# ITERATED RESIDUE, TORIC FORMS AND WITTEN GENUS 

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#### Abstract

We introduce the notion of iterated residue to study generalized Bott manifolds. When applying the iterated residues to compute the Borisov-Gunnells toric form and the Witten genus of certain toric varieties as well as complete intersections, we obtain interesting vanishing results and some theta function identities, one of which is a twisted version of a classical Rogers-Ramanujan type formula.


## 1. Introduction

The residue of a meromorphic function $f$ at an isolated singularity $a$, often denoted by $\operatorname{Res}_{a} f$, is the coefficient $a_{-1}$ of its Laurent series expansion at $a$. According to the Cauchy integral theorem, we have

$$
\operatorname{Res}_{a} f=\frac{1}{2 \pi i} \oint_{\gamma} f(z) d z
$$

where $\gamma$ is a circle small enough around $a$ in a counterclockwise manner. This residue has an interesting simple topological application on complex projective spaces. Actually, as $H^{*}\left(\mathbb{C} P^{n}, \mathbb{C}\right) \cong \mathbb{C}[z] /\left\langle z^{n+1}\right\rangle$, for any cohomology class $g$ in $H^{*}\left(\mathbb{C} P^{n}, \mathbb{C}\right)$, we have

$$
\left\langle g,\left[\mathbb{C} P^{n}\right]\right\rangle=\operatorname{Res}_{0} \frac{g}{z^{n+1}}=\frac{1}{2 \pi i} \oint_{\gamma} \frac{g}{z^{n+1}} d z
$$

For the case of multiple complex variables, there exists similar story. Let $U$ be the ball $\left\{\mathbf{z} \in \mathbb{C}^{n}:|\mathbf{z}|<\epsilon\right\}$ and $f_{1}, \cdots, f_{n} \in \mathcal{O}(\bar{U})$ be holomorphic functions in a neighborhood of the closure $\bar{U}$. Assume $f_{i}{ }^{\prime}$ 's have the origin as the isolated common zero and their Jacobian determinant is nonzero at the origin. The residue of meromorphic $n$-form

$$
\omega=\frac{g(\mathbf{z}) d z_{1} \wedge \cdots \wedge d z_{n}}{f_{1}(\mathbf{z}) \cdots f_{n}(\mathbf{z})} \quad(g \in \mathcal{O}(\bar{U}))
$$

is defined ([11]) by

$$
\begin{equation*}
\operatorname{Res}_{0} \omega=\frac{1}{(2 \pi i)^{n}} \int_{\Gamma} \omega \tag{1.1}
\end{equation*}
$$

where $\Gamma$ is the real $n$-cycle defined by $\Gamma=\left\{\mathbf{z}:\left|f_{i}(\mathbf{z})\right|<\epsilon\right\}$. Multiple residue also has nice topological applications. Let $X$ be an oriented closed manifold such that $H^{*}(X, \mathbb{C}) \cong \mathbb{C}\left[z_{1}, \cdots, z_{n}\right] /\left\langle f_{1}, \cdots, f_{n}\right\rangle$. Then for any cohomology class $g$ in $H^{*}(X, \mathbb{C})$, one has

$$
\begin{equation*}
\langle g,[X]\rangle=\operatorname{Res}_{0} \frac{g(\mathbf{z}) d z_{1} \wedge \cdots \wedge d z_{n}}{f_{1}(\mathbf{z}) \cdots f_{n}(\mathbf{z})} \tag{1.2}
\end{equation*}
$$

Typical examples of such $X$ are Grassmannian manifolds (where the ideal arises from the Landau-Ginzburg potential, c.f. [21]).

Nevertheless when $\left\{f_{i}\right\}$ 's are degenerated, namely their Jacobian determinant vanishes, the form $\omega$ does not admit a well defined residue as (1.1). In this paper, we look at the "iterated residue" (Definition (3) under this degenerate situation,

$$
\operatorname{Res}_{0}\left\{\cdots \operatorname{Res}_{0}\left\{\frac{g(\mathbf{z})}{f_{1}(\mathbf{z}) \cdots f_{n}(\mathbf{z})} d z_{1}\right\} \cdots\right\} d z_{n}
$$

and apply it to compute the Borisov-Gunnells toric form [4] and the Witten genus.
Remark 1. "Iterated residue" is similar to iterated integral in multiple variable calculus and its value usually depends on the order of variables. The "iterated residue" coincides with usual residue when $\left\{f_{i}\right\}$ 's are non-degenerated.

Our motivation for looking at the iterated residue arises from generalized Bott manifolds, which are analogous to the relation between the multiple residue (1.1) and Grassmannians.

A generalized Bott manifold $B_{n}$ (the total spaces of iterated projective bundle over projective space, which will be reviewed in Section 2) has cohomology ring

$$
H^{*}\left(B_{n}\right) \cong \mathbb{Z}\left[u_{1}, \cdots, u_{n}\right] /\left\langle f_{i}\left(u_{1}, \cdots, u_{n}\right): i=1, \cdots, n\right\rangle,
$$

where $f_{i}\left(u_{1}, \cdots, u_{n}\right)=u_{i} \prod_{j=1}^{n_{i}}\left(u_{i}+x_{i j}\right)$ with $\left\{x_{i j}\right\}$ being the formal Chern roots. We will prove that
Theorem 1.1 (Theorem 2.3). For any top cohomology class $g$, one has

$$
\left\langle g,\left[B_{n}\right]\right\rangle=\operatorname{Res}_{0}\left\{\cdots \operatorname{Res}_{0}\left\{\frac{g(\boldsymbol{u})}{f_{1}(\boldsymbol{u}) \cdots f_{n}(\boldsymbol{u})} d u_{1}\right\} \cdots\right\} d u_{n}
$$

where the order of $u_{i}$ coincides with their position in the generalized Bott tower.
Remark 2. When the iterated projective bundles are trivial, generalized Bott manifolds $B_{n}$ are simply products of projective spaces, and iterated residue reduces to multiple residue.

For the genera of complete intersections in generalized Bott manifolds, we have
Theorem 1.2 (Theorem 2.4). For a generalized Bott manifold $B_{n}$

$$
B_{n} \xrightarrow{p_{n}} B_{n-1} \xrightarrow{p_{n-1}} \cdots \longrightarrow B_{2} \xrightarrow{p_{2}} B_{1} \longrightarrow p t,
$$

let $-u_{k}$ be the first Chern class of the tautological line bundle over $B_{k}=\mathbb{C} P\left(\xi_{k} \oplus \mathbb{\mathbb { C }}\right)$ and $\left\{x_{k 1}, x_{k 2}, \cdots, x_{k n_{k}}\right\}$ the Chern roots of $\xi_{k}$. Denote by $X$ the submanifold of $B_{n}$ Poincaré dual to $a_{1} u_{1}+\cdots+a_{n} u_{n} \in H^{2}\left(B_{n} ; \mathbb{Z}\right)$. For any genus $\psi$ with the characteristic power series $Q(x)=x / f(x)$, we have
$\psi(X)=\operatorname{Res}_{0}\left\{\cdots\left\{\operatorname{Res}_{0} \frac{f\left(a_{1} u_{1}+\cdots+a_{n} u_{n}\right)}{f\left(u_{1}\right)^{n_{1}+1}} \cdot \prod_{i=1}^{n} \frac{1}{f\left(u_{i}\right)} \cdot \prod_{j=1}^{n_{i}} \frac{1}{f\left(u_{i}+x_{i j}\right)} d u_{1}\right\} \cdots\right\} d u_{n}$.
We will present two applications of above formula in this paper. One concerns the Borisov-Gunnells toric form ([4]) and the other one concerns the Witten genus [21].

Toric varieties are very important objects in both algebraic geometry and geometric topology. Generalized Bott manifolds are examples of toric varieties. In [4], given a toric variery $X$ and a "degree function" deg, Borisov and Gunnells introduced a function $f_{N, \operatorname{deg}}$ on the upper half plane $\mathbb{H}$, called toric form, and proved that it is a modular
form when deg satisfies certain natural condition. In Section 3, applying our iterated residue formula to the toric forms of some concrete generalized Bott manifolds (Hirzebruch surfaces and certain 4 -folds), we obtain interesting theta function identities. Let $\theta(z, \tau)$ denote the classical Jacobi theta function (c.f. [7])

$$
\theta(z, \tau)=2 q^{1 / 8} \sin (\pi z) \prod_{j=1}^{\infty}\left[\left(1-q^{j}\right)\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\right] .
$$

Theorem 1.3 (Proposition (3.1)). When $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4} \notin \mathbb{Z}$ are complex numbers, one has

$$
\begin{align*}
& \sum_{a, b \in \mathbb{Z}} \frac{\left(1-e^{2 \pi i\left(\alpha_{3}+\alpha_{4}\right)}\right)\left(1-e^{2 \pi i\left(\alpha_{1}+\alpha_{2}\right)} q^{k a}\right)}{\left(1-e^{2 \pi i \alpha_{4}} q^{a}\right)\left(1-e^{2 \pi i \alpha_{1}} q^{b}\right)\left(1-e^{2 \pi i \alpha_{3}} q^{-a}\right)\left(1-e^{2 \pi i \alpha_{2}} q^{k a-b}\right)}  \tag{1.3}\\
= & \frac{1}{(2 \pi i)^{2}}\left\{\left(\frac{\theta^{\prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}+\frac{\theta^{\prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{4}\right)}\right)\left(\frac{\theta^{\prime}\left(-\alpha_{1}\right)}{\theta\left(-\alpha_{1}\right)}+\frac{\theta^{\prime}\left(-\alpha_{2}\right)}{\theta\left(-\alpha_{2}\right)}\right)+\frac{k}{2}\left(\frac{\theta^{\prime \prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}-\frac{\theta^{\prime \prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{4}\right)}\right)\right\} .
\end{align*}
$$

Theorem 1.4 (Proposition (3.2)). When $j, k \in \mathbb{Z}$, one has

$$
\begin{align*}
& (2 \pi i)^{4} \cdot \sum_{a, b, c, d \in \mathbb{Z}} \frac{2\left(1+q^{j c+k d}\right)}{\left(1+q^{a}\right)\left(1+q^{b}\right)\left(1+q^{c}\right)\left(1+q^{d}\right)\left(1+q^{j c+k d-a-b}\right)\left(1+q^{-c-d}\right)}  \tag{1.4}\\
= & \left\{\left(\frac{3 \theta^{\prime \prime}\left(\frac{1}{2}\right)}{2 \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(3)}(0)}{2 \theta^{\prime}(0)}\right)^{2}+\left(j^{2}+k^{2}-j k\right)\left(\frac{\theta^{\prime \prime 2}\left(\frac{1}{2}\right)}{4 \theta^{2}\left(\frac{1}{2}\right)}+\frac{5 \theta^{(4)}\left(\frac{1}{2}\right)}{24 \theta\left(\frac{1}{2}\right)}-\frac{7 \theta^{(3)}(0) \theta^{\prime \prime}\left(\frac{1}{2}\right)}{12 \theta^{\prime}(0) \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(5)}(0)}{24 \theta^{\prime}(0)}+\frac{\left(\theta^{(3)}(0)\right)^{2}}{6 \theta^{\prime 2}(0)}\right)\right\} .
\end{align*}
$$

In the special case $j=k=0$, this formula actually gives the classical Rogers-Ramanujan type formula (c.f. [2, 3])

$$
\begin{equation*}
\sum_{m, n \geq 1} \frac{q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(1+q^{m+n}\right)}=\sum_{r \geq 1} \sigma_{1}(r) q^{2 r} \tag{1.5}
\end{equation*}
$$

which was obtained by Borisov and Libgober using toric form [5]. In Section 3.1, we will review the proof. Our formula (1.4) can be viewed as a twisted version of the classical formula (1.5), see Section 3.3 for details.
Concerning toric forms, using iterated residue formula, we are also able to obtain the following vanishing result.

Theorem 1.5 (Theorem 3.3). Consider toric variety $V:=\mathbb{C} P\left(\eta^{\otimes i_{1}} \oplus \eta^{\otimes i_{2}} \oplus \eta^{\otimes i_{3}} \oplus \mathbb{C}\right)$ over $\mathbb{C} P^{n_{1}}$, where $\eta$ denote the tautological bundle of $\mathbb{C} P^{n_{1}}$. Assume $\left(i_{1}, i_{2}, i_{3}\right)$ are coprime, $\left(i_{1}, i_{1}-i_{2}, i_{1}-i_{3}\right),\left(i_{2}, i_{2}-i_{1}, i_{2}-i_{3}\right),\left(i_{3}, i_{3}-i_{1}, i_{3}-i_{2}\right)$ are coprime respectively, $\sum_{j=0}^{3} \alpha_{n_{1}+2+j} \in \mathbb{Z}, \sum_{i=1}^{n_{1}+1} \alpha_{i}-\sum_{j=1}^{3} i_{j} \alpha_{n_{1}+2+j} \in \mathbb{Z}$, then the toric form $f_{N, \operatorname{deg}}(q)=0$.
Remark 3. The coprime condition seems to be a bit strict. We will provide an example to explain it. Let $\left(i_{1}, i_{2}, i_{3}\right)=(1,3,4)$, we can check that $(1,3,4),(1,2,3),(3,2,1)$ and $(4,3,1)$ are coprime respectively. It is possible to transfer iterated residue into residues at simple poles by residue theorem when coprime condition hold.

Another application is about Witten genus, which plays an important role in index theory. Let $M$ be a $4 m$ dimensional compact oriented smooth manifold. Let
$\left\{ \pm 2 \pi \sqrt{-1} z_{j}, 1 \leq j \leq 2 m\right\}$ denote the formal Chern roots of $T_{\mathbb{C}} M$, the complexification of the tangent vector bundle $T M$ of $M$. Then the famous Witten genus of $M$ can be written as (c.f. [17])

$$
\varphi_{W}(M)=\left\langle\left(\prod_{j=1}^{2 m} z_{j} \frac{\theta^{\prime}(0, \tau)}{\theta\left(z_{j}, \tau\right)}\right),[M]\right\rangle \in \mathbb{Q}[[q]],
$$

with $\tau \in \mathbb{H}$, the upper half-plane, and $q=e^{\pi \sqrt{-1} \tau}$. The Witten genus was first introduced in [21] and can be viewed as the loop space analogue of the $\widehat{A}$-genus. It can be expressed as a $q$-deformed $\widehat{A}$-genus as

$$
\varphi_{W}(M)=\left\langle\widehat{A}(T M) \operatorname{ch}\left(\Theta\left(T_{\mathbb{C}} M\right)\right),[M]\right\rangle,
$$

where

$$
\Theta\left(T_{\mathbb{C}} M\right)={\underset{n=1}{\infty} S_{q^{2 n}}\left(\widetilde{T_{\mathbb{C}} M}\right), \text { with } \widetilde{T_{\mathbb{C}} M}=T_{\mathbb{C}} M-\mathbb{C}^{4 m}, \text {, }, ~}
$$

is the Witten bundle defined in [21]. When the manifold $M$ is Spin, according to the Atiyah-Singer index theorem [1], the Witten genus can be expressed analytically as the index of twisted Dirac operators, $\varphi_{W}(M)=\operatorname{ind}\left(D \otimes \Theta\left(T_{\mathbb{C}} M\right)\right) \in \mathbb{Z}[[q]]$, where $D$ is the Atiyah-Singer Spin Dirac operator on $M$ (c.f. [13]). Moreover, if $M$ is String, i.e. $\frac{1}{2} p_{1}(T M)=0$, or even weaker, if $M$ is Spin and the first rational Pontryagin class of $M$ vanishes, then $\varphi_{W}(M)$ is a modular form of weight $2 k$ over $S L(2, \mathbb{Z})$ with integral Fourier expansion (c.f. [22]). The homotopy theoretical refinements of the Witten genus on String manifolds leads to the theory of $\operatorname{tmf}$ (topological modular form) developed by Hopkins and Miller [14]. The String condition is the orientablity condition for this generalized cohomology theory.

The Lichnerowicz theorem[16] asserts that if a closed spin manifold carries a Riemannian metric of positive scalar curvature, then its $\widehat{A}$ genus vanishes. Along this line, Stolz conjectured [19] that the Witten genus of a string manifold carrying positive Ricci curvature metric vanishes. There are two kinds of vanishing results to support the Stolz's conjecture. One is the theorem asserting that every string manifold carrying a nontrivial action of a semi-simple Lie group $G$ has vanishing Witten genus ([10], [18]). The other is the Landweber-Stong vanishing theorem (c.f. page 89 in [13]) asserting that a string complete intersection in complex projective spaces has vanishing Witten genus. The proof uses the calculation of residues. The Landweber-Stong type vanishing results were also obtained for the following objects: string complete intersections in products of complex projective spaces ([8], [9]), string complete intersections in products of Grassmannians and flag manifolds ([25, 26]).

In this paper, by applying the iterated residue, we can prove a Landweber-Stong type vanishing theorem for the Witten genus complete intersections in two-staged generalized Bott manifolds.

Theorem 1.6 (Theorem 4.1). For a string complete intersection $H_{n_{1}, 3}^{\mathbf{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)$, if $\mathbf{I}=$ $(i, j, k)$ are coprime, $(i, i-j, i-k),(j, j-i, j-k),(k, k-i, k-j)$ are coprime respectively, then the Witten genus

$$
\varphi_{W}\left(H_{n_{1}, 3}^{\mathbf{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)\right)=0
$$

Furthermore, it is also possible to calculate the mod 2 Witten genus of generalized Bott manifolds by making use of Rokhlin congruence formula in [23, 24]. We will study this in a forthcoming project.

The paper is organised as follow. In Section 2, we introduce the concept of "iterated residue" and give an explicit expression of the genus of complete intersections in generalized Bott manifolds. In Section 3, we apply "iterated residue" to toric forms and get interesting theta function identities. In Section 4, we will use "iterated residue" to discuss when Witten genus of complete intersections in generalized Bott manifolds vanishes and give the proof of Theorem 1.6.

## 2. Iterated residue in generalized Bott manifolds

In the study of symmetric spaces, Bott and Samelson [6] first introduced a family of toric manifolds obtained as the total spaces of iterated bundles over $\mathbb{C} P^{1}$ with fibre $\mathbb{C} P^{1}$. Grossberg and Karshon [12] showed that these manifolds form an important family of smooth projective toric varieties, and called them Bott manifolds.

Definition 1. A generalized Bott tower of height $n$ is a tower of projective bundles

$$
B_{n} \xrightarrow{p_{n}} B_{n-1} \xrightarrow{p_{n-1}} \cdots \longrightarrow B_{2} \xrightarrow{p_{2}} B_{1} \longrightarrow p t,
$$

where $B_{1}=\mathbb{C} P^{n_{1}}$ and each $B_{k}$ is the complex projectivisation of sum of $n_{k}$ complex line bundles and one trivial line bundle over $B_{k-1}$. The fibre of the bundle $p_{k}: B_{k} \longrightarrow B_{k-1}$ is $\mathbb{C} P^{n_{k}}$.

The last stage $B_{n}$ in a generalized Bott tower is called generalized Bott manifold.
The topology of generalized Bott manifolds is very clear. Its cohomology ring relies on the following result:

Theorem 2.1. (c.f. [20, Chapter V]) Let $\xi$ be a complex $n$-dimensional vector bundle over a finite cell complex $X$, the complex projectivisation of $\xi$ is $\mathbb{C} P(\xi)$. Let $-u \in H^{2}(\mathbb{C} P(\xi))$ be the first Chern class of the tautological line bundle over $\mathbb{C} P(\xi) . H^{*}(\mathbb{C} P(\xi) ; \mathbb{Z})$ is the quotient of polynomial ring $H^{*}(X)$ [u] by single relation

$$
u^{n}+c_{1}(\xi) u^{n-1}+\cdots+c_{n}(\xi)=0 .
$$

Corollary 1. Let $-u_{k}$ be the first Chern class of tautological line bundle $\eta_{k}$ over $B_{k}=\mathbb{C} P\left(\xi_{k} \oplus\right.$ $\mathbb{C})$, suppose the formal Chern roots of $\xi_{k}$ are $\left\{x_{k 1}, x_{k 2}, \cdots, x_{k_{k} k}\right\}$. Then

$$
H^{*}\left(B_{n}\right) \cong \mathbb{Z}\left[u_{1}, \cdots, u_{n}\right] /\left\langle f_{i}\left(u_{1}, \cdots, u_{n}\right): i=1, \cdots, n\right\rangle,
$$

where $f_{i}\left(u_{1}, \cdots, u_{n}\right)=u_{i} \prod_{j=1}^{n_{i}}\left(u_{i}+x_{i j}\right)$.
Moreover, its tangential bundle is clear,

$$
T B_{k+1} \oplus \underline{\mathbb{C}} \cong p^{*}\left(T B_{k}\right) \oplus\left(\bar{\eta}_{k} \otimes p^{*}\left(\xi_{k} \oplus \underline{\mathbb{C}}\right)\right) .
$$

The generalized Bott manifolds provide a rich family of toric varieties on which we can calculate all kinds of characteristic numbers and study their geometry and topology.

In [15], the authors introduced the concept of twisted Milnor hypersurfaces, which is a generalization of Milnor hypersurfaces (Milnor hypersurfaces can be used as the
representatives of unitary bordism). Consider the projective bundle over $\mathbb{C} P^{n_{1}}$ with fiber $\mathbb{C} P^{n_{2}}$, i.e.

$$
V=\mathbb{C} P\left(\eta^{\otimes i_{1}} \oplus \cdots \oplus \eta^{\otimes i_{n_{2}}} \oplus \underline{\mathbb{C}}\right) \rightarrow \mathbb{C} P^{n_{1}}
$$

where $\eta$ is the tautological line bundle over $\mathbb{C} P^{n_{1}}$ and $\mathbb{C}$ is the trivial line bundle.
Let $\gamma$ be the vertical tautological line bundle over $V, u=c_{1}(\bar{\eta}), v=c_{1}(\bar{\gamma}) \in H^{2}(V ; \mathbb{Z})$. Denote $\mathbf{I}=\left(i_{1}, \cdots, i_{n_{2}}\right)$ be the index.
Definition 2. We call the smooth hypersurface Poincare dual to $d_{1} u+d_{2} v$ in $V$ twisted Milnor hypersurface, denoted by $H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)$.
Remark 4. We can also consider hypersurfaces or complete intersections on generalized Bott manifolds, such as $H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)$ denote the complete intersection Poincaré dual to ( $d_{1} u+$ $\left.d_{2} v\right) \cdot\left(d_{3} u+d_{4} v\right)$ in $V$.

As we all know, generalized Bott manifolds are toric varieties, but their complete intersections are not necessarily algebraic.

Denote $X$ be the submanifold of $B_{n}$ Poincaré dual to $a_{1} u_{1}+\cdots+a_{n} u_{n} \in H^{2}\left(B_{n} ; \mathbb{Z}\right)$. Suppose $\nu$ denotes the normal bundle of inclusion $i: X \hookrightarrow B_{n}$. Then $c_{1}(\nu)=i^{*}\left(a_{1} u_{1}+\cdots+a_{n} u_{n}\right)$, while $i^{*}\left(T B_{n}\right) \cong T X \oplus \nu$.

For any genus $\psi$ with the charateristic power series $Q(x)=x / f(x)$, we have

$$
\begin{equation*}
\psi(X)=\left\langle\left(\frac{u_{1}}{f\left(u_{1}\right)}\right)^{n_{1}+1} \cdot f\left(a_{1} u_{1}+\cdots+a_{n} u_{n}\right) \cdot \prod_{i=1}^{n} \frac{u_{i}}{f\left(u_{i}\right)} \cdot \prod_{j=1}^{n_{i}} \frac{u_{i}+x_{i j}}{f\left(u_{i}+x_{i j}\right)},\left[B_{n}\right]\right\rangle . \tag{2.1}
\end{equation*}
$$

If $B_{n}$ is a product of projective spaces, $\psi(X)$ can be simplified into a very neat expression. As to general $B_{n}$, the idea $f_{i}{ }^{\prime}$ s in $H^{*}\left(B_{n} ; \mathbb{Z}\right)$ will be the biggest obstacle in calculating genus, so we have to cope with the relation between $u_{i}$ 's carefully.

Motivated by the idea of Witten [21], we try to reduce $\psi(X)$ to multiple residue. Recall the global residue theorem
Theorem 2.2. (Global Residue Theorem [11, Chapter 5]) Let $M$ be a compact complex $n$-manifold, $D=D_{1}+\cdots+D_{n}$ be a divisor on $M$ such that $D_{1}, \cdots, D_{n}$ are effective divisors on $M$ and the intersection $D_{1} \cap \cdots \cap D_{n}$ is discrete, hence finite-set of points in $M$. Then for any $\omega \in H^{0}\left(M, \Omega^{n}(D)\right)$,

$$
\sum_{P \in\left\{D_{1} \cap \cdots \cap D_{n}\right\}} \operatorname{Res}_{P} \omega=0 .
$$

Global Residue Theorem does not apply to generalized Bott manifolds, since the ideas $f_{i}$ 's are not necessarily non-degenerated, thus multiple residue at point $(0, \cdots, 0)$ can not be well defined.

To overcome this obstacle, we introduce the concept "iterated residue".
Definition 3. Let $U$ be the ball $\left\{\boldsymbol{z} \in \mathbb{C}^{n}:|\boldsymbol{z}|<\epsilon\right\}$ and $f_{1}, \cdots, f_{m} \in \mathcal{O}(\bar{U})$ be functions holomorphic in a neighborhood of the closure $\bar{U}$ of $U$. Then "iterated residue" of a meromorphic $n$-form

$$
\omega=\frac{g(\boldsymbol{z}) d z_{1} \wedge \cdots \wedge d z_{n}}{f_{1}(\boldsymbol{z}) \cdots f_{m}(\boldsymbol{z})} \quad(g \in \mathcal{O}(\bar{U}))
$$

is defined by

$$
\operatorname{Res}_{0}\left\{\cdots \operatorname{Res}_{0}\left\{\frac{g(\boldsymbol{z})}{f_{1}(\boldsymbol{z}) \cdots f_{m}(\boldsymbol{z})} d z_{1}\right\} \cdots\right\} d z_{n}
$$

Remark 5. "Iterated residue" is similar to iterated integral in multiple calculus, its value depends on the order of variables. Of course, "iterated residue" coincides with usual residue when $f_{i}(z)$ 's are non-degenerated.
"Iterated residue" admits a topological interpretation via characteristics of generalized Bott manifolds.

Theorem 2.3. For any $F \in H^{n_{1}+\cdots+n_{n}}\left(B_{n} ; \mathbb{Z}\right)$, we have

$$
\left\langle F,\left[B_{n}\right]\right\rangle=\operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \cdots\left\{\operatorname{Res}_{0} \frac{F}{\prod_{j=1}^{n_{i}} f_{j}\left(u_{1}, \cdots, u_{n}\right)} d u_{1}\right\} \cdots d u_{n-1}\right\} d u_{n}
$$

where the order of $u_{i}$ coincides with their position in the generalized Bott tower.
Proof. Since $\left\langle F,\left[B_{n}\right]\right\rangle$ and $\operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \cdots\left\{\operatorname{Res}_{0} \frac{F}{\prod_{j=1}^{n_{i} f_{j}\left(u_{1}, \cdots, u_{n}\right)}} d u_{1}\right\} \cdots d u_{n-1}\right\} d u_{n}$ are both linear forms, $H^{n_{1}+\cdots+n_{n}}\left(B_{n} ; \mathbb{Z}\right) \cong \mathbb{Z}\left\langle u_{1}^{n_{1}} u_{2}^{n_{2}} \cdots u_{n}^{n_{n}}\right\rangle$, it remains to prove

$$
\left\langle u_{1}^{n_{1}} u_{2}^{n_{2}} \cdots u_{n}^{n_{n}},\left[B_{n}\right]\right\rangle=\operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \cdots\left\{\operatorname{Res}_{0} \frac{u_{1}^{n_{1}} u_{2}^{n_{2}} \cdots u_{n}^{n_{n}}}{\prod_{j=1}^{n_{i}} f_{j}\left(u_{1}, \cdots, u_{n}\right)} d u_{1}\right\} \cdots d u_{n-1}\right\} d u_{n} .
$$

Obviously, $\left\langle u_{1}^{n_{1}} u_{2}^{n_{2}} \cdots u_{n}^{n_{n}},\left[B_{n}\right]\right\rangle=1$, while

$$
\begin{aligned}
& \operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \cdots\left\{\operatorname{Res}_{0} \frac{u_{1}^{n_{1}} u_{2}^{n_{2}} \cdots u_{n}^{n_{n}}}{\prod_{j=1}^{n_{i}} f_{j}\left(u_{1}, \cdots, u_{n}\right)} d u_{1}\right\} \cdots d u_{n-1}\right\} d u_{n} \\
= & \operatorname{Res}_{0}\left\{\cdots\left\{\operatorname{Res}_{0} \frac{u_{2}^{n_{2}} \cdots u_{n}^{n_{n}}}{\prod_{j=2}^{n_{i}} f_{j}\left(u_{2}, \cdots, u_{n}\right)} d u_{2}\right\} \cdots\right\} d u_{n} \\
= & 1 .
\end{aligned}
$$

For a generalized Bott manifold $B_{n}$

$$
B_{n} \xrightarrow{p_{n}} B_{n-1} \xrightarrow{p_{n-1}} \cdots \longrightarrow B_{2} \xrightarrow{p_{2}} B_{1} \longrightarrow \mathrm{pt} .
$$

Let $-u_{k}$ be the first Chern class of the tautological line bundle over $B_{k}=\mathbb{C} P\left(\xi_{k} \oplus \mathbb{C}\right)$ over $B_{k-1}$, the Chern roots of $\xi_{k}$ be $\left\{x_{k 1}, x_{k 2}, \cdots, x_{k_{k} k}\right\}$. Denote $X$ be the submanifold of $B_{n}$ Poincaré dual to $a_{1} u_{1}+\cdots+a_{n} u_{n} \in H^{2}\left(B_{n} ; \mathbb{Z}\right)$.

Applying Theorem [2.3 to formula (2.1), we get our main result
Theorem 2.4. For any genus $\psi$ with the charateristic power series $Q(x)=x / f(x)$, we have (2.2)

$$
\psi(X)=\operatorname{Res}_{0}\left\{\cdots\left\{\operatorname{Res}_{0} \frac{f\left(a_{1} u_{1}+\cdots+a_{n} u_{n}\right)}{f\left(u_{1}\right)^{n_{1}+1}} \cdot \prod_{i=1}^{n} \frac{1}{f\left(u_{i}\right)} \cdot \prod_{j=1}^{n_{i}} \frac{1}{f\left(u_{i}+x_{i j}\right)} d u_{1}\right\} \cdots\right\} d u_{n} .
$$

Formula (2.2) kills the ideals in $H^{*}\left(B_{n} ; \mathbb{Z}\right)$ and only $f$ is left. If power series $f$ admits good property, it is possible to give explicit formula of $\psi(X)$.

## 3. Borisov-Gunnellels Toric forms: theta functions identities and VANISHING RESULTS

Toric variety is a very important object in both algebraic geometry and geometric topology. Borisov and Libgober[5] gave the explicit formula of elliptic genus of a toric variety by its combinatorial data, and showed that elliptic genera of a Calabi-Yau hypersurface in a toric variety and its mirror coincide up to sign.
In [4], Borisov and Gunnells defined algebraic toric form which is motivated from the expression of normalized ellptic genus of toric variety[5].

Let $N \in \mathbb{R}^{r}$ be a lattice, $M$ be its dual lattice. For a complete rational polyhedral fan $\Sigma \subset N \otimes \mathbb{R}$. A degree function $\operatorname{deg}: N \longrightarrow \mathbb{C}$ is a piecewise linear function on the cones of $\Sigma$.

For every cone $C \in \Sigma$, they defined a map

$$
f_{C}: \mathbb{H} \times M_{\mathbb{C}} \longrightarrow \mathbb{C}
$$

as follows. Write $q=e^{2 \pi i \tau}, \tau \in \mathbb{H}$, the upper halfplane. If $m \in M_{\mathbb{C}}$ satisfies

$$
m \cdot(C \backslash\{0\})>0,
$$

for all $\tau$ with sufficiently large imaginart part, they set

$$
f_{C}(q, m):=\sum_{n \in C \cap N} q^{m \cdot n} e^{2 \pi i \operatorname{deg}(n)} .
$$

The toric form associated to ( $N, \mathrm{deg}$ ) is the function $f_{N, \text { deg }}: \mathbb{H} \longrightarrow \mathbb{C}$ defined by

$$
f_{N, \operatorname{deg}}(q):=\sum_{m \in M}\left(\sum_{C \in \Sigma}(-1)^{\left.\operatorname{codim} C_{a . c .}\left(\sum_{n \in C} q^{m \cdot n} e^{2 \pi i \operatorname{deg}(n)}\right)\right),, ~}\right.
$$

here "a.c." denotes analytic continuation of $f_{C}$.
The definition of $f_{N, \text { deg }}$ is well-defined. More precisely, there exists $\epsilon>0$ such that the sum over $M$ converges absolutely and uniformly for all $|q|<\epsilon$.
Remark 6. Suppose that $\operatorname{deg}(d)=1 / 2$ for all generators $d$ of one-dimensional cones of $\Sigma$, and toric variety $X$ associated to $\Sigma$ is nonsingular. Then the function $f_{N, \mathrm{deg}}(q)$ is the normalized elliptic genus of $X$. This example is the main motivation of toric forms [5].

Furthermore they proved that the toric forms are modular forms under certain conditions on deg.

Theorem 3.1 (Borisov, Gunnells). Suppose deg fucntion takes values in $\frac{1}{l} \mathbb{Z}$, and $\operatorname{deg}$ is not integral valued on the primitive generator of any 1-cone of $\Sigma$. Then toric form $f_{N, \operatorname{deg}}(q)$ is holomorphic modular form of weight $r$ for the congruence subgroup $\Gamma_{1}(l)$.

Borisov and Gunnells gave a topological interpretation of toric forms by Hirzebruch-Riemann-Roch theorem. Let $\left\{d_{i}\right\}$ be the set of primitive generator of 1-cone of $\Sigma, X$ be toric variety associated to $\Sigma$, and for each $d_{i}, D_{i} \subset X$ be the corresponding toric divisor. In the following, we abuse $D_{i}$ to mean either the divisor or its cohomology class. Recall that

$$
\theta(z, \tau)=2 q^{1 / 8} \sin (\pi z) \prod_{j=1}^{\infty}\left[\left(1-q^{j}\right)\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\right]
$$

Theorem 3.2 ( Borisov, Gunnells). Assume that the toric variety $X$ is nonsingular, and that $\alpha_{i} \notin \mathbb{Z}$ for all primitive generator of 1-cone of $\Sigma$. Then

$$
f_{N, \operatorname{deg}}(q)=\int_{X} \prod_{i} \frac{\left(D_{i} / 2 \pi i\right) \theta\left(D_{i} / 2 \pi i-\alpha_{i}\right) \theta^{\prime}(0)}{\theta\left(D_{i} / 2 \pi i\right) \theta\left(-\alpha_{i}\right)} .
$$

Our "iterated residue" can be applied to calculate above Euler characteristic for some generalized Bott manifolds, thus getting interesting theta function identities regarding to toric forms.

### 3.1. Projective plane $\mathbb{C} P^{2}$.

Let $N=\mathbb{Z}^{2}$, and $\Sigma$ be the fan in Figure 1. Then the corresponding toric variety is projective plane $\mathbb{C} P^{2}$.


Figure 1
Assume that deg takes $\alpha, \beta, \gamma$ on the generators $\mathbf{e}_{1}, \mathbf{e}_{2},-\mathbf{e}_{1}-\mathbf{e}_{2}$ respectively. Then the toric form is

$$
\begin{aligned}
& f_{N, \operatorname{deg}}(q) \\
= & \sum_{a, b \in \mathbb{Z}} \frac{1}{\left(1-e^{2 \pi i \alpha} q^{a}\right)\left(1-e^{2 \pi i \beta} q^{b}\right)}+\frac{1}{\left(1-e^{2 \pi i \beta} q^{b}\right)\left(1-e^{2 \pi i \gamma} q^{-a-b}\right)} \\
& +\frac{1}{\left(1-e^{2 \pi i \alpha} q^{a}\right)\left(1-e^{2 \pi i \gamma} q^{-a-b}\right)}-\frac{1}{1-e^{2 \pi i \alpha} q^{a}}-\frac{1}{1-e^{2 \pi i \beta} q^{b}}-\frac{1}{1-e^{2 \pi i \gamma} q^{-a-b}}+1 \\
= & \sum_{a, b \in \mathbb{Z}} \frac{1-e^{2 \pi i(\alpha+\beta+\gamma)}}{\left(1-e^{2 \pi i \alpha} q^{a}\right)\left(1-e^{2 \pi i \beta} q^{b}\right)\left(1-e^{2 \pi i \gamma} q^{-a-b}\right)} .
\end{aligned}
$$

In the case $\alpha=\beta=\gamma=\frac{1}{2}$, Borisov and Libgober [5] deduced the famous identity

$$
\sum_{m, n \geq 1} \frac{q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(1+q^{m+n}\right)}=\sum_{r \geq 1} q^{2 r} \sum_{k \mid r} k=\sum_{r \geq 1} \sigma_{1}(r) q^{2 r}
$$

The deduction is the following. By Theorem 3.2,

$$
\begin{aligned}
& \sum_{m, n \in \mathbb{Z}} \frac{2 q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(1+q^{m+n}\right)} \\
& =\int_{X} \prod_{i} \frac{\left(D_{i} / 2 \pi i\right) \theta\left(D_{i} / 2 \pi i-\frac{1}{2}\right) \theta^{\prime}(0)}{\theta\left(D_{i} / 2 \pi i\right) \theta\left(-\frac{1}{2}\right)} \\
& =\frac{1}{(2 \pi i)^{2}} \cdot \frac{\theta^{\prime 3}(0)}{\theta^{2}\left(\frac{1}{2}\right)} \cdot \operatorname{Res}_{0} \frac{1}{\theta^{3}(u)} d u \\
& =\frac{1}{(2 \pi i)^{2}}\left\{\frac{3 \theta^{\prime \prime}\left(\frac{1}{2}\right)}{2 \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(3)}(0)}{2 \theta^{\prime}(0)}\right\} .
\end{aligned}
$$

## Then we see

## LHS

$$
\begin{aligned}
& =4 \sum_{m, n \geq 1}\left\{\frac{q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(1+q^{m+n}\right)}+\frac{q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(q^{m}+q^{n}\right)}\right\}+4 \sum_{m=1}^{\infty} \frac{q^{m}}{\left(1+q^{m}\right)^{2}}+\frac{1}{4} \\
& =12 \sum_{m, n \geq 1} \frac{q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(1+q^{m+n}\right)}+6 \sum_{m=1}^{\infty} \frac{q^{m}}{\left(1+q^{m}\right)^{2}}+\frac{1}{4} .
\end{aligned}
$$

On the other side, we list the derivatives of theta function first.

$$
\begin{aligned}
\theta^{\prime}(z)= & 2 q^{1 / 8} \prod_{j=1}^{\infty}\left[\left(1-q^{j}\right)\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\right] \cdot\left\{\pi \cos (\pi z)-\sin (\pi z) \sum_{j=1}^{\infty} \frac{2 \pi i q^{j}\left(e^{2 \pi i z}-e^{-2 \pi i z}\right)}{\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)}\right\}, \\
\theta^{\prime \prime}(z)= & 2 q^{1 / 8} \prod_{j=1}^{\infty}\left[\left(1-q^{j}\right)\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\right] \cdot\left\{-2 \pi \cos (\pi z) \sum_{j=1}^{\infty} \frac{2 \pi i q^{j}\left(e^{2 \pi i z}-e^{-2 \pi i z}\right)}{\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)}\right. \\
& -\pi^{2} \sin (\pi z)+\sin (\pi z)\left(\sum_{j=1}^{\infty} \frac{2 \pi i q^{j}\left(e^{2 \pi i z}-e^{-2 \pi i z}\right)}{\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)}\right)^{2} \\
& \left.-\sin (\pi z) \sum_{j=1}^{\infty} \frac{(2 \pi i)^{2} q^{j}\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\left(e^{2 \pi i z}+e^{-2 \pi i z}\right)-\left(2 \pi i q^{j}\right)^{2}\left(e^{2 \pi i z}-e^{-2 \pi i z}\right)^{2}}{\left(1-e^{2 \pi i z} q^{j}\right)^{2}\left(1-e^{-2 \pi i z} q^{j}\right)^{2}}\right\} .
\end{aligned}
$$

Then

## RHS

$$
\begin{aligned}
& =\frac{1}{(2 \pi i)^{2}}\left\{\frac{3}{2}\left(-\pi^{2}+\sum_{j=1}^{\infty} \frac{(2 \pi i)^{2} 2 q^{j}}{\left(1+q^{j}\right)^{2}}\right)-\frac{1}{2}\left(-\pi^{2}-3 \sum_{j=1}^{\infty} \frac{(2 \pi i)^{2} 2 q^{j}}{\left(1-q^{j}\right)^{2}}\right)\right\} \\
& =\frac{1}{4}+3 \sum_{j=1}^{\infty}\left\{\frac{q^{j}}{\left(1+q^{j}\right)^{2}}+\frac{q^{j}}{\left(1-q^{j}\right)^{2}}\right\} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \sum_{m, n \geq 1} \frac{q^{m+n}}{\left(1+q^{m}\right)\left(1+q^{n}\right)\left(1+q^{m+n}\right)} \\
= & \frac{1}{4} \sum_{j=1}^{\infty}\left\{\frac{q^{j}}{\left(1-q^{j}\right)^{2}}-\frac{q^{j}}{\left(1+q^{j}\right)^{2}}\right\} \\
= & \sum_{j=1}^{\infty} \sum_{a>b, a+b o d d} q^{j} \cdot q^{a j} \cdot q^{b j} \\
= & \sum_{r \geq 1} q^{2 r} \sum_{k \mid r} k \\
= & \sum_{r \geq 1} \sigma_{1}(r) q^{2 r} .
\end{aligned}
$$

### 3.2. Hirzebruch surface.

Hirzebruch surface $F_{k}=\mathbb{C} P(\mathbb{C} \oplus \mathcal{O}(k))$ is a $\mathbb{C} P^{1}$ bundle over $\mathbb{C} P^{1}$. Its corresponding fan is spanned by four primitive vectors $\mathbf{e}_{1}, \mathbf{e}_{2},-\mathbf{e}_{1}+k \mathbf{e}_{2},-\mathbf{e}_{2}$, see Figure 2.


Figure 2
Assume that deg takes $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$ on the generators $\mathbf{e}_{1},-\mathbf{e}_{1}+k \mathbf{e}_{2},-\mathbf{e}_{2}, \mathbf{e}_{2}$, . Then the toric form is

$$
\begin{aligned}
& f_{N, \operatorname{deg}}(q) \\
= & \sum_{a, b \in \mathbb{Z}} \frac{1}{\left(1-e^{2 \pi i \alpha_{1}} q^{b}\right)\left(1-e^{2 \pi i \alpha_{4}} q^{a}\right)}+\frac{1}{\left(1-e^{2 \pi i \alpha_{1}} q^{b}\right)\left(1-e^{2 \pi i \alpha_{3}} q^{-a}\right)} \\
& +\frac{1}{\left(1-e^{2 \pi i \alpha_{3}} q^{-a}\right)\left(1-e^{2 \pi i \alpha_{2}} q^{k a-b}\right)}+\frac{1}{\left(1-e^{2 \pi i \alpha_{4}} q^{a}\right)\left(1-e^{2 \pi i \alpha_{2}} q^{k a-b}\right)} \\
& -\frac{1}{1-e^{2 \pi i \alpha_{4}} q^{a}}-\frac{1}{1-e^{2 \pi i \alpha_{1}} q^{b}}-\frac{1}{1-e^{2 \pi i \alpha_{2}} q^{k a-b}}-\frac{1}{1-e^{2 \pi i \alpha_{3}} q^{-a}}+1 \\
= & \sum_{a, b \in \mathbb{Z}} \frac{\left(1-e^{2 \pi i\left(\alpha_{3}+\alpha_{4}\right)}\right)\left(1-e^{2 \pi i\left(\alpha_{1}+\alpha_{2}\right)} q^{k a}\right)}{\left(1-e^{2 \pi i \alpha_{4}} q^{a}\right)\left(1-e^{2 \pi i \alpha_{1}} q^{b}\right)\left(1-e^{2 \pi i \alpha_{3}} q^{-a}\right)\left(1-e^{2 \pi i \alpha_{2}} q^{k a-b}\right)} .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& f_{N, \operatorname{deg}}(q) \\
= & \int_{X} \prod_{i=1}^{4} \frac{\left(D_{i} / 2 \pi i\right) \theta\left(D_{i} / 2 \pi i-\alpha_{i}\right) \theta^{\prime}(0)}{\theta\left(D_{i} / 2 \pi i\right) \theta\left(-\alpha_{i}\right)} \\
= & \frac{\theta^{\prime}(0)^{4}}{(2 \pi i)^{4} \prod_{i=1}^{4} \theta\left(-\alpha_{i}\right)} \operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \frac{\theta\left(\frac{u}{2 \pi i}-\alpha_{1}\right) \theta\left(\frac{u}{2 \pi i}-\alpha_{2}\right) \theta\left(\frac{v}{2 \pi i}-\alpha_{3}\right) \theta\left(\frac{v-k u}{2 \pi i}-\alpha_{4}\right)}{\theta^{2}\left(\frac{u}{2 \pi i}\right) \theta\left(\frac{v}{2 \pi i}\right) \theta\left(\frac{v-k u}{2 \pi i}\right)} d u\right\} d v \\
= & \frac{\theta^{\prime}(0)^{4}}{(2 \pi i)^{2} \prod_{i=1}^{4} \theta\left(-\alpha_{i}\right)} \operatorname{Res}_{0} \frac{\theta\left(v-\alpha_{3}\right)}{\theta(v)}\left\{\operatorname{Res}_{0} \frac{\theta\left(u-\alpha_{1}\right) \theta\left(u-\alpha_{2}\right) \theta\left(v-k u-\alpha_{4}\right)}{\theta^{2}(u) \theta(v-k u)} d u\right\} d v \\
= & \frac{\theta^{\prime}(0)^{2}}{(2 \pi i)^{2} \theta\left(-\alpha_{3}\right) \theta\left(-\alpha_{4}\right)} \operatorname{Res}_{0} \frac{\theta\left(v-\alpha_{3}\right) \theta\left(v-\alpha_{4}\right)}{\theta^{2}(v)}\left\{\frac{\theta^{\prime}\left(-\alpha_{1}\right)}{\theta\left(-\alpha_{1}\right)}+\frac{\theta^{\prime}\left(-\alpha_{2}\right)}{\theta\left(-\alpha_{2}\right)}-k \frac{\theta^{\prime}\left(v-\alpha_{4}\right)}{\theta\left(v-\alpha_{4}\right)}+k \frac{\theta^{\prime}(v)}{\theta(v)}\right\} d v .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\operatorname{Res}_{0} \frac{\theta\left(v-\alpha_{3}\right) \theta\left(v-\alpha_{4}\right)}{\theta^{2}(v)} d v & =\frac{\theta\left(-\alpha_{3}\right) \theta\left(-\alpha_{4}\right)}{\theta^{\prime 2}(0)}\left\{\frac{\theta^{\prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}+\frac{\theta^{\prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{4}\right)}\right\}, \\
\operatorname{Res}_{0} \frac{\theta\left(v-\alpha_{3}\right) \theta^{\prime}\left(v-\alpha_{4}\right)}{\theta^{2}(v)} d v & =\frac{\theta\left(-\alpha_{3}\right) \theta^{\prime}\left(-\alpha_{4}\right)}{\theta^{\prime 2}(0)}\left\{\frac{\theta^{\prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}+\frac{\theta^{\prime \prime}\left(-\alpha_{4}\right)}{\theta^{\prime}\left(-\alpha_{4}\right)}\right\} .
\end{aligned}
$$

and

$$
\begin{aligned}
& \operatorname{Res}_{0} \frac{\theta\left(v-\alpha_{3}\right) \theta\left(v-\alpha_{4}\right) \theta^{\prime}(v)}{\theta^{3}(v)} d v \\
= & \operatorname{Res}_{0} \frac{\left(\theta\left(-\alpha_{3}\right)+\theta^{\prime}\left(-\alpha_{3}\right) v+\frac{\theta^{\prime \prime}\left(-\alpha_{3}\right) v^{2}}{2}\right)\left(\theta\left(-\alpha_{4}\right)+\theta^{\prime}\left(-\alpha_{4}\right) v+\frac{\theta^{\prime \prime}\left(-\alpha_{4}\right) v^{2}}{2}\right)\left(\theta^{\prime}(0)+\frac{\theta^{(3)}(0) v^{2}}{2}\right)}{\left(\theta^{\prime}(0) v+\theta^{(3)}(0) v^{3} / 6\right)^{3}} d v \\
= & \operatorname{Res}_{0} \frac{\left(\theta\left(-\alpha_{3}\right)+\theta^{\prime}\left(-\alpha_{3}\right) v+\frac{\theta^{\prime \prime}\left(-\alpha_{3}\right) v^{2}}{2}\right)\left(\theta\left(-\alpha_{4}\right)+\theta^{\prime}\left(-\alpha_{4}\right) v+\frac{\theta^{\prime \prime}\left(-\alpha_{4}\right) v^{2}}{2}\right)}{\theta^{\prime 2}(0) v^{3}} d v \\
= & \frac{\theta\left(-\alpha_{3}\right) \theta\left(-\alpha_{4}\right)}{2 \theta^{\prime 2}(0)}\left\{\left(\frac{\theta^{\prime \prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}+\frac{\theta^{\prime \prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{4}\right)}+2 \frac{\theta^{\prime}\left(-\alpha_{3}\right) \theta^{\prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{3}\right) \theta\left(-\alpha_{4}\right)}\right)\right\} .
\end{aligned}
$$

Proposition 3.1. For Hirzebruch surface $F_{k}=\mathbb{C} P(\underline{\mathbb{C}} \oplus \mathcal{O}(k))$, we have theta function identity

$$
\begin{aligned}
& \sum_{a, b \in \mathbb{Z}} \frac{\left(1-e^{2 \pi i\left(\alpha_{3}+\alpha_{4}\right)}\right)\left(1-e^{2 \pi i\left(\alpha_{1}+\alpha_{2}\right)} q^{k a}\right)}{\left(1-e^{2 \pi i \alpha_{4}} q^{a}\right)\left(1-e^{2 \pi i \alpha_{1}} q^{b}\right)\left(1-e^{2 \pi i \alpha_{3}} q^{-a}\right)\left(1-e^{2 \pi i \alpha_{2}} q^{k a-b}\right)} \\
= & \frac{1}{(2 \pi i)^{2}}\left\{\left(\frac{\theta^{\prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}+\frac{\theta^{\prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{4}\right)}\right)\left(\frac{\theta^{\prime}\left(-\alpha_{1}\right)}{\theta\left(-\alpha_{1}\right)}+\frac{\theta^{\prime}\left(-\alpha_{2}\right)}{\theta\left(-\alpha_{2}\right)}\right)+\frac{k}{2}\left(\frac{\theta^{\prime \prime}\left(-\alpha_{3}\right)}{\theta\left(-\alpha_{3}\right)}-\frac{\theta^{\prime \prime}\left(-\alpha_{4}\right)}{\theta\left(-\alpha_{4}\right)}\right)\right\} .
\end{aligned}
$$

It is easy to check that if $\alpha_{3}+\alpha_{4} \in \mathbb{Z}$, both sides equal zero.

## 3.3. $\mathbb{C} P^{2}$ bundle over $\mathbb{C} P^{2}$.

Consider a $\mathbb{C} P^{2}$ bundle over $\mathbb{C} P^{2}$, its corresponding fan is spanned by five primitive vectors $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}, \mathbf{e}_{4},-\mathbf{e}_{1}-\mathbf{e}_{2}+j \mathbf{e}_{3}+k \mathbf{e}_{4},-\mathbf{e}_{3}-\mathbf{e}_{4}$ in $\mathbb{Z}^{4}$.

Assume that deg takes $\frac{1}{2}$ on all generators $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}, \mathbf{e}_{4},-\mathbf{e}_{1}-\mathbf{e}_{2}+j \mathbf{e}_{3}+k \mathbf{e}_{4},-\mathbf{e}_{3}-\mathbf{e}_{4}$ in $\mathbb{Z}^{4}$. Then the toric form is

$$
f_{N, \operatorname{deg}}(q)=\sum_{a, b, c, d \in \mathbb{Z}} \frac{2\left(1+q^{j c+k d}\right)}{\left(1+q^{a}\right)\left(1+q^{b}\right)\left(1+q^{c}\right)\left(1+q^{d}\right)\left(1+q^{j c+k d-a-b}\right)\left(1+q^{-c-d}\right)} .
$$

On the other hand,

$$
\begin{aligned}
& f_{N, \operatorname{deg}}(q) \\
= & \int_{X} \prod_{i=1}^{6} \frac{\left(D_{i} / 2 \pi i\right) \theta\left(D_{i} / 2 \pi i-\alpha_{i}\right) \theta^{\prime}(0)}{\theta\left(D_{i} / 2 \pi i\right) \theta\left(-\alpha_{i}\right)} \\
= & \frac{\theta^{\prime}(0)^{6}}{(2 \pi i)^{6} \theta^{6}\left(\frac{1}{2}\right)} \operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \frac{\theta^{3}\left(\frac{u}{2 \pi i}-\frac{1}{2}\right) \theta\left(\frac{v}{2 \pi i}-\frac{1}{2}\right) \theta\left(\frac{v-j u}{2 \pi i}-\frac{1}{2}\right) \theta\left(\frac{v-k u}{2 \pi i}-\frac{1}{2}\right)}{\theta^{3}\left(\frac{u}{2 \pi i}\right) \theta\left(\frac{v}{2 \pi i}\right) \theta\left(\frac{v-j u}{2 \pi i}\right) \theta\left(\frac{v-k u}{2 \pi i}\right)} d u\right\} d v \\
= & \frac{\theta^{\prime}(0)^{6}}{(2 \pi i)^{4} \theta^{6}\left(\frac{1}{2}\right)} \operatorname{Res}_{0} \frac{\theta\left(v-\frac{1}{2}\right)}{\theta(v)}\left\{\operatorname{Res}_{0} \frac{\theta^{3}\left(u-\frac{1}{2}\right) \theta\left(v-j u-\frac{1}{2}\right) \theta\left(v-k u-\frac{1}{2}\right)}{\theta^{3}(u) \theta(v-j u) \theta(v-k u)} d u\right\} d v \\
= & \frac{\theta^{\prime}(0)^{3}}{(2 \pi i)^{4} \theta^{3}\left(\frac{1}{2}\right)} \operatorname{Res}_{0} \frac{\theta^{3}\left(v+\frac{1}{2}\right)}{\theta^{3}(v)}\left\{\frac{3 \theta^{\prime \prime}\left(\frac{1}{2}\right)}{2 \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(3)}(0)}{2 \theta^{\prime}(0)}+\frac{j^{2}+k^{2}}{2}\left(\frac{\theta^{\prime \prime}\left(v+\frac{1}{2}\right)}{\theta\left(v+\frac{1}{2}\right)}-\frac{\theta^{\prime \prime}(v)}{\theta(v)}\right)\right. \\
& \left.+j k\left(\frac{\theta^{\prime 2}\left(v+\frac{1}{2}\right)}{\theta^{2}\left(v+\frac{1}{2}\right)}-\frac{\theta^{\prime 2}(v)}{\theta^{2}(v)}\right)+(j+k)^{2} \frac{\theta^{\prime 2}(v)}{\theta^{2}(v)}-(j+k)^{2} \frac{\theta^{\prime}(v) \theta^{\prime}\left(v+\frac{1}{2}\right)}{\theta(v) \theta\left(v+\frac{1}{2}\right)}\right\} d v \\
= & \frac{1}{(2 \pi i)^{4}}\left\{\left(\frac{3 \theta^{\prime \prime}\left(\frac{1}{2}\right)}{2 \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(3)}(0)}{2 \theta^{\prime}(0)}\right)^{2}+\left(j^{2}+k^{2}-j k\right)\left(\frac{\theta^{\prime \prime 2}\left(\frac{1}{2}\right)}{4 \theta^{2}\left(\frac{1}{2}\right)}+\frac{5 \theta^{(4)}\left(\frac{1}{2}\right)}{24 \theta\left(\frac{1}{2}\right)}-\frac{7 \theta^{(3)}(0) \theta^{\prime \prime}\left(\frac{1}{2}\right)}{12 \theta^{\prime}(0) \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(5)}(0)}{24 \theta^{\prime}(0)}+\frac{\left(\theta^{(3)}(0)\right)^{2}}{6 \theta^{\prime 2}(0)}\right)\right\} .
\end{aligned}
$$

Proposition 3.2. We have the following theta function identity

$$
\begin{aligned}
& \sum_{a, b, c, d \in \mathbb{Z}} \frac{2\left(1+q^{j c+k d}\right)}{\left(1+q^{a}\right)\left(1+q^{b}\right)\left(1+q^{c}\right)\left(1+q^{d}\right)\left(1+q^{j c+k d-a-b}\right)\left(1+q^{-c-d}\right)} \\
= & \frac{1}{(2 \pi i)^{4}}\left\{\left(\frac{3 \theta^{\prime \prime}\left(\frac{1}{2}\right)}{2 \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(3)}(0)}{2 \theta^{\prime}(0)}\right)^{2}+\left(j^{2}+k^{2}-j k\right)\left(\frac{\theta^{\prime \prime 2}\left(\frac{1}{2}\right)}{4 \theta^{2}\left(\frac{1}{2}\right)}+\frac{5 \theta^{(4)}\left(\frac{1}{2}\right)}{24 \theta\left(\frac{1}{2}\right)}-\frac{7 \theta^{(3)}(0) \theta^{\prime \prime}\left(\frac{1}{2}\right)}{12 \theta^{\prime}(0) \theta\left(\frac{1}{2}\right)}-\frac{\theta^{(5)}(0)}{24 \theta^{\prime}(0)}+\frac{\left(\theta^{(3)}(0)\right)^{2}}{6 \theta^{\prime 2}(0)}\right)\right\} .
\end{aligned}
$$

In special case $j=k=0, f_{N, \operatorname{deg}}(q)$ is exactly the square of normalized elliptic genus of $\mathbb{C} P^{2}$ which we have already discussed in Example 3.1.

### 3.4. Vanishing result.

Besides the accurate calculation in low dimensions, we can also get a LandweberStong type vanishing result for a family of generalized Bott manifolds.

Theorem 3.3. Consider toric variety $V:=\mathbb{C} P\left(\eta^{\otimes i_{1}} \oplus \eta^{\otimes i_{2}} \oplus \eta^{\otimes i_{3}} \oplus \mathbb{C}\right)$ over $\mathbb{C} P^{n_{1}}$, where $\eta$ denote the tautological bundle of $\mathbb{C} P^{n_{1}}$.

If $\left(i_{1}, i_{2}, i_{3}\right)$ are coprime, $\left(i_{1}, i_{1}-i_{2}, i_{1}-i_{3}\right),\left(i_{2}, i_{2}-i_{1}, i_{2}-i_{3}\right),\left(i_{3}, i_{3}-i_{1}, i_{3}-i_{2}\right)$ are coprime respectively, $\sum_{j=0}^{3} \alpha_{n_{1}+2+j} \in \mathbb{Z}, \sum_{i=1}^{n_{1}+1} \alpha_{i}-\sum_{j=1}^{3} i_{j} \alpha_{n_{1}+2+j} \in \mathbb{Z}$, then the toric form $f_{N, \operatorname{deg}}(q)=0$.

The proof of Theorem 3.3 is similar to that of Theorem4.1.

## 4. Witten genus: A vanishing result

Let $M$ be a $4 k$ dimensional closed oriented smooth manifold, $E$ be a complex vector bundle over $M$. For any complex number $t$, set

$$
S_{t}(E)=\underline{\mathbb{C}}+t E+t^{2} S^{2}(E)+\cdots,
$$

where $S^{j}(E)$ is the $j$-th symmetric power of $E$.
Let $q=e^{2 \pi i \tau}$ with $\tau \in \mathbb{H}$, the upper half plane. Witten defined

$$
\Theta_{q}(E)=\bigotimes_{n \geq 1} S_{q^{n}}(E)
$$

then defined the Witten genus

$$
\varphi_{W}(M)=\left\langle\widehat{A}(M) \operatorname{ch}\left(\Theta_{q}\left(T M \otimes \mathbb{C}-\underline{\mathbb{C}}^{4 k}\right)\right),[M]\right\rangle .
$$

Let $\pm 2 \pi i x_{j}(1 \leq j \leq 2 k)$ be the formal Chern roots of $T M \otimes \mathbb{C}$. The Witten genus can be rewritten as (c.f. [17])

$$
\varphi_{W}(M)=\left\langle\left(\prod_{j=1}^{2 m} z_{j} \frac{\theta^{\prime}(0, \tau)}{\theta\left(z_{j}, \tau\right)}\right),[M]\right\rangle,
$$

where $\theta(z, \tau)$ is the Jacobi theta function

$$
\theta(z, \tau)=2 q^{1 / 8} \sin (\pi z) \prod_{j=1}^{\infty}\left[\left(1-q^{j}\right)\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\right], q=e^{2 \pi i \tau}
$$

The manifold $M$ is called spin if $\omega_{1}(M)=0, \omega_{2}(M)=0$, where $\omega_{1}(M), \omega_{2}(M)$ are the first and second Stiefel-Whitney classes of $M$. According to Atiyah-Singer index theorem, when $M$ is spin, $\varphi_{W}(M) \in \mathbb{Z}[[q]]$.
A spin manifold $M$ is called string if $\frac{1}{2} p_{1}(M)=0$, where $p_{1}(M)$ is the first Pontrjagin class of $M$. It is well known that if $M$ is string, $\varphi_{W}(M)$ is a modular form of weight $2 k$ over $S L(2, \mathbb{Z})$, see [13, Hirzebruch]. In this section, we mainly discuss the Witten genus of string manifolds.

### 4.1. String complete intersection in generalized Bott manifolds.

Consider a two staged generalized Bott manifold $V:=\mathbb{C} P\left(\eta^{\otimes i_{1}} \oplus \cdots \eta^{\otimes i_{n_{2}}} \oplus \mathbb{C}\right)$ over $\mathbb{C} P^{n_{1}}$, where $\eta$ denotes the tautological bundle of $\mathbb{C} P^{n_{1}}$.

Let $\mathbf{I}=\left(i_{1}, \cdots, i_{n_{2}}\right)$ denote the index of projective bundle and $H_{n_{1}, n_{2}}^{\mathbf{I}}\left(d_{1}, d_{2}\right)$ denote the submanifold Poincaré dual to $d_{1} u+d_{2} v \in H^{2}(V ; \mathbb{Z})$. Let $i: H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right) \longrightarrow V$ be the natural embedding, $\nu$ denote the normal bundle of this embedding. We have

$$
i^{*}(T V) \cong i^{*}(\nu) \oplus T H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)
$$

Thus

$$
c_{1}\left(H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)\right)=i^{*}\left(\left(n_{1}+1\right) u+v+\sum_{j=1}^{n_{2}}\left(v-i_{j} u\right)-\left(d_{1} u+d_{2} v\right)\right) .
$$

and
$p_{1}\left(H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)\right)=i^{*}\left\{\left(n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}-d_{1}^{2}\right) u^{2}+\left(1+n_{2}-d_{2}^{2}\right) v^{2}-2\left(d_{1} d_{2}+\sum_{j=1}^{n_{2}} i_{j}\right) u v\right\}$.
Lemma 4.1. Twisted Milnor hypersurfaces $H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)$ can't be string for $n_{2} \geq 3$.
Proof. Let $i_{!}: H^{*}\left(H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)\right) \longrightarrow H^{*}(V)$ be the pushforward map.
First we assume

$$
\left\{\begin{array}{l}
n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}=d_{1}^{2} \\
1+n_{2}=d_{2}^{2} \\
d_{1} d_{2}+\sum_{j=1}^{n_{2}} i_{j}=0
\end{array}\right.
$$

Thus $\left(n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}\right)\left(1+n_{2}\right)=\left(d_{1} d_{2}\right)^{2}=\left(\sum_{j=1}^{n_{2}} i_{j}\right)^{2}$, which is impossible. Thus $\left(n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}-d_{1}^{2}\right) u^{2}+\left(1+n_{2}-d_{2}^{2}\right) v^{2}+\left(-2 \sigma_{1}-2 d_{1} d_{2}\right) u v \neq 0$.

On the other hand, if $p_{1}\left(H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)\right)=0$, then

$$
\begin{aligned}
& i_{!} p_{1}\left(H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2}\right)\right) \\
= & i_{!} i^{*}\left\{\left(n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}-d_{1}^{2}\right) u^{2}+\left(1+n_{2}-d_{2}^{2}\right) v^{2}+\left(-2 \sigma_{1}-2 d_{1} d_{2}\right) u v\right\} \\
= & \left(d_{1} u+d_{2} v\right)\left\{\left(n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}-d_{1}^{2}\right) u^{2}+\left(1+n_{2}-d_{2}^{2}\right) v^{2}+\left(-2 \sigma_{1}-2 d_{1} d_{2}\right) u v\right\} \\
= & 0 .
\end{aligned}
$$

Since $d_{1} u+d_{2} v \neq 0$, as long as $n_{2} \geq 3$, it is impossible for $p_{1}=0$.

Remark 7. It is possible for twisted Milnor hypersurface $H_{n_{1}, n_{2}}^{\mathbf{I}}\left(d_{1}, d_{2}\right)$ to be string when $n_{2}<3$. For example, choose $n_{2}=1, d_{2}=0$, we have

$$
p_{1}=i^{*}\left\{\left(n_{1}+1+i_{1}^{2}-d_{1}^{2}\right) u^{2}+2 v^{2}-2 i_{1} u v\right\}
$$

where $2 v^{2}-2 i_{1} u v$ is killed by the relation in the cohomology ring of 2 staged generalized Bott manifold. Thus if $n_{1}+1+i_{1}^{2}-d_{1}^{2}=0$, we have $p_{1}=0$. For example, twisted Milnor hypersurface $H_{12,1}^{ \pm 6}( \pm 7,0)$ is the required 24-dim string manifold.

Due to the restriction of Lemma 4.1, we proceed to investigate string complete intersections. The submanifold $H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)$ Poincaré dual to cohomology class $\left(d_{1} u+d_{2} v\right) \cdot\left(d_{3} u+d_{4} v\right) \in H^{4}(V ; \mathbb{Z})$ is string as long as the equation set $\left\{\begin{array}{l}n_{1}+1+\sum_{j=1}^{n_{2}} i_{j}^{2}=d_{1}^{2}+d_{3}^{2} \\ 1+n_{2}=d_{2}^{2}+d_{4}^{2} \\ d_{1} d_{2}+d_{3} d_{4}+\sum_{j=1}^{n_{2}} i_{j}=0\end{array}\right.$ has integer solutions. Actually, there exist plenty of string complete intersections.
4.2. Witten genus. Consider the Jacobi theta function

$$
\theta(z, \tau)=2 q^{1 / 8} \sin (\pi z) \prod_{j=1}^{\infty}\left[\left(1-q^{j}\right)\left(1-e^{2 \pi i z} q^{j}\right)\left(1-e^{-2 \pi i z} q^{j}\right)\right],
$$

where $\theta$ admits periodicity, $\theta(z+1, \tau)=-\theta(z, \tau), \theta(z+b \tau, \tau)=(-1)^{b} e^{-\pi i\left(2 b z+b^{2} \tau\right)} \theta(z, \tau)$. Clearly $\theta$ has simple zeros on lattice $\mathbb{Z}+\mathbb{Z} \tau$ and no pole. Applying our iterated residue, the Witten genus can be reformulated as

$$
\begin{aligned}
& \varphi_{W}\left(H_{n_{1}, n_{2}}^{\mathrm{I}}\right) \\
= & \theta^{\prime n_{1}+n_{2}}(0)\left\langle\left(\frac{u}{\theta(u)}\right)^{n_{1}+1} \frac{v}{\theta(v)} \prod_{j=1}^{n_{2}} \frac{v-i_{j} u}{\theta\left(v-i_{j} u\right)} \frac{\theta\left(d_{1} u+d_{2} v\right)}{d_{1} u+d_{2} v} \frac{\theta\left(d_{3} u+d_{4} v\right)}{d_{3} u+d_{4} v},\left[H_{n_{1}, n_{2}}^{\mathrm{I}}\right]\right\rangle \\
= & \frac{\theta^{\prime n_{1}+n_{2}}(0)}{(2 \pi i)^{-2}}\left\langle\left(\frac{u}{\theta(u)}\right)^{n_{1}+1} \frac{v}{\theta(v)} \prod_{j=1}^{n_{2}} \frac{v-i_{j} u}{\theta\left(v-i_{j} u\right)} \theta\left(d_{1} u+d_{2} v\right) \cdot \theta\left(d_{3} u+d_{4} v\right),[V]\right\rangle \\
= & \frac{\theta^{\prime n_{1}+n_{2}}(0)}{(2 \pi i)^{n_{1}+n_{2}-2}} \operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \frac{\theta\left(d_{1} u+d_{2} v\right) \cdot \theta\left(d_{3} u+d_{4} v\right)}{\theta^{n_{1}+1}(u) \theta(v) \prod_{j=1}^{n_{2}} \theta\left(v-i_{j} u\right)} d u\right\} d v .
\end{aligned}
$$

Remark 8. If meromorphic function $g$ has pole of order $n$ at point $c$, then

$$
\operatorname{Res}_{c} g=\frac{1}{(n-1)!} \cdot \lim _{z \rightarrow c} \frac{d^{n-1}}{d z^{n-1}}\left((z-c)^{n} g(z)\right)
$$

It is usually difficult to calculate the residue at high order poles for both variables $u$ and $v$ directly, thus we shall make use of residue theorem to reduce iterated residue at high order poles to simple poles. This is the common operation for Landweber-Stong type vanishing results.

Under certain condition, we obtain a Landweber-Stong type vanishing result for string complete intersections in generalized Bott manifolds. We add the coprime condition to make sure that the poles except for 0 are simple.
Theorem 4.1. For a string complete intersection $H_{n_{1}, 3}^{\mathbf{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)$, if $\mathbf{I}=(i, j, k)$ are coprime, $(i, i-j, i-k),(j, j-i, j-k),(k, k-i, k-j)$ are coprime respectively, then the Witten genus

$$
\varphi_{W}\left(H_{n_{1}, 3}^{\mathbf{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)\right)=0 .
$$

Proof.

$$
\varphi_{W}\left(H^{\mathbf{I}}\right)=\frac{\theta^{n_{1}+n_{2}}(0)}{(2 \pi i)^{n_{1}+n_{2}-2}} \operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \frac{\theta\left(d_{1} u+d_{2} v\right) \theta\left(d_{3} u+d_{4} v\right)}{\theta^{n_{1}+1}(u) \theta(v) \theta(v+i u) \theta(v+j u) \theta(v+k u)} d u\right\} d v
$$

Denote $\omega(u)=\frac{\theta\left(d_{1} u+d_{2} v\right) \cdot \theta\left(d_{3} u+d_{4} v\right)}{\theta^{n_{1}+1}(u) \theta(v) \prod_{j=1}^{n_{2}} \theta\left(v-i_{j} u\right)} d u$, we check its periodicity

$$
\begin{gathered}
\omega(u+1)=(-1)^{d_{1}+d_{3}-n_{1}-1-\sum i_{j}} \omega(u)=\omega(u) ; \\
\omega(u+\tau)=\frac{(-1)^{d_{1}+d_{3}} e^{-\pi i\left(2 d_{1}\left(d_{1} u+d_{2} v\right)+2 d_{3}\left(d_{3} u+d_{4} v\right)+\left(d_{1}^{2}+d_{3}^{2}\right) \tau\right.} \theta\left(d_{1} u+d_{2} v\right) \cdot \theta\left(d_{3} u+d_{4} v\right)}{(-1)^{n_{1}+1+\sum i_{j}} e^{-\pi i\left(\left(n_{1}+1\right)(2 u+\tau)+\sum\left(-2 i_{j}\left(v-i_{j} u\right)+i_{j}^{2} \tau\right)\right)} \theta^{n_{1}+1}(u) \theta(v) \prod \theta\left(v-i_{j} u\right)} d u .
\end{gathered}
$$

Since $H_{n_{1}, n_{2}}^{\mathrm{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)$ is string, we have $\omega(u+\tau)=\omega(u)$. We can claim that $\omega(u)$ admits double periodicity on lattice $L:=\mathbb{Z}+\mathbb{Z} \tau$. Let $T^{2} \cong \mathbb{C} / L$, thus $\omega(u)$ can be
defined on closed surface $T^{2}$. Then we can apply residue theorem on $\omega(u)$. Since $\theta$ has simple zeros on $\mathbb{Z}+\mathbb{Z} \tau$ and no poles, the possible poles of $\omega$ are

$$
0, \frac{a+b \tau-v}{i}(1 \leq a, b \leq i), \frac{c+d \tau-v}{j}(1 \leq c, d \leq j), \frac{m+n \tau-v}{k}(1 \leq m, n \leq k) .
$$

Applying residue theorem on $u$, we get
$\left\{\operatorname{Res}_{0}+\operatorname{Res}_{1+\tau-\frac{v}{i}}+\operatorname{Res}_{1+\tau-\frac{v}{j}}+\operatorname{Res}_{1+\tau-\frac{v}{k}}+\operatorname{Res}_{\frac{a+b \tau-v}{i}}+\operatorname{Res}_{\frac{c+d \tau-v}{j}}+\operatorname{Res}_{\frac{m+n \tau-v}{k}}\right\} \omega(u)=0$.
Moreover

$$
\begin{gathered}
\operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{i}} \omega(u)\right\} d v=\frac{1}{\theta^{\prime}(0)} \cdot \operatorname{Res}_{0} \frac{\theta\left(\left(d_{2} i-d_{1}\right) v\right) \theta\left(\left(d_{4} i-d_{3}\right) v\right)}{\theta^{n_{1}+1}(-v) \theta(i v) \theta((i-j) v) \theta((i-k) v)} d v, \\
\operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{j}} \omega(u)\right\} d v=\frac{1}{\theta^{\prime}(0)} \cdot \operatorname{Res}_{0} \frac{\theta\left(\left(d_{2} j-d_{1}\right) v\right) \theta\left(\left(d_{4} j-d_{3}\right) v\right)}{\theta^{n_{1}+1}(-v) \theta(j v) \theta((j-i) v) \theta((j-k) v)} d v, \\
\operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{k}} \omega(u)\right\} d v=\frac{1}{\theta^{\prime}(0)} \cdot \operatorname{Res}_{0} \frac{\theta\left(\left(d_{2} k-d_{1}\right) v\right) \theta\left(\left(d_{4} k-d_{3}\right) v\right)}{\theta^{n_{1}+1}(-v) \theta(k v) \theta((k-i) v) \theta((k-j) v)} d v, \\
\operatorname{Res}_{0}\left\{\operatorname{Res}_{\frac{a+b \tau-v}{i}} \omega(u)\right\} d v=\frac{(-1)^{a+b+1}}{i \cdot \theta^{\prime 2}(0)} e^{2 \pi i b^{2} \tau} \cdot \frac{\theta\left(\frac{d_{1}(a+b \tau)}{i}\right) \theta\left(\frac{d_{3}(a+b \tau)}{i}\right)}{\theta^{n_{1}+1}\left(\frac{a+b \tau}{i}\right) \theta\left(\frac{j(a+b \tau)}{i}\right) \theta\left(\frac{k(a+b \tau)}{i}\right)}, \\
\operatorname{Res}_{0}\left\{\operatorname{Res}_{\frac{c+d \tau-v}{j}}^{j} \omega(u)\right\} d v=\frac{(-1)^{c+d+1}}{j \cdot \theta^{\prime 2}(0)} e^{2 \pi i d^{2} \tau} \cdot \frac{\theta\left(\frac{d_{1}(c+d \tau)}{j}\right) \theta\left(\frac{d_{3}(c+d \tau)}{j}\right)}{\theta^{n_{1}+1}\left(\frac{c+d \tau}{j}\right) \theta\left(\frac{i(c+d \tau)}{j}\right) \theta\left(\frac{k(c+d \tau)}{j}\right)}, \\
\operatorname{Res}_{0}\left\{\operatorname{Res}_{\frac{m+n \tau-v}{k}}^{k} \omega(u)\right\} d v=\frac{(-1)^{m+n+1}}{k \cdot \theta^{\prime 2}(0)} e^{2 \pi i n^{2} \tau} \cdot \frac{\theta\left(\frac{d_{1}(m+n \tau)}{k}\right) \theta\left(\frac{d_{3}(m+n \tau)}{k}\right)}{\theta^{n_{1}+1}\left(\frac{m+n \tau}{k}\right) \theta\left(\frac{i(m+n \tau)}{k}\right) \theta\left(\frac{j(m+n \tau)}{k}\right)} .
\end{gathered}
$$

Apply residue theorem again on $v$ to the followings

$$
\operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{i}} \omega(u)\right\} d v, \operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{j}} \omega(u)\right\} d v, \operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{k}} \omega(u)\right\} d v .
$$

Here we take $\operatorname{Res}_{0}\left\{\operatorname{Res}_{1+\tau-\frac{v}{i}} \omega(u)\right\} d v$ for example. Denote

$$
\omega(v)=\frac{\theta\left(\left(d_{2} i-d_{1}\right) v\right) \theta\left(\left(d_{4} i-d_{3}\right) v\right)}{\theta^{n_{1}+1}(-v) \theta(i v) \theta((i-j) v) \theta((i-k) v)} d v .
$$

Since $i,(i-j), i-k$ are coprime, it is easy to check $\omega(v)$ can also be defined on $T^{2} \cong \mathbb{C} / L$. The possible poles of $\omega(v)$ include $0, \frac{a+b \tau}{i}, \frac{a+b \tau}{i-j}, \frac{a+b \tau}{i-k}$. We have

$$
\operatorname{Res}_{\frac{a+b \tau}{i}} \omega(v)=\frac{(-1)^{a+b}}{i \cdot \theta^{\prime}(0)} e^{2 \pi i b^{2} \tau} \cdot \frac{\theta\left(\frac{\left(d_{2} i-d_{1}\right)(a+b \tau)}{i}\right) \theta\left(\frac{\left(d_{4} i-d_{3}\right)(a+b \tau)}{i}\right)}{\theta^{n_{1}+1}\left(-\frac{a+b \tau}{i}\right) \theta\left(\frac{(i-j)(a+b \tau)}{i}\right) \theta\left(\frac{(i-k)(a+b \tau)}{i}\right)} .
$$

$$
\text { Since } \theta(z+b \tau)=(-1)^{b} e^{-2 \pi i b z-\pi i b^{2} \tau} \theta(z)
$$

$$
\begin{aligned}
& \left.\frac{\operatorname{Res}_{\frac{a+b \tau}{i}} \omega(v)}{\operatorname{Res}_{0}\left\{\operatorname{ReS}_{\frac{a+b \tau-v}{i}}^{i}\right.} \omega(u)\right\} d v \\
= & (-1)^{n_{1}+1+\left(d_{2}+d_{4}\right)(a+b)} \cdot e^{-2 \pi i\left(-d_{2} d_{1}-d_{4} d_{3}+j+k\right) b\left(\frac{a+b \tau}{i}\right)-\pi i b^{2}\left(d_{2}^{2}+d_{4}^{2}-2\right) \tau} \\
= & 1
\end{aligned}
$$

Similarly,

$$
\begin{gathered}
\operatorname{Res}_{\frac{c+d \tau}{j}}\left\{\operatorname{Res}_{1+\tau-\frac{v}{j}} \omega(u)\right\} d v=\operatorname{Res}_{0}\left\{\operatorname{Res}_{\frac{c+d \tau-v}{j}} \omega(u)\right\} d v \\
\operatorname{Res}_{\frac{m+n \tau}{k}}\left\{\operatorname{Res}_{1+\tau-\frac{v}{k}} \omega(u)\right\} d v=\operatorname{Res}_{0}\left\{\operatorname{Res}_{\frac{m+n \tau-v}{k}} \omega(u)\right\} d v
\end{gathered}
$$

and

$$
\begin{aligned}
& \frac{\operatorname{Res}_{\frac{a+b \tau}{i-j}} \omega(v)}{\operatorname{Res}_{\frac{a+b \tau}{i-j}}\left\{\operatorname{Res}_{1+\tau-\frac{v}{j}} \omega(u)\right\} d v} \\
= & (-1)^{1+\left(d_{2}+d_{4}\right)(a+b)} \cdot e^{-2 \pi i\left(d_{2}\left(d_{2} j-d_{1}\right)+d_{4}\left(d_{4} j-d_{3}\right)-2 j+k\right) b \frac{a+b \tau}{i-j}-\pi i b^{2}\left(d_{2}^{2}+d_{4}^{2}-2\right) \tau} \\
= & -1 .
\end{aligned}
$$

Similarly,

$$
\operatorname{Res}_{\frac{a+b \tau}{i-k}} \omega(v)=-\operatorname{Res}_{\frac{a+b \tau}{i-k}}\left\{\operatorname{Res}_{1+\tau-\frac{v}{k}} \omega(u)\right\} d v
$$

and

$$
\operatorname{Res}_{\frac{a+b \tau}{j-k}}\left\{\operatorname{Res}_{1+\tau-\frac{v}{j}} \omega(u)\right\} d v=-\operatorname{Res}_{\frac{a+b \tau}{j-k}}\left\{\operatorname{Res}_{1+\tau-\frac{v}{k}} \omega(u)\right\} d v .
$$

Glue all the calculations together, we have $\operatorname{Res}_{0}\left\{\operatorname{Res}_{0} \omega(u)\right\} d v=0$, i.e.,

$$
\varphi_{W}\left(H_{n_{1}, 3}^{\mathrm{I}}\left(d_{1}, d_{2} ; d_{3}, d_{4}\right)\right)=0 .
$$

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