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Almost nonnegative Ricci curvature and new vanishing theorems for genera

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Abstract
We derive new vanishing theorems for genera under almost nonnegative Ricci curvature and infinite fundamental group. A vanishing theorem of Euler characteristic number for almost nonnegatively curved Alexandrov spaces is also proved.

Contents

1 Introduction

A classical theorem of Lichnerowicz asserts that a compact spin manifold carrying a Riemannian metric of positive scalar curvature has vanishing $\widehat{A}$-genus. In a different direction, Lott conjectured that a compact spin manifold with *almost nonnegative sectional curvature* has vanishing $\widehat{A}$-genus (Prop.1 in [26]). In this paper we derive new vanishing theorems for genera under almost nonnegative Ricci curvature condition. We start from the Todd genus.

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Theorem 1.1 *Let M be a compact complex manifold with infinite fundamental group. If M admits a sequence of Kähler metrics with almost nonnegative Ricci curvature, then the Todd genus of M vanishes.*

If a sequence of Riemannian metrics $\{g_i\}_{i\in\mathbb{N}}$ on a smooth manifold M satisfies

$$\mathrm{Ric}(g_i) \geq -1 \quad \text{and} \quad \mathrm{diam}(g_i) \leq \frac{1}{i},$$

for all $i \in \mathbb{N}$, then we say that $\{g_i\}_{i\in\mathbb{N}}$ have almost nonnegative Ricci curvature. Here $\mathrm{Ric}(g_i)$ and $\mathrm{diam}(g_i)$ stand for the Ricci curvature and diameter of g_i, respectively. If g has nonnegative Ricci curvature, it is clear that $g_i = \epsilon_i g$, $\epsilon_i = \frac{1}{i^2 \,\mathrm{diam}(g)^2}$ have almost nonnegative Ricci curvature.

Given a compact complex manifold M^n of complex dimension n, its Todd genus (or holomorphic Euler number) is defined to be

$$\sum_{p=0}^{n} (-1)^p \dim H^{0,p}(M, \mathbb{C}),$$

where $H^{0,p}(M, \mathbb{C})$ is the Dolbeault cohomology group of M.

Theorem 1.1 can be viewed as a complex analogue of a theorem of Fukaya–Yamaguchi (Corollary 0.12, [13]), which says that a compact Riemannian manifold with *almost nonnegative sectional curvature* and infinite fundamental group has vanishing topological Euler number. However, a compact Riemannian manifold with almost nonnegative Ricci curvature and infinite fundamental group may have nonzero topological Euler number, see the example constructed by Anderson in [1].

Example 1.2 Let X^{n+1} be a smooth abelian variety of complex dimension $(n+1) \geq 3$ embedded in some complex projective space $\mathbb{CP}^N$. Let M^n be the intersection of X with some generic $\mathbb{CP}^{N-1}$. Then M^n is a smooth hypersurface of X with positive holomorphic normal bundle L. By Lefschetz hyperplane theorem, the fundamental group of M is an infinite abelian group. Moreover, the Todd genus of M is nonzero. In fact, by Hirzebruch–Riemann–Roch theorem [2, 19], the Todd genus of M is computed by $\int_M \mathrm{td}(TM)$, where $\mathrm{td}(TM)$ is the Todd class of the tangent bundle of M. Let d be the first Chern class of L. As X is an abelian variety, then $\mathrm{td}(TX) = 1$. Moreover, by properties of Todd class [19], we have

$$\mathrm{td}(TM)\,\mathrm{td}(L) = \mathrm{td}(TX)|_M,$$
$$\mathrm{td}(L) = \frac{d}{1 - e^{-d}}, \quad d = c_1(L).$$

Then

$$\mathrm{td}(TM) = \frac{1 - e^{-d}}{d}.$$

As L is a positive line bundle, we see that

$$\int_M \mathrm{td}(TM) = (-1)^n \int_M \frac{d^n}{(n+1)!} \neq 0.$$

Hence the Todd genus of M is nonzero. It follows that $M \times \mathbb{CP}^k$ has nonzero Todd genus for any $k \geq 1$. Therefore by Theorem 1.1, $M \times \mathbb{CP}^k$ does not admit a sequence of Kähler metrics with almost nonnegative Ricci curvature. For $k \geq 2$, it seems that all previous known obstructions to almost nonnegative Ricci curvature do not apply to $M \times \mathbb{CP}^k$.

Riemannian manifolds with almost nonnegative Ricci curvature have been studied extensively [4, 12, 21, 23, 33]. We briefly recall some previously known results (here $m = \dim M$).

- The fundamental group of M has a nilpotent subgroup of finite index [23].
- The first Betti number of M is bounded above by m [14] with equality being achieved if and only if M is diffeomorphic to a torus [8, 11].
- If M is spin and m is divisible by 4, then its $\widehat{A}$-genus is bounded from above by $2^{\frac{m}{2}}$. See Theorem E, page 294 in [16] and [14].

Compact Kähler manifolds with almost nonnegative Ricci curvature have been studied in [12, 33]. Under the additional assumptions that M has bounded sectional curvature and other things, it was shown in [33] that M has finite fundamental group. Moreover, M^n is diffeomorphic to a complex manifold X such that the universal covering of X has a decomposition: $\widetilde{X} = X_1 \times \cdots \times X_s$, where X_i is a Calabi–Yau manifold, or a hyperKähler manifold, or X_i satisfies $H^{p,0}(X_i, \mathbb{C}) = 0,\ p > 0$. If M has infinite fundamental group, under the stronger assumption that M has almost nonnegative bisectional curvature and bounded sectional curvature, it was shown in [12] that there is a holomorphic fibration $M \to J(M)$, where $J(M)$ is the Jacobian of M, a complex torus of dimension $\frac{1}{2}b_1(M)$ and b_1 is the first Betti number of M.

Compared with the works in [12, 33], we do *not* assume boundedness of sectional curvature in Theorem 1.1.

Our method can also be used to prove the following vanishing result for $\widehat{A}$-genus.

Theorem 1.3 *Let M be a $4m$-dimensional compact spin manifold with infinite fundamental group. If M admits a sequence of Riemannian metrics with almost nonnegative Ricci curvature, then its $\widehat{A}$-genus vanishes.*

Lott [26] conjectured that a compact spin manifold M with *almost nonnegative sectional curvature* has vanishing $\widehat{A}$-genus. Theorem 1.3 weakens the assumption from almost nonnegative sectional curvature to almost nonnegative Ricci curvature under the additional assumption that M has infinite fundamental group. We emphasize that this additional assumption is necessary. In fact, the $K3$ surface is a simply connected spin manifold which admits a Ricci flat metric. However, its $\widehat{A}$-genus is nonzero.

As an application of Theorem 1.3, we give the following example, where all previously known obstructions to almost nonnegative Ricci curvature do not apply.

Example 1.4 Let B^8 be a Bott manifold, which is spin, simply connected and $\widehat{A}(B^8) = 1$ (c.f. Section 4 in [31]). Let $M^8 = (\mathbb{T}^2 \times S^6)\sharp B^8$. Then M^8 is a spin manifold with infinite fundamental group and

$$\widehat{A}(M^8) = \widehat{A}(\mathbb{T}^2 \times S^6) + \widehat{A}(B^8) = 1.$$

By Theorem 1.3, M^8 can not admit a sequence of Riemannian metrics with almost nonnegative Ricci curvature.

Some special cases of Theorems 1.1 and 1.3 are easy to prove: If M has infinite fundamental group and admits a Riemannian metric with nonnegative Ricci curvature, then by the Splitting Theorem in [9], some finite cover of M is diffeomorphic to a product manifold $\mathbb{T}^k \times N$ for some $k \geq 1$. It follows that the genera of M appearing in Theorems 1.1, 1.3 are zero.

On the other hand, based on a fibration theorem proved in [22], a vanishing theorem on signature is proved in [3] for Riemannian manifolds with *almost nonnegative sectional curvature* and infinite fundamental group. However, no such fibration theorem exists under almost nonnegative Ricci curvature condition, by the counterexample constructed in [1]. In order to prove our vanishing results on genera, we employ an analytic method based on the Bochner technique. Via the Atiyah–Singer index theorem, our strategy is to show that the indices of certain Dirac operators are zero. Using the method in [4], we can derive a bound of the indices of those Dirac operators. In the presence of infinite fundamental group, our crucial contribution is to show that the indices are in fact zero. In this step the multiplicity property of genera under a finite covering is used in an essential way. See Sect. 2 for the details of proof.

In [10], we also obtained several vanishing theorems for genera under different curvature assumptions. We emphasize that the methods used in the proof in [10] and this paper are different.

2 Proof of Theorems 1.1 and 1.3

In this section we prove Theorems 1.1 and 1.3.

Firstly we prove Theorem 1.1. Let M be a compact complex manifold of complex dimension n with infinite fundamental group. We are going to show that the Todd genus of M vanishes if M admits a sequence of Kähler metrics g_i satisfying

$$\mathrm{Ric}(g_i) \geq -1, \quad \mathrm{diam}(g_i) \leq \frac{1}{i}.$$

Then the fundamental group of M contains a nilpotent subgroup of finite index [5, 23]. Let M_1 be a finite cover of M such that $\pi_1(M_1)$ is nilpotent. Then we have the following algebraic lemma where the assumption that M has infinite fundamental group will be used.

Lemma 2.1 *There exists a sequence of normal subgroups of finite index*

$$\pi_1(M_1) = G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_j \supset \cdots$$

such that $\lim_{j\to+\infty}[G_1 : G_j] = \infty$.

Proof If $\pi_1(M_1)$ is free abelian, i.e. $\pi_1(M_1) \simeq \mathbb{Z}^k$ for some $k \geq 1$, we can simply take $G_j = (2^{j-1}\mathbb{Z})^k$, $j \geq 1$. In general, we will apply a theorem of Hirsch [18]. Since $\pi_1(M_1)$ is a finitely generated nilpotent group, by Hirsch's theorem (Theorem 3, page 84 in [18]), it is residually finite, i.e. the intersection of all normal subgroups of finite index is the unit element. Let G_j be the intersection of all normal subgroups of $\pi_1(M_1)$ with index $\leq j$, $j = 1, 2, \ldots$. By a theorem of Hall [17] (see Property 4, page 128 in [17]), in every finitely generated group, the number of subgroups of given finite index is finite. Then we see that G_j is a normal subgroup of $\pi_1(M_1)$ with finite index. From the definition of G_j, we have

$$\pi_1(M_1) = G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_j \supset \cdots,$$
$$\bigcap_j G_j = \{1\}.$$

Moreover, we have

$$\lim_{j\to+\infty}[G_1 : G_j] = \infty.$$

Otherwise, there exists some constant $C > 0$ such that $[G_1 : G_j] \leq C$ for all j. As $[G_1 : G_j]$ is a monotone increasing sequence taking values in $\mathbb{Z}$, there must exist some j_0 such that $[G_1 : G_j] = [G_1 : G_{j_0}]$ for all $j \geq j_0$. It turns out that $G_j = G_{j_0}$ for each $j \geq j_0$. Hence

$\bigcap_j G_j = G_{j_0}$. Since $\bigcap_j G_j = \{1\}$, then $G_{j_0} = \{1\}$. As $[G_1 : G_{j_0}] < \infty$, it follows that $\pi_1(M_1) = G_1$ is a finite group, which contradicts that M has infinite fundamental group. □

The Dolbeault cohomology groups of a compact Kähler manifold M satisfy dim $H^{0,p}(M, \mathbb{C}) = \dim H^{p,0}(M, \mathbb{C})$ for any p. Let

$$D_i = \partial + \partial^* : \oplus_k \wedge^{2k,0}(M) \to \oplus_k \wedge^{2k+1,0}(M)$$

be the first order elliptic operators on M determined by g_i. Then the Todd genus of M is equal to the index of D_i for any i by Hodge theory. Denote $\widehat{g_i}^j$, $\widehat{D_i}^j$ be the pulling backs of g_i, D_i to M_j, where M_j is the finite covering of M_1 with $\pi_1(M_j) = G_j$. Then we can compare the diameter of $\widehat{g_i^j}$ with that of $\widehat{g_i^1}$:

Lemma 2.2 (Ivanov [20])

$$\operatorname{diam}(\widehat{g_i^j}) \leq [G_1 : G_j] \operatorname{diam}(\widehat{g_i^1}).$$

Proof We include the proof given by Sergei Ivanov [20] on Mathoverflow for completeness. Denote $[G_1 : G_j] = N$. Suppose the contrary, let $\gamma^j : [0, \ell] \to M^j$ be a geodesic realizing the diameter of $(M^j, \widehat{g_i^j})$ with $\ell > N \operatorname{diam}(\widehat{g_i^1})$. Let γ be the image of γ^j in M_1 under the canonical projection. We break γ into N shorter geodesics of equal length ℓ/N. Namely

$$\gamma := a_1 a_2 \cdots a_N.$$

Since $\ell/N > \operatorname{diam}(\widehat{g_i^1})$, there must exist a strictly shorter geodesic b_i connecting the two ends of a_i for each $i \in \{1, \cdots, N\}$. Set $\bar{p} = \gamma^j(0)$, $p = \gamma(0)$ and $\bar{q} = \gamma^j(\ell)$, $q = \gamma(\ell)$. Let H be the subgroup of $\pi_1(M_1, p)$ whose elements(loops) lift to closed loops in M_j. Clearly $[\pi_1(M_1, p) : H] = N$. Now consider the following $(N+1)$ loops based at p:

$$s_0 = e, s_1 = b_1 a_1^{-1}, s_2 = b_1 b_2 (a_1 a_2)^{-1}, \cdots, s_N = b_1 \cdots b_N (a_1 \cdots a_N)^{-1}.$$

There must exist two elements s_k and s_r with $k < r$ such that $(s_k)^{-1} s_r \in H$. Now the path $(s_k)^{-1} s_r \gamma$ still lifts p and q to $\bar{p}$ and $\bar{q}$, yet it is strictly shorter than γ^j, a contradiction. □

Now we derive a bound of the index of $\widehat{D_i}^j$.

Lemma 2.3 *There exists a positive constant $C(n)$ depending only on the dimension n such that for each i, j, we have*

$$|\operatorname{ind}(\widehat{D_i}^j)| \leq C(n).$$

Remark 2.4 It is essential for us that the constant $C(n)$ is independent of i, j.

Proof We will prove Lemma 2.3 using the method in [4]. It is based on the following lemmas.

Lemma 2.5 *For a smooth section s of an Euclidean vector bundle E over a Riemannian manifold (M, g), let*

$$\|s\|_\infty = \sup\{|s|_x : x \in M\},$$
$$\|s\|_2^2 = \int_M |s|_x^2 dV_g,$$
$$L(s) = \frac{V_g \|s\|_\infty^2}{\|s\|_2^2},$$

where V_g is the volume of (M, g). Given a finite dimensional subspace F of $C^\infty(E)$, we have

$$\dim F \leq l \sup\{L(s) : s \in F - 0\},$$

where l is the rank of E.

Lemma 2.5 is essentially due to Peter Li (Lemma 11 in [24]). We refer the readers to [4], page 387 for its proof.

Lemma 2.6 *Let (M, g) be a closed m-dimensional smooth Riemannian manifold such that for some constant $b > 0$,*

$$r_{\min}(g)(\mathrm{diam}(g))^2 \geq -(m-1)b^2.$$

If $f \in W^{1,2}(M)$ is a nonnegative continuous function such that $f\Delta f \geq -hf^2$ (here Δ is a negative operator) in the sense of distribution for some nonnegative continuous function h then

$$\max_{x \in M} |f|^2(x) \leq C(m, p, R, \Lambda) \frac{\int_M f^2 dV}{V_g},$$

where $C(m, p, R, \Lambda)$ is some constant depending only on m, p, $R = \frac{\mathrm{diam}(g)}{bC(b)}$ and

$$\Lambda = \frac{\int_M h^p dV}{V_g}, \; p > \frac{m}{2}.$$

We refer the readers to see Theorem 3.3 in [10] or Theorem 3, page 395 in [4] for the proof of Lemma 2.6. Several notations in the above lemma need to be clarified here:

(1) $r_{\min}(g) = \inf\{\mathrm{Ric}(g)(u, u) : u \in TM, g(u, u) = 1\}$.
(2) V_g is the volume of (M, g).
(3) $C(b)$ is the unique positive root of the equation

$$x \int_0^b (cht + xsht)^{m-1} dt = \int_0^\pi \sin^{m-1} t dt.$$

The following explicit expression of the constant $C(m, p, R, \Lambda)$ derived in [10] is important for us (see Theorem 3.3 and its proof in [10]). Let $v = \frac{m}{2}$ if $m > 2$ and $1 < v < p$ be arbitrary for $m = 2$. Let μ be the conjugate of v such that

$$\frac{1}{v} + \frac{1}{\mu} = 1.$$

Then

$$C(m, p, R, \Lambda) = \mu^{2K_1 \frac{p(\mu-1)}{\mu(p-1)-p}} B^{2K_2} \tag{2.1}$$

$$B = C(m, p) \Lambda^{\frac{1}{2} \frac{\mu-1}{\mu(p-1)-p}} R^{\frac{p(\mu-1)}{\mu(p-1)-p}} + 2 \tag{2.2}$$

$$K_1 = \sum i \mu^{-i}, \; K_2 = \sum \mu^{-i}, \tag{2.3}$$

where $C(m, p)$ is some positive constant depending only on m, p.

Now we apply Lemmas 2.5 and 2.6 to prove Lemma 2.3, namely,

$$|\mathrm{ind}(\widehat{D_i}^j)| \leq C(n), \; n = \dim_{\mathbb{C}} M.$$

Let $E_j = \oplus_k \wedge^{2k,0}(M_j)$, $F_{i,j} = \ker(\widehat{D_i}^j)$. Then for any $s \in \ker(\widehat{D_i}^j)$, applying the Bochner formula to s, we get

$$\frac{1}{2}\Delta|s|^2 = |\nabla s|^2 + \langle \mathcal{F}s, s\rangle.$$

As g_i is a Kähler metric for each i, it is crucial here that the curvature term $\langle \mathcal{F}s, s\rangle$ is controlled by the Ricci curvature, see for example Page 84 in [2], for the general, see Chapter 7, Section 3.2 in [30]. By assumption, we have

$$\mathrm{Ric}(g_i) \geq -1,\ \mathrm{diam}(g_i) \leq \frac{1}{i}.$$

Then $\mathrm{Ric}(\widehat{g_i}^j) \geq -1$ and there exists some positive constant $C(n)$ depending only on n such that

$$\frac{1}{2}\Delta|s|^2 \geq |\nabla s|^2 - C(n)|s|^2.$$

By Kato's inequality (c.f. page 380 in [4]), we have

$$|\nabla s| \geq |\nabla |s||.$$

Then we get

$$|s|\Delta|s| \geq -C(n)|s|^2.$$

Let $G_j = \pi_1(M_j)$, $G_0 = \pi_1(M)$. By Lemma 2.2, we get

$$\mathrm{diam}(\widehat{g_i}^j) \leq [G_1 : G_j]\,\mathrm{diam}(\widehat{g_i}^1) \leq [G_1 : G_j]\,[G_0 : G_1]\,\mathrm{diam}(g_i).$$

As $\mathrm{diam}(g_i) \leq \frac{1}{i}$, we get

$$\mathrm{diam}(\widehat{g_i}^j) \leq \frac{[G_1 : G_j]\,[G_0 : G_1]}{i}.$$

For a fixed j, then

$$\lim_{i\to\infty} \mathrm{diam}(\widehat{g_i}^j) = 0.$$

Applying Lemma 2.6 to $|s|$ for any $p > n$ (for example take $p = n + 1$), we get

$$|s|_\infty^2 =: \max_{x\in M} |s|^2(x) \leq C(n, i, j)\frac{\int_M |s|^2 dV_i}{V(g_i)},$$

where $C(n, i, j)$ is some positive constant depending on n, i, j (which can be described explicitly via Eq. 2.1).

Applying Lemma 2.5 to $F_{i,j}$, we get

$$\dim(\ker(\widehat{D_i}^j)) \leq C(n, i, j).$$

By a similar argument applied to the adjoint operator of $\widehat{D_i}^j$, we can also show that

$$\dim(\mathrm{coker}(\widehat{D_i}^j)) \leq C'(n, i, j),$$

where $C'(n, i, j)$ is some positive constant depending on n, i, j (which can be described explicitly via Eq. 2.1).

Then

$$|\operatorname{ind}(\widehat{D_i}^j)| \le \max(C(n,i,j), C'(n,i,j)).$$

For a fixed j, we have that $\operatorname{ind}(\widehat{D_i}^j)$ is independent of i. Since $\operatorname{Ric}(\widehat{g_i}^j) \ge -1$ and $\lim_{i\to\infty} \operatorname{diam}(\widehat{g_i}^j) = 0$ for a fixed j, then we get

$$|\operatorname{ind}(\widehat{D_i}^j)| = \lim_{i\to\infty} |\operatorname{ind}(\widehat{D_i}^j)| \le C(n),$$

where $C(n)$ is some positive constant depending only on n. It is essential for us that the constant $C(n)$ is independent of i, j. In fact, for a fixed j, as $\operatorname{Ric}(\widehat{g_i}^j) \ge -1$ and $\lim_{i\to\infty} \operatorname{diam}(\widehat{g_i}^j) = 0$, then we get $\lim_{i\to\infty} R_i^j = 0$, where $R_i^j = \frac{\operatorname{diam}(\widehat{g_i}^j)}{b_i^j C(b_i^j)}$ are the constants as defined in Lemma 2.6. By Eq. (2.1), we get

$$\lim_{i\to\infty} |\operatorname{ind}(\widehat{D_i}^j)| \le C(n).$$

□

Now we are going to finish the proof of Theorem 1.1. As Todd genus is multiplicative under a finite covering [19], we have

$$|\operatorname{ind}(\widehat{D_i}^1)|\,[G_1 : G_j] = |\operatorname{ind}(\widehat{D_i}^j)|.$$

By Lemmas 2.1 and 2.3, we get

$$|\operatorname{ind}(\widehat{D_i}^j)| \le C(n), \quad \lim_{j\to\infty} [G_1 : G_j] = \infty.$$

Hence for each i, we must have

$$|\operatorname{ind}(\widehat{D_i}^1)| = 0.$$

It follows that M_1 has vanishing Todd genus, so does M.

The above argument in fact yields the following Theorem (Here we do not assume that M has infinite fundamental group).

Theorem 2.7 *There exists some constant $C(n)$ depending only n such that:*

(i) *let M be a compact n-dimensional complex manifold admitting a sequence of Kähler metrics with almost nonnegative Ricci curvature, then the Todd genus of M is bounded above by $C(n)$.*

(ii) *Furthermore, if the order of $\pi_1(M)$ is bigger than $C(n)$, then the Todd genus of M vanishes.*

Proof The first part follows from Lemma 2.3. Then it suffices to show if $\pi_1(M)$ is a finite group such that $|\pi_1(M)| > C(n)$, then the Todd genus of M vanishes. This again follows from the multiplicative property of Todd genus under a finite covering (compare the Todd genus of M with its universal covering). □

The proof of Theorem 1.3 is similar to that of Theorem 1.1. Here we look at the Atiyah–Singer spin Dirac operator D. The curvature term of the Bochner formula applied to the kernel of D is bounded from below by the scalar curvature. Then we can apply Lemma 2.5 and Lemma 2.6 to get a bound of $\hat{A}$-genus. This in fact was already proved in [14]. In the presence of infinite fundamental group, then our arguments given above can be used to show that the $\hat{A}$-genus is in fact zero.

3 Vanishing theorems on Euler number

The Euler number of a manifold is not a genus. However, certain vanishing results about Euler number have been obtained under *almost nonnegative sectional curvature* assumption. Recall that a smooth Riemannian manifold M has almost nonnegative sectional curvature if it admits a sequence of Riemannian metrics $\{g_i\}_{i\in\mathbb{N}}$ such that

$$\sec(g_i) \geq -1, \quad \mathrm{diam}(g_i) \leq \frac{1}{i},$$

where $\sec(g_i)$ denotes the sectional curvatures of g_i.

Fukaya–Yamaguchi proved that a closed manifold with almost nonnegative sectional curvature and infinite fundamental group has vanishing Euler number. Their proof is based on a fibration theorem [13] characterizing these manifolds. We are going to give a new proof of this result based on the idea developed in previous proofs. In fact our proof works for a larger class of spaces: almost nonnegatively curved Alexandrov spaces. Alexandrov spaces are generalizations of Riemannian manifolds with lower sectional curvature bound. They could have topological singularities, for example the spherical cone over $\mathbb{CP}^n$ with Fubini-Study metric is an Alexandrov space. For background of Alexandrov spaces, cf [6, 7]. Note that due to the possible singularities on Alexandrov spaces, there is no fibration theorem available for almost nonnegatively curved Alexandrov spaces. Yet, we still have:

Theorem 3.1 *Let X be an m-dimensional almost nonnegatively curved Alexandrov space with infinite fundamental group, then the Euler number of X vanishes.*

Proof By [32], the fundamental group of X contains a nilpotent subgroup of finite index. By passing to a finite cover, we can assume that $\pi_1(X)$ is nilpotent. By Lemma 2.1, there is a sequence of subgroups G_j of $\pi_1(X)$ with finite index such that $\lim_{j\to+\infty}[G_1 : G_j] = \infty$. Then $\widetilde{X}/G_j$ is a finite cover of X, where $\widetilde{X}$ is the universal covering of X. Clearly, $\widetilde{X}/G_j$ is almost nonnegatively curved by pulling back the metrics. By the extension of Gromov's Betti number estimate to Alexandrov spaces (see the main Theorem of [25] and [15]), we get that for any j,

$$\sum_{p=0}^{m} b_p(\widetilde{X}/G_j) \leq C(m).$$

This in particular implies that the absolute value of Euler number of $\widetilde{X}/G_j$ is bounded above by some constant depending only on m. As $\lim_{j\to+\infty}[G_1 : G_j] = \infty$, we see that the Euler number of X is zero by the multiplicity property of Euler number under a finite covering. □

The same idea can also be used to prove a vanishing theorem about L^2 Betti numbers under almost nonnegative sectional curvature condition:

Theorem 3.2 *Let M be a compact manifold (or more generally a compact Alexandrov space) with almost nonnegative sectional curvature and infinite fundamental group, then all the L^2 Betti numbers of its universal covering $\widetilde{M}$ vanish.*

We refer the reader to [28] for the precise definition of L^2-Betti numbers. We only recall the following two facts on L^2-Betti numbers. See pages 37, 51 and 476 in [28] for details.

Lemma 3.3 ([27]) *Let M be a finite connected CW-complex with fundamental group G. Suppose there exists a sequence of normal subgroups of finite index*

$$G = G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_j \supset \cdots$$

such that $\bigcap_j G_j = \{1\}$, *then the* p*-th* L^2 *Betti number of* $\widetilde{M}$ *is equal to*

$$\lim_{j\to+\infty} \frac{b_p(\widetilde{M}/G_j)}{[G_1 : G_j]}.$$

Lemma 3.4 *The Euler number of* M *is equal to the alternative sum of* L^2 *Betti numbers of* $\widetilde{M}$.

Proof of Theorem 3.2 By passing to a finite cover, we can assume that $\pi_1(M)$ is nilpotent. By Lemma 2.1, there is a sequence of subgroups G_j of $\pi_1(M)$ with finite index such that $\lim_{j\to+\infty}[G_1 : G_j] = \infty$. Since the Betti numbers $b_p(\widetilde{M}/G_j)$ is bounded above by some constant depending only on dim M, then we have

$$\lim_{j\to+\infty} \frac{b_p(\widetilde{M}/G_j)}{[G_1 : G_j]} = 0.$$

By Lemma 3.3, we see that the p-th L^2 Betti number of $\widetilde{M}$ vanishes. □

Remark 3.5 The above proof also works for Alexandrov spaces. In fact by [29], any compact Alexandrov space X admits a good cover, it follows that X is homotopic to the nerve of a good cover of X. Then Lück's Lemma 3.3 still applies.

Remark 3.6 It follows from Theorem 3.2 and Lemma 3.4 that the Euler number of M vanishes. Therefore this provides a second proof of Theorem 3.1 without using fibration theorem for almost nonnegatively curved manifolds.

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