from (a), we will concentrate on (a). Let S be a consistent recursive extension of PA. Find countable Scott sets $\mathscr{X}_1, \mathscr{X}_2$ such that $(\omega, \mathscr{X}_1) \vDash \Gamma$ and $(\omega, \mathscr{X}_2) \nvDash \Gamma$. By Fact 3.4, there are completions $T_1, T_2 \supseteq S + \neg \mathrm{Con}(S)$ such that $\mathscr{X}_1 = \mathrm{Rep}(T_1)$ and $\mathscr{X}_2 = \mathrm{Rep}(T_2)$. $\square$

A great deal more can be said about the standard system $\mathrm{SSy}(M)$ in the language $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$, and indeed the second-order structure for arithmetic $(\omega, \mathrm{SSy}(M))$ can be interpreted in each (M, ω). In particular, Keisler [11, 12] found, for each of the Big Five theories in reverse mathematics, a natural $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-theory that corresponds to it.

Because $\mathrm{SSy}(M)$ can be defined using a single formula θ important closure conditions on $\mathrm{SSy}(M)$ can be expressed in $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ as axioms for (M, ω). Perhaps the most important of these is closure under jump. The following fact takes this much further and is essentially due to Kirby and Paris [13].

Fact 3.6. For nonstandard $M \vDash \mathrm{PA}$ the following are equivalent:

(a) $\mathrm{SSy}(M)$ is closed under jump;

(b) $(\omega, \mathrm{SSy}(M)) \vDash \mathrm{ACA}_0$;

(c) ω is *strong* in M i.e. for each $a \in M$, there is a nonstandard b such that for all $n \in \omega$, $(a)_n > \omega$ iff $(a)_n > b$;

(d) For all $\theta(x,y)$ of $\mathscr{L}_{\mathrm{A}}$ there is a canonical choice of $\phi(x,y)$ in $\mathscr{L}_{\mathrm{A}}$ such that,

$$(M, \omega) \vDash \forall a\ \exists b\ \forall n{\in}\mathsf{N}\ (\phi(b,n) \leftrightarrow \exists m{\in}\mathsf{N}\ \theta(a, \langle n, m\rangle)).$$

There are models of arithmetic (indeed, elementarily equivalent ones) in which ω is strong in one but not the other—see, for example, Chapter 13 of Kaye [8]. This was how we proved $\mathrm{Th}(M, \omega) \neq \Omega(\mathrm{PA}^{\mathrm{cut}} + \mathrm{Th}(M))$ for every $M \vDash \mathrm{PA}$ in Corollary 2.8. Given a (false) completion $T \supseteq \mathrm{PA}$, the most obvious completion of $\Omega(\mathrm{PA}^{\mathrm{cut}} + T)$ is $\mathrm{Th}(P_T, \omega)$, where P_T denotes the prime model of T. Here, ω is strong in P_T if and only if $\mathrm{Rep}(T)$ is closed under jump (where $\mathrm{Rep}(T)$ is by definition $\mathrm{Rep}(P_T)$, also equal to $\mathrm{SSy}(P_T)$), and by Fact 3.4, for any countable Scott set $\mathscr{X}$ (including ones closed and not closed under jump) there is a completion $T \supseteq \mathrm{PA} + \neg\,\mathrm{Con}(\mathrm{PA})$ such that $\mathscr{X} = \mathrm{Rep}(T)$.

For $\mathrm{SSy}(M, \omega)$ the situation appears to be more complicated since we know of no analogue to the β function that gives a canonical way of encoding reals $A \in \mathrm{SSy}(M, \omega)$. Some limited progress can be made by assuming the model M is 'full', i.e. that $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$, and then use the usual Gödel β function to code reals. Such models do exist, and will be discussed below.

Another useful simplification that can be made is in assuming some saturation properties of M. Countable recursively saturated models M of PA are characterised exactly by their complete theory and their standard system, and the only restriction on pairs $\mathrm{Th}(M)$, $\mathrm{SSy}(M)$ that can occur is that $\mathrm{Th}(M) \in \mathrm{SSy}(M)$ relative to some suitable Gödel numbering [8]. It is convenient (but arguably not always correct) to think of recursively saturated models as being 'more saturated' the larger their standard system. Perhaps the most useful notion of this type is that of *arithmetical saturation* which for PA means that the model M is recursively saturated and $\mathrm{SSy}(M)$ is closed under jump.

It is also clear that for any nonstandard model $M \vDash \mathrm{PA}$ both $\mathrm{Rep}(M, \omega)$ and $\mathrm{SSy}(M, \omega)$ are closed under jump. Thus models M with $\mathrm{Rep}(M) \neq \mathrm{Rep}(M, \omega)$ and $\mathrm{SSy}(M) \neq \mathrm{SSy}(M, \omega)$ are easy to find. The following beautiful result, due to Kanovei, takes this observation further and settles the question of which (countable) Scott sets occur as $\mathrm{Rep}(M, \omega)$ where M is a nonstandard model of true arithmetic.

Fact 3.7 (Kanovei [7]). (a) For all $M \succ \mathbb{N}$, $\mathrm{Rep}(M, \omega)$ is closed under jump and contains $\mathbf{0}^{(\omega)}$, the set of Gödel numbers of sentences in TA.

(b) Suppose $\mathscr{X}$ is a countable Scott set that is closed under jump and which contains $\mathbf{0}^{(\omega)}$. Then there is an elementary extension M of $\mathbb{N}$ such that $\mathrm{Rep}(M, \omega) = \mathscr{X}$.

We will provide a different argument for Fact 3.7(a) in Section 4 below. The best known proof of part (b) of this result is currently rather long and technical.

Since $\mathrm{Rep}(M, \omega)$ depends only on $\mathrm{Th}(M, \omega)$, Fact 3.7 immediately tells us how many $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$-theories of standard cuts there are.

Corollary 3.8. There are continuum many theories of structures (M, ω), where $M \vDash \mathrm{TA}$.

From Fact 3.7, we might expect that some notion of 'super saturation', meaning here that the model $M \vDash$ PA is recursively saturated and $\mathrm{SSy}(M)$ contains $\mathbf{0}^{(\omega)}$ and is closed under jump, would give interesting consequences. We are unaware of any such results though.

We have already considered the possibility that $\mathrm{SSy}(M, \omega)$ and $\mathrm{SSy}(M)$ may be the same. Another possible meaning of 'super saturation' is recursive saturation plus the condition $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$.

Definition 3.9. Let $M \vDash$ PA be nonstandard. We say M is *full* iff $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$ and *fully saturated* if it is full and recursively saturated.

Notice that the property of fullness of M can be expressed in $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$, and it is straightforward to check that fullness of M implies $(\omega, \mathrm{SSy}(M)) \vDash \mathrm{CA}_0$, the full second-order comprehension axiom scheme. Kaye (to appear in a forthcoming paper) has recently shown a converse to the assertion here: that if $M \vDash$ PA is nonstandard and $(\omega, \mathrm{SSy}(M)) \vDash \mathrm{CA}_0$ then $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$. It follows that if M is full and $N \vDash PA$ is an end extension of M, then N is full as well, and this in turn implies that $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(N, \omega)$. The fact that the standard system is preserved by end extensions is a very useful property of models of PA. We suspect the corresponding result does not hold for $\mathrm{SSy}(M, \omega)$, but at the moment it is an open problem.

Question 3.10. Let M be a nonstandard models of PA.

  (a) Is $\mathrm{SSy}(M, \omega)$ preserved under end extensions of M?

  (b) Is $\mathrm{SSy}(M, \omega)$ preserved under elementary end extensions of M?

If M is countable and recursively saturated and N is a countable recursively saturated elementary end extension of M, then $(M, \omega) \cong (N, \omega)$, hence (b) has positive answer for such extensions.

Fullness is related to work by Engström and Kaye [2] who introduce the notion of *transplendent* models, which are essentially like resplendent models but (under an appropriate consistency condition) allow one to expand a model M to a larger language without changing the domain, satisfying some recursive or coded theory, and simultaneously omit a recursive or coded type. Countable transplendent models exist, but necessarily have very rich standard systems. A 'nice' characterisation of the standard systems of transplendent models of PA is not known but it is known that if $M \vDash$ PA is transplendent and $\mathscr{X} = \mathrm{SSy}(M)$ then $(\omega, \mathscr{X})$ is an elementarily submodel of $(\omega, \mathscr{P}(\omega))$.

The following is a sample application of the theory of transplendency.

Proposition 3.11. Suppose M is a countable transplendent model of PA. Then M is full, i.e. $\mathrm{SSy}(M, \omega) = \mathrm{SSy}(M)$.

Proof. Given a real $A \in \mathrm{SSy}(M, \omega)$ defined by $A = \{n \in \omega : (M, \omega) \vDash \theta(n, b, \omega)\}$ we consider the theory in $\mathscr{L}_{\mathrm{A}}^{\mathrm{cut}}$ together with additional constants a, b stating that the subset of N coded by a (using the β-function) is exactly the set of $n \in \mathsf{N}$ such that $(M, \mathsf{N}) \vDash \theta(n, b, \mathsf{N})$. We also omit the type $p_\omega(x) = \{x \in \mathsf{N}\} + \{x \neq \underline{k} : k \in \omega\}$, so that N represents ω. By transplendency there is an expansion of (M, b) satisfying all this, which suffices, so $A \in \mathrm{SSy}(M)$. $\square$

It follows that whenever $\mathscr{X}$ is a countable Scott set which is the standard system of a transplendent model there is a countable $M \vDash$ PA with $\mathrm{SSy}(M, \omega) = \mathscr{X}$. This result has been improved recently by Kaye (forthcoming) who pointed out that any countable Scott set $\mathscr{X}$ such that $(\omega, \mathscr{X})$ satisfies the full comprehension scheme is the standard system of a full model of PA.

These results on transplendent models and full models at least provide some conditions on a Scott set $\mathscr{X}$ which allow us to realise $\mathscr{X}$ as $\mathrm{SSy}(M, \omega)$ for some $M \vDash$ PA, but are rather unsatisfactory as they rely on very strong hypotheses. Work is currently underway on reducing these hypotheses, and we would make the following conjecture.

Conjecture 3.12. Let $\mathscr{X}$ be a countable Scott set.

(a) $\mathscr{X}$ is realised as $\mathrm{SSy}(M, \omega)$ for some $M \vDash$ PA if and only if $\mathscr{X}$ is closed under jump.

(b) $\mathscr{X}$ is realised as $\mathrm{SSy}(M, \omega)$ for some model of true arithmetic $M \succ \omega$ if and only if $\mathscr{X}$ contains $\mathbf{0}^{(\omega)}$ and is closed under jump.

The necessity of the condition concerning $\mathbf{0}^{(\omega)}$ in part (b) will be shown in the following section. It would follow from this, for example, that not all nonstandard models of PA are full. We hope that similar additional information on the Scott sets $\mathrm{Rep}(M, \omega)$ might be obtained.

4 Truth definitions

Kanovei's observation, Fact 3.7(a), that $\mathbf{0}^{(\omega)}$ is in $\mathrm{Rep}(M, \omega)$ for any nonstandard model of true arithmetic M says that there is a truth definition for ω definable in the pair (M, ω). This is of particular interest because of the connections between truth definitions and recursive saturation which we would like to explore further. But before we do that we present in more detail an argument why it is true. The following argument is somewhat different from Kanovei's but useful as it generalises in a number of interesting directions.

Let $M \vDash$ PA be nonstandard and $K \subseteq M$ a substructure. We shall attempt to give a definition of truth in K. (One could consider, more generally, the case when K is an interpreted structure over M, but we see no obvious application in our context here.)

Definition 4.1. An element $c \in M$ is a *certificate* (of truth of certain formulas in K) if it is an M-finite set of triples $\langle \phi, a, t \rangle$ where ϕ is a formula (in the sense of M) of the language $\mathscr{L}_{\mathrm{A}}$, a is an element of M (thought of as coding a sequence $(a_0, a_1, \ldots)$) and t is 0 or 1, such that the following are all satisfied for all ϕ, ψ and a.

1. $\langle \phi, a, 0 \rangle \in c \Rightarrow \langle \phi, a, 1 \rangle \notin c$

2. $\langle \phi, a, 1 \rangle \in c \Rightarrow \langle \phi, a, 0 \rangle \notin c$

3. $\langle \neg\phi, a, 1 \rangle \in c \Rightarrow \langle \phi, a, 0 \rangle \in c$

4. $\langle \neg\phi, a, 0 \rangle \in c \Rightarrow \langle \phi, a, 1 \rangle \in c$

5. $\langle \phi \wedge \psi, a, 1 \rangle \in c \Rightarrow \langle \phi, a, 1 \rangle \in c$ and $\langle \psi, a, 1 \rangle \in c$

6. $\langle \phi \wedge \psi, a, 0 \rangle \in c \Rightarrow \langle \phi, a, 0 \rangle \in c$ or $\langle \psi, a, 0 \rangle \in c$

7. $\langle \forall x_i \ \phi, a, 1 \rangle \in c \Rightarrow \forall b \in K \ \langle \phi, a[b/i], 1 \rangle \in c$

8. $\langle \forall x_i \ \phi, a, 0 \rangle \in c \Rightarrow \exists b \in K \ \langle \phi, a[b/i], 0 \rangle \in c$

9. $\langle \phi, a, 1 \rangle \in c \Rightarrow K \vDash \phi[a]$, when ϕ is atomic and $a_i \in K$ for all i such that x_i is free in ϕ

10. $\langle \phi, a, 0 \rangle \in c \Rightarrow K \nvDash \phi[a]$, when ϕ is atomic and $a_i \in K$ for all i such that x_i is free in ϕ

Here, $a[b/i]$ is the code for the sequence obtained from a by replacing a_i by b, the free variables in our language are x_i for various i, and $\phi[a]$ is the statement obtained by replacing in ϕ each free variable x_i with a_i.

When c is a certificate, we write $\vDash_c \phi[a]$ for $\langle \phi, a, 1 \rangle \in c$ and $\nvDash_c \phi[a]$ for $\langle \phi, a, 0 \rangle \in c$. The reader can add clauses for $\vee$ and $\exists$ if he/she wishes, following the above model.

Definition 4.2. The certificate c *decides* $\phi[a]$ if $\vDash_c \phi[a]$ or $\nvDash_c \phi[a]$.

Definition 4.3. The certificate c is K-*complete* if given a, ϕ, if c decides $\phi[a]$ then c decides $\phi[b]$ for all b such that $b_i \in K$ for all i such that x_i is free in ϕ.

The following is a straightforward induction (in the meta-theory) on the complexity of ϕ.

Lemma 4.4. Suppose M, K are as above, c_1, c_2 are K-complete certificates, ϕ is a standard formula, $a_i \in K$ for each x_i free in ϕ, and both c_1, c_2 decide $\phi[a]$. Then

$$\vDash_{c_1} \phi[a] \ \Leftrightarrow \vDash_{c_2} \phi[a].$$

Definition 4.5. Given M, K as above, we define S to be the parameter-free definable set of pairs $\langle \phi, a \rangle$ (parameter-free definable in (M, K, ω)) such that $\phi \in \omega$, for all i with x_i free in ϕ we have $a_i \in K$, and

$$\forall c \ (c \text{ is a } K\text{-complete certificate deciding } \phi[a] \text{ implies } \vDash_c \phi[a]).$$

We write $\vDash_S \phi[a]$ for $\langle \phi, a \rangle \in S$.

As we shall show shortly, under certain circumstances the set S so defined is a definition of full truth for K in (M, K, ω) and S is also defined by

$$\exists c \ (c \text{ is a } K\text{-complete certificate deciding } \phi[a] \text{ and } \vDash_c \phi[a]).$$

By the previous lemma, the main thing to check is of course that for all standard ϕ and all appropriate a there is a K-complete certificate deciding $\phi[a]$. This works in the following two cases.

Proposition 4.6. Suppose $K \prec M$ and is not cofinal in M. Then S is a definition of truth for K.

Proof. That K-complete certificates exist for each $\phi[a]$ is proved by an induction (in the meta-theory) on ϕ. Note that the definition of certificate is not in $\mathscr{L}_A$ but uses K as a predicate, so an internal induction is not available. The only tricky step is the case for the quantifier, as the other cases can be dealt with by taking the union of certificates, and/or the union of a certificate and a set obtained from the separation principle for M-finite sets, provable in PA. (It is this separation principle that requires K to be not cofinal in M. See Kaye–Wong [9] for the details.)

Given a K-complete certificate c deciding $\phi[a]$ for all appropriate a, we need to observe that the set A of appropriate a such that $\exists b \in K \; \langle \phi, a[b/i], 0 \rangle \in c$ is coded in M, for then

$$\{ \langle \forall x_i \; \phi, a, 1 \rangle : a \notin A \} \cup \{ \langle \forall x_i \; \phi, a, 0 \rangle : a \in A \} \cup c$$

would be coded by a certificate c' deciding $\forall x_i \; \phi$. But this follows from $K \prec M$ since for any a with $a_j \in K$ for x_j free in $\forall x_i \; \phi$ we have

$$\exists b \in K \; \neg\phi[a[b/i]] \Leftrightarrow \exists b \in M \; \neg\phi[a[b/i]] \Leftrightarrow \exists b \in M \; \langle \phi, [a[b/i]], 1 \rangle \in c$$

by elementarity, and the last statement is definable in $\mathscr{L}_A$ so the set of such $a[b/i]$ is coded. $\qquad\square$

If a set parameter $A \subseteq \omega$ or a parameter $a \in M$ coding such a real $A \in \mathrm{SSy}(M, \omega)$ were present, we would require a corresponding predicate $A^* \subseteq M$ such that $(\omega, A) \prec (M, A^*)$ for the above proof to work to answer question 4.10.

The previous result proves Kanovei's observation about models of true arithmetic. For models of false arithmetic we have the following.

Proposition 4.7. Suppose $M \vDash \mathrm{PA}$, ω is strong in M and $K = \omega$. Then S is a definition of truth for ω.

Proof. We deal with the quantifier step in a different way, using strength of ω instead of elementarity. Given an ω-complete certificate c deciding $\phi[a]$ for all appropriate a, we need the set A of appropriate a such that $\exists b \in \omega \; \langle \phi, a[b/i], 0 \rangle \in c$ to be coded in M. This can be shown by using the strength of ω on the coded function $f : \omega^2 \to M$ defined by

$$f(\phi, a) = \begin{cases} (\min b)(\langle \phi, a[b/i], 0 \rangle \in c), & \text{if it exists;} \\ d, & \text{otherwise,} \end{cases}$$

where d is some fixed nonstandard number. $\qquad\square$

Note that any countable arithmetically saturated model M satisfied the hypothesis of this theorem.

Corollary 4.8. Let $M \vDash \mathrm{PA}$ be a nonstandard model, and suppose either that $M \vDash \mathrm{TA}$ or ω is strong in M. Let (K, N) be a model of $\mathrm{Th}(M, \omega)$ where N is nonstandard. Then N has a full inductive satisfaction class. In particular, N is recursively saturated.

Proof. By Proposition 4.7, the set defined by the definition of S in (K, N) is a full inductive satisfaction class for N. Since N is nonstandard, it must be recursively saturated. $\qquad\square$

In general, we cannot expect this N to be any more than recursively saturated. For example, given $T = \text{Th}(M, \omega)$ there is a countable Scott set $\mathscr{X}$ containing T and not closed under jump. (Indeed there are continuum-many such $\mathscr{X}$.) Let K be a countable recursively saturated model of $\text{Th}(M)$ such that $\text{SSy}(K) = \mathscr{X}$. Since K is resplendent, there is N such that $(K, N) \vDash T$ and (K, N) is recursively saturated. In particular, N is recursively saturated with $\text{SSy}(K) = \text{SSy}(N) = \mathscr{X}$, and as $\mathscr{X}$ is not closed under jump N is not arithmetically saturated.

Using a theorem of Smith's [21, Theorem 3.14], one can find models of PA that contain a nonstandard pseudo-standard cut, but are not recursively saturated themselves.

More generally, Proposition 4.6 gives other variants of Corollary 4.8.

Corollary 4.9. Suppose $M \vDash \text{PA}$ is nonstandard, and $K \prec M$ is not cofinal. Then for any $(M', K', N) \vDash \text{Th}(M, K, \omega)$ with N nonstandard the substructure K' is recursively saturated.

We are unable to show that $\text{SSy}(M, \omega)$ is closed under the ωth jump without any additional hypothesis. It is tempting to conjecture from Kanovei's theorem and Proposition 4.7 that this is at least true when ω is strong or when M satisfies true arithmetic.

Question 4.10. Suppose $M \vDash \text{PA}$ in which ω is strong, or $M \vDash \text{TA}$. Is it the case that $\text{SSy}(M, \omega)$ is closed under the ωth-jump $\mathbf{a} \mapsto \mathbf{a}^{(\omega)}$?

Kotlarski [15] defined the *recursively saturated part* of a model to be its largest elementary cut that is recursively saturated. In connection with Corollary 4.8 we can ask:

Question 4.11. Can a pseudo-standard cut in a model of TA be its recursively saturated part?

5 Conclusions

It is perhaps surprising that the model theory of structures of the form (M, ω) is so difficult compared to that of M. It seems that this may reflect some intrinsic difficulty, or else model theoretic tools are not available (or are not known by the present authors).

There are many questions one can ask about the model theory of pairs (M, N), where M is a countable model of PA and N is a pseudo-standard cut. Using the omitting types theorem, Tin Lok Wong managed to prove several results on existence of elementary end extensions and cofinal extensions of such structures. The fact that N is pseudo-standard, rather than just a cut in M, plays a crucial role in the proofs. It is interesting that none of the result seem to generalize to the uncountable case. One area we have not addressed in this short paper is definability in models (M, ω). Smith [21] makes a start in this direction, and this suggests that expressive power (for such models when the predicate N is interpreted as ω) is not much stronger. But there is still much work to do in these directions, in particular in unravelling the distinctions between what is definable by single first-order formula and what is definable by a simple infinitary formula, such as an infinitary conjunction of first-order formulas.

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