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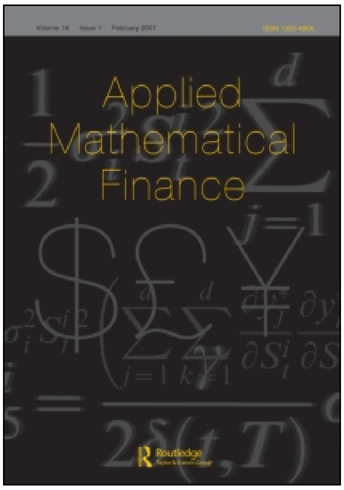

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A Semi-Explicit Approach to Canary Swaptions in HJM One-Factor Model

Marc Henrard [a]
[a] Derivatives Group, Bank for International Settlements, CH-4002 Basel, Switzerland

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A Semi-Explicit Approach to Canary Swaptions in HJM One-Factor Model

MARC HENRARD

Derivatives Group, Bank for International Settlements, Centralbahnplatz 2, CH-4002 Basel, Switzerland

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ABSTRACT *Leveraging the explicit formula for European swaptions and coupon-bond options in the HJM one-factor model, a semi-explicit formula for 2-Bermudan options (also called Canary options) is developed. The European swaption formula is extended to future times. So equipped, one is able to reduce the valuation of a 2-Bermudan swaption to a single numerical integration at the first expiry date. In that integration the most complex part of the embedded European swaptions valuation has been simplified to perform it only once and not for every point. In a special but very common in practice case, a semi-explicit formula is provided. Those results lead to a significantly faster and more precise implementation of swaption valuation. The improvements extend even more favourably to sensitivity calculations.*

KEY WORDS: Bermudan swaption, HJM one-factor model, Hull–White model, explicit formula, numerical integration

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Introduction

This article is devoted to Bermudan swaptions, more precisely to 2-Bermudan swaptions (swaptions with two exercise dates). Those swaptions are also called Canary swaptions as the Canary Islands are halfway between Bermuda and Europe.

We leverage the explicit formula for European swaptions and coupon-bonds in the HJM one-factor model presented in Henrard (2003). This is done by first calculating the value of European option at any future point of time. The value of an European option is a random process which is function of the fundamental random processes of the problem: the price of zero-coupon bonds. Such a formula is required as we need to compare a swap with the remaining European swaption at the first expiry date. We have in mind the application of the result to the Hull and White (1990) or extended Vasicek model. Too keep the formula as explicit as possible we restrict the form of the volatility function. We allow ourself those restrictions as long as the standard time-dependent Hull and White model is still covered. The restriction we impose is a separability condition on the volatility (see condition (H2) below).

Correspondence Address: Marc Henrard, Derivatives Group, Bank for International Settlements, Centralbahnplatz 2, CH-4002 Basel, Switzerland. Email: Marc.Henrard@bis.org

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Using the explicit formula we are able to reduce the valuation of a 2-Bermudan option to a single expectation. This is an improvement with respect to a direct or tree approach as, even if there are two expiry dates, the numerical process is done only at one date. Also the computation is done only at the expiry date and not at intermediary points. In that sense our results are related to the one of Gandhi and Hunt (1997) who also study Bermudan swaptions using numerical integration technique. Like in the tree approach or in Jamshidian (1989) paper on swaption, Gandhi and Hunt construction is based on the short rate. Our approach is more direct as we consider only discount bonds.

The most time consuming part of the European swaption computation is to solve a non-linear one-dimensional equation. We are able to reduce the computation time by solving it once and then reusing the solution for all the other points of the integration.

Finally we propose a semi-explicit formula for cancelable swaps or options on underlying with similar cash-flows after the second expiry date. The formula is explicit for the valuation of the part corresponding to the exercise at the first expiry date and still written as an expectation for the rest. The size of the interval on which the expectation has to be computed is reduced by the probability of the exercise at the first expiry date. In other words, for first expiry at-the-money options, the number of points in the numerical integration is divided by (around) two.

Those results lead to several possible implementations of valuation formulas. They all perform significantly better than a double integration and standard tree implementation both in term of speed and precision. The precision improvement in particular is striking when sensitivities (delta and gamma) are computed. The sensitivities stability is of particular importance in practice as sensitivities are (partly) hedged. If they are unstable (or worst, plainly wrong) it will generate unnecessary transactions and costs.

The HJM one-factor model and hypothesis used are described in the next section. Then we present some preliminary results before presenting the main results in the fourth section and the simplified formulas in the fifth section. Numerical implementation results are then presented.

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Model and Hypothesis

The model and main hypothesis used in this paper are the same as in Henrard (2003).

We use a model for $P(t, u)$, the price at t of the zero-coupon bond paying 1 in u. We will describe this for all $0 \leq t$, $u \leq T$, where T is some fixed constant.

When the discount curve $P(t, .)$ is absolutely continuous and positive, which is something that is always the case in practice as the curve is constructed from rates and by some kind of interpolation, there exists $f(t, u)$ such that

$$P(t, u) = \exp\left(- \int_t^u f(t,s) \mathrm{d}s \right) \tag{1}$$

The idea of Heath *et al.* (1992) was to exploit this property by modelling f as

$$\mathrm{d}f(t, u) = \mu(t, u)\mathrm{d}t + \sigma(t, u)\mathrm{d}W_t$$

for some suitable (possibly stochastic) μ and σ.

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Here we use a similar model, but we restrict ourself to non-stochastic coefficients. The exact hypothesis on the volatility term σ is described by (H2). We don't need all the technical refinement used in their paper or similar one, like the one described in Hunt and Kennedy (2000) in the chapter on *dynamical term structure model*. So instead of describing the conditions that lead to such a model, we assume that the *conclusions* of such a model are true. By this we mean we have a model, that we call a *HJM one-factor model*, with the following properties.

Let $A=\{(s,u)\in\mathbb{R}^2 : u\in[0,T] \text{ and } s\in[0,u]\}$. We work in a filtered probability space $(\Omega, F, \mathbb{P}^{\text{real}}, (\mathcal{F}_t))$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W^{\text{real}})_{0\le t\le T}$.

H1 There exists $\sigma:[0,T]^2\to\mathbb{R}^+$ measurable and bounded[1] with $\sigma=0$ on $[0,T]^2\backslash A$ such that for some process $(r_s)_{0\le t\le T}$, $N_t=\exp\left(\int_0^t r(s)\mathrm{d}s\right)$ forms, with some measure N, a numeraire pair[2] (with Brownian motion W_t),

$$\mathrm{d}f(t,u)=\sigma(t,u)\int_t^u\sigma(t,s)\mathrm{d}s\,\mathrm{d}t-\sigma(t,u)\mathrm{d}W_t$$

$$\mathrm{d}P^N(t,u)=P^N(t,u)\int_t^u\sigma(t,s)\mathrm{d}s\,\mathrm{d}W_t$$

and $r(t)=f(t,t)$.

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s)=N_t^{-1}P(t,s)$. To simplify the writing in the rest of the paper, we use the notation

$$v(t,u)=\int_t^u\sigma(t,s)\mathrm{d}s$$

Note that v is increasing in u, measurable and bounded.

To be able to use the explicit formula for the valuation of the European swaptions, we will also use the following hypothesis.

H2 The function σ satisfies $\sigma(t,u)=g(t)h(u)$ for some positive function g and h.

Note that this condition is essentially equivalent to the condition (H2) of Henrard (2003) but written on σ instead of on v. The condition on v was $v(s,t_2)-v(s,t_1)=f(t_1,t_2)g(s)$.

Example. The Ho and Lee (1986) volatility model and the Hull and White (1990) volatility model satisfy the condition (H2). For Ho and Lee one has $v(s,t)=\sigma(t-s)$ and $\sigma(s,t)=\sigma$; for Hull and White one has $v(s,t)=(1-\exp(-a(t-s)))\sigma/a$ and $\sigma(s,t)=\sigma\exp(-a(t-s))$. The volatility time-dependent versions of the models also satisfy the conditions.

Preliminary Results

We want to price some option in this model. For this we recall the generic pricing theorem (for example Theorem 7.33–7.34 in Hunt and Kennedy, 2000).

Theorem 1. Let V_T be some $\mathcal{F}_T$−measurable random variable. If V_T is attainable, then the time-t value of the derivative is given by $V_t^N = V_0^N + \int_0^t \phi_s \, dP_s^N$ where ϕ_t is the strategy and

$$V_t = N_t \mathrm{E}^{\mathrm{N}}\left[V_T N_T^{-1} \middle| \mathcal{F}_t\right].$$

We now state two technical lemmas that generalize the lemmas presented in Henrard (2003). Similar formulas can be found in Brody and Hughston (2004) in a different framework.

Lemma 1. Let $0 \le t \le u \le v$. In a HJM one-factor model, the price of the zero coupon bond can be written has,

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left(-\frac{1}{2}\int_t^u \left(v^2(s,v) - v^2(s,u)\right) ds + \int_t^u (v(s,v) - v(s,u)) \, dW_s\right)$$

Proof. By definition of the forward rate and its equation,

$$\begin{aligned} P(u,v) &= \exp\left(-\int_u^v f(u,\tau) d\tau\right) \\ &= \exp\left(-\int_u^v \left[f(t,\tau) + \int_t^u v(s,\tau) D_2 v(s,\tau) ds - \int_t^u D_2 v(s,\tau) dW_s\right] d\tau\right) \end{aligned}$$

Then using again the definition of forward rates and the Fubini theorem on inversion of iterated integrals, we have

$$\begin{aligned} P(u,v) &= \frac{P(t,v)}{P(t,u)} \exp\left(-\int_t^u \int_u^v v(s,\tau) D_2 v(s,\tau) d\tau \, ds + \int_t^u \int_u^v D_2 v(s,\tau) d\tau \, dW_s\right) \\ &= \frac{P(t,v)}{P(t,u)} \exp\left(-\frac{1}{2}\int_t^u \left(v^2(s,v) - v^2(s,u)\right) ds + \int_t^u v(s,v) - v(s,u) dW_s\right) \end{aligned}$$

Lemma 2. In the HJM one-factor model, we have

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s \, ds\right) = P(u,v) \exp\left(\int_u^v v(s,v) dW_s - \frac{1}{2}\int_u^v v^2(s,v) ds\right)$$

Proof. By definition of r,

$$\begin{aligned} r_\tau &= f(\tau,\tau) = f(t,\tau) + \int_t^\tau df(s,\tau) ds \\ &= f(t,\tau) + \int_t^\tau v(s,\tau) D_2 v(s,\tau) ds + \int_t^\tau D_2 v(s,\tau) dW_s \end{aligned}$$

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Then using Fubini, we have

$$\begin{aligned}\int_u^v r(\tau)\mathrm{d}\tau &= \int_u^v f(t,\tau)\mathrm{d}\tau + \int_u^v\int_s^v \nu(s,\tau)D_2\nu(s,\tau)\mathrm{d}\tau\,\mathrm{d}s - \int_u^v\int_s^v D_2\nu(s,\tau)\mathrm{d}\tau\,\mathrm{d}W_s \\ &= \int_u^v f(t,\tau)\mathrm{d}\tau + \frac{1}{2}\int_u^v \nu^2(s,v)\mathrm{d}s + \int_u^v \nu(s,v)\mathrm{d}W_s\end{aligned}$$

We give the pricing formula for swaptions for a future time. This is essentially the Theorem 3.1 of Henrard (2003) but written for any future time $t \geq 0$. Jamshidian (1989) also provides an *exact* solution for European swaption at time 0. His approach requires to solve a non-linear equation with respect to the instantaneous short rate r. Even if it is also based on the one-factor model, its approach is less *explicit* and as such more difficult to implement.

We represent swaps by their *cash-flow equivalent* representation. A swap receiving coupon $R\delta_i$ at time t_i and starting at t_0 is equivalent (in the pricing sense) to the set of cash-flows $c_0 = -1$ at t_0 (initial nominal), $c_i = R\delta_i$ at t_i ($1 \leq i \leq n-1$) and $c_n = R\delta_n + 1$ (nominal and coupon at maturity).

Theorem 2. Suppose we work in the HJM one-factor model with a volatility term of the form (H2). Let $\theta \leq t_0 < \ldots < t_n$, $c_0 < 0$ and $c_i \geq 0$ ($1 \leq i \leq n$). The price of a European receiver swaption, with expiry θ on a swap with cash-flows c_i and cash-flow dates t_i is given at time t by the $\mathcal{F}_t$-measurable random variable

$$\sum_{i=0}^{n} c_i P(t,t_i) N(\kappa + \alpha_i)$$

where κ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of

$$\sum_{i=0}^{n} c_i P(t,t_i) \exp\left(-\frac{1}{2}\alpha_i^2 - \alpha_i\kappa\right) = 0 \tag{2}$$

and

$$\alpha_i^2 = \int_t^\theta (\nu(s,t_i) - \nu(s,\theta))^2\,\mathrm{d}s$$

The price of the payer swaption is

$$-\sum_{i=0}^{n} c_i P(t,t_i) N(-\kappa - \alpha_i)$$

Proof. Let $\mu(s,\theta) = \nu(s,\theta)$ if $s \geq t$ and 0 if $s < t$. We define $W_\tau^\# = W_\tau - \int_0^\tau \mu(s,\theta)\mathrm{d}s$. By Girsanov's theorem (Lamberton and Lapeyre, 1997, Section 4.2.2, p. 72), the process $W^\#$ is a standard Brownian motion with respect to the probability $\mathbb{P}^\#$ of density

$$L_\theta = \exp\left(\int_0^\theta \mu(s,\theta)\mathrm{d}W_s - \frac{1}{2}\int_0^\theta \mu^2(s,\theta)\mathrm{d}s\right)$$

Using Lemma 1 and rewriting $v^2(s, t_i)-v^2(s, \theta)$ as $(v(s, t_i)-v(s, \theta))^2+2v(s, \theta)(v(s, t_i)-v(s, \theta))$, we have

$$P(\theta, t_i)=\frac{P(t, t_i)}{P(t, \theta)}\exp\left(-\frac{1}{2}\alpha_i^2-\alpha_i X\right)$$

where $-\alpha_i X=\int_t^\theta v(s, t_i)-v(s, \theta)\mathrm{d}W_s^\#$ and X is a standard normally distributed with respect to $\mathbb{P}^\#$. The hypothesis (H2) is used here to prove that the random variable X is the same for all i.

Using Lemma 2, we have $N_t N_\theta^{-1}=P(t, \theta)L_\theta$. By the generic pricing Theorem 1, the price of the option is

$$V_t=\mathrm{E}^\#\left[\max\left(\sum_{i=0}^{n} c_i P(t, t_i)\exp\left(-\frac{1}{2}\alpha_i^2-\alpha_i X\right), 0\right)\middle|\mathcal{F}_t\right].$$

Note that $P(t, t_i)$ is $\mathcal{F}_t$–measurable and X is independent of $\mathcal{F}_t$. Using a property of the conditional expectation (Lamberton and Lapeyre, 1997, Proposition A.2.5, p. 166), we can do this computation in two parts.

Let's fix $P(t, t_i)=P_i$. Like in the proof for $t=0$, we have $\Sigma\, c_i P_i \exp\left(-\frac{1}{2}\alpha_i^2-\alpha_i y\right) > 0$ if and only if $y<\kappa$ where κ is the unique solution of $\Sigma\, c_i P_i \exp\left(-\frac{1}{2}\alpha_i^2-\alpha_i y\right)=0$.

So we have $V_t=\phi(P)$ where $\phi(p)=\Sigma\, c_i p_i N(\kappa+\alpha_i)$. Or more explicitly

$$V_t=\sum c_i P(t, t_i)N(\kappa+\alpha_i)$$

where $P(t, t_i)$ and κ are $\mathcal{F}_t$–measurable and κ is implicitly defined by the equation (2)

2-Bermudan swaption

We are now in a position to state and prove the main theoretical result of this article concerning 2-Bermudan swaptions.

Theorem 3. Let $\theta_1<\theta_2$, $t_{i,j}$ ($i=1$, 2, $j=0$, …, n_i) be such that $\theta_i \le t_{i,0} < t_{i,1} < \cdots < t_{i,n_i}$ and $c_{i,j}$ ($i=1$, 2, $j=0$, …, n_i) be such that $c_{i,0}<0$ and $c_{i,j}\ge 0$ ($j>0$). In the HJM one-factor model, when the volatility term has the form (H2), the price of a 2-Bermudan receiver swaption with expiries θ_i and underlying swaps with cash-flow $c_{i,j}$ and cash-flow dates $t_{i,j}$ is given by

$$\begin{aligned} V_0=\mathrm{E}\Bigg(\max\Bigg(&\sum_{j=0}^{n_1} c_{1,j}P(0, t_{1,j})\exp\left(-\frac{1}{2}\alpha_{1,j}^2(0, \theta_1)-\alpha_{1,j}(0, \theta_1)X\right)\\ &\sum_{j=0}^{n_2} c_{2,j}P(0, t_{2,j})\exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_1)-\alpha_{2,j}(0, \theta_1)X\right)\\ &N\left(\kappa(X)+\alpha_{2,j}(\theta_1, \theta_2)\right)\Bigg)\Bigg) \end{aligned} \tag{3}$$

where $\kappa(X)$ is the unique solution of

$$\sum_{j=0}^{n_2} c_{2,j}P(0, t_{2,j})\exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_2)-\alpha_{2,j}(0, \theta_1)X-\alpha_{2,j}(\theta_1, \theta_2)\kappa\right)=0 \tag{4}$$

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X is a standard normally distributed random variable with respect to E and

$$\alpha_{i,j}^2(u, v) = \int_u^v \left(v\left(s, t_{i,j}\right) - v(s, v)\right)^2 \, \mathrm{d}s$$

The price of the payer swaption is

$$\begin{aligned} V_0 = \mathrm{E}\Bigg(\max\Bigg(&-\sum_{j=0}^{n_1} c_{1,j} P\left(0, t_{1,j}\right)\exp\left(-\frac{1}{2}\alpha_{1,j}^2(0, \theta_1) - \alpha_{1,j}(0, \theta_1)X\right), \\ &-\sum_{j=0}^{n_2} c_{2,j} P\left(0, t_{2,j}\right)\exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_1) - \alpha_{2,j}(0, \theta_1)X\right) \\ &N\left(-\kappa(X) - \alpha_{2,j}(\theta_1, \theta_2)\right)\Bigg)\Bigg) \end{aligned} \tag{5}$$

Proof. In θ_1 the price of the swaption is given by the maximum of the price of the first swap and the price of the European swaption on the second swap.

We define $W_t^\# = W_t - \int_0^t v(s, \theta_1)\mathrm{d}s$. By Girsanov theorem, the process $W^\#$ is a standard Brownian motion with respect to the probability $P^\#$ of density

$$L_{\theta_1} = \exp\left(\int_0^{\theta_1} v(s, \theta_1)\mathrm{d}W_s - \frac{1}{2}\int_0^{\theta_1} v^2(s, \theta_1)\mathrm{d}s\right)$$

Using Lemma 1, we have

$$P\left(\theta, t_{i,j}\right) = \frac{P\left(0, t_{i,j}\right)}{P(0, \theta_1)}\exp\left(-\frac{1}{2}\alpha_{i,j}^2(0, \theta_1) - \alpha_{i,j}(0, \theta_1)X\right) \tag{6}$$

where $-\alpha_{i,j}(0, \theta_1)X = \int_0^{\theta_1} v\left(s, t_{i,j}\right) - v(s, \theta_1)\mathrm{d}W_s^\#$ and X is a random variable with standard normal distribution with respect to $\mathrm{P}^\#$. As in Theorem 2, we use the hypothesis (H2) to prove that the random variable X is the same for all i and j.

By Lemma 2, $N_{\theta_1}^{-1} = P(0, \theta_1)L_{\theta_1}$ and so using the generic pricing Theorem 1 and the swaption pricing Theorem 2, we then have the Equation (3) where κ is the solution of

$$\sum_{j=0}^{n_2} P\left(\theta_1, t_{2,j}\right)\exp\left(-\frac{1}{2}\alpha_{2,j}^2(\theta_1, \theta_2) - \alpha_{2,j}(\theta_1, \theta_2)\kappa\right) = 0$$

By using the Equation (6) we obtain the described result for the value of κ.

Remark. The same approach is also valid for one-payer-one-receiver swaptions, choice swaption (at first expiry the holder has the choice between a payer and a receiver swaption, possibly with different expiry dates) or any combination of swaps and swaptions. For choice swaption where the choice is between different swaps (or set of cash-flows), as indicated in Henrard (2003), an explicit formula can be obtained.

Simplified Formulas

This result can be written in a form easier to compute.

Theorem 4. Under the conditions of Theorem 3, if the volatility structure satisfies (H2), then the value of the 2-Bermudan receiver swaption is given by the same formula (3) (payer given by (5)) but with $\kappa(X)=\left(\Lambda-\sqrt{G(\theta_2)-G(\theta_1)}X\right)/\sqrt{G(\theta_1)-G(0)}$ where Λ is the unique solution of

$$\sum_{j=0}^{n_2} c_{2,j}P(0,t_{2,j})\exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_2)-H(t_{2,j})\Lambda\right)=0, \tag{7}$$

$G(t)=\int_0^t g^2(s)ds$ and $H(t)=\int_0^t h(s)ds$ (with g and h described in (H2)).

Proof. Using condition (H2) we can write

$$\alpha_{i,j}^2(u,v)=\left(H(t_{i,j})-H(v)\right)^2(G(v)-G(u))$$

As g and h are positive, G and H are increasing and all the factors in the previous formula are positive. If we inject that description in (4) and simplify some factors, we have as equation for κ

$$\sum_{j=0}^{n_2} c_{2,j}P(0,t_{2,j})\exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_2)-H(t_{2,j})\left(\sqrt{G(\theta_1)-G(0)}X+\sqrt{G(\theta_2)-G(\theta_1)}\kappa\right)\right)=0$$

If we denote by Λ the term $\sqrt{G(\theta_1)-G(0)}X+\sqrt{G(\theta_2)-G(\theta_1)}\kappa$, which is independent of j, we have the result.

In the case where the underlying swaps have the same cash-flows after the settlement of the second swap, the formula can be simplified further. The simplification consists in the analytical solution of the expected value of the first swap in case of exercise in θ_1. This is applicable in particular for cancellable swaps and bonds with embedded 2-Bermudan options.

Theorem 5. Let $0<k$, $\theta_1\leq t_0<t_1<\ldots t_{k-1}<\theta_2\leq t_k<\ldots<t_n$, $c_j>0$, $(j=1,\ldots,n)$, $c_0<0$ and $d_k<0$. We consider two receiver swaps which are represented by the cash-flows $(c_0, c_1,\ldots,c_n)$ on dates $(t_0,\ldots,t_n)$ and by the cash-flows $(d_k,d_{k+1},\ldots,d_n)=(d_k,c_{k+1},\ldots,c_n)$ on dates $(t_k,\ldots,t_n)$.

In the HJM one-factor model, when the volatility term has the form (H2), the price of a 2-Bermudan receiver swaption with expiries θ_i and underlying swap described above is given by

$$\begin{aligned}V_0=&\sum_{j=0}^{n} c_jP(0,t_j)N\left(\mu+\alpha_j(0,\theta_1)\right)\\&+\mathrm{E}\left(\mathbb{1}(X\geq\mu)\max\left(\sum_{j=0}^{n} c_jP(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)-\alpha_j(0,\theta_1)X\right)\right.\right.\\&\left.\left.\sum_{j=k}^{n} d_jP(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)-\alpha_j(0,\theta_1)X\right)N\left(\kappa(X)+\alpha_j(\theta_1,\theta_2)\right)\right)\right)\end{aligned}$$

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where μ is the smallest solution of

$$\sum_{j=0}^{n} c_j P(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)-\alpha_j(0,\theta_1)\mu\right)$$
$$-\sum_{j=k}^{n} c_j P(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)-\alpha_j(0,\theta_1)\mu\right)N\big(\kappa(\mu)+\alpha_j(\theta_1,\theta_2)\big)=0$$

and κ is the function defined by (4).

The price of the 2-Bermudan payer swaption is

$$V_0=-\sum_{j=0}^{n} c_j P(0,t_j)N\big(-\mu-\alpha_j(0,\theta_1)\big)$$
$$+\mathrm{E}\Bigg(1\!\!1(X\leq\mu)\max\Bigg(-\sum_{j=0}^{n} c_j P(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)-\alpha_j(0,\theta_1)X\right)$$
$$-\sum_{j=k}^{n} d_j P(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)-\alpha_j(0,\theta_1)X\right)N\big(-\kappa(X)-\alpha_j(\theta_1,\theta_2)\big)\Bigg)\Bigg)$$

Proof. By the implicit function theorem applied to Equation (4), κ is continuous (as a function of X). Let $Q_j=c_jP(0,t_j)\exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1)\right)$. Note that $Q_0<0$ and $Q_j>0$ ($j=1,\ldots,n$). The difference between the value of the first swap and the swaption can be written as

$$\exp(-\alpha_0X)\left(Q_0+\sum_{j=1}^{k-1}Q_j\exp\big((\alpha_0-\alpha_j)X\big)+\sum_{j=k}^{n}Q_j\left(1-\frac{d_j}{c_j}N\big(\kappa(X)+\alpha_j\big)\right)\exp\big((\alpha_0-\alpha_j)X\big)\right)$$

As $\alpha_0-\alpha_j<0$, (see Henrard (2003) for a proof of it) all the coefficient in the exponentials are negative. Moreover as $d_j=c_j$ for $j>k$, $d_k<0$ and $0<N<1$, all the factors of the exponential are positive. The only negative term is Q_0.

Using all those elements, the term within the parenthesis tends to $+\infty$ as X tends to $-\infty$ and converges to $Q_0<0$ when X tends to $+\infty$.

By continuity at least one point exists for which the difference is 0. Also as it converges to $+\infty$ in $-\infty$, the set of zeros is bounded from below. This proves that the set of solutions, which is closed, has a finite minimum.

Remark. It is not clear if it is possible to have an equation for μ with several solutions. If the solution is unique, the price formula can be simplified further by removing the first term of the max.

Remark. For the result validity, the *existence* of μ is sufficient. Imposing the cash-flows to be the same after the second expiry is one way to achieve this but there are a lot of other cases for which this is true. But it is not obvious which one would be of practical relevance.

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Numerical Implementation

Expected values are usually computed through a numerical integration. The expected value we have to compute is the one of a random variable written as the function of a standard normal random variable. It means that we know quite well the weight of the distribution underlying our expected value. We can use points for the numerical integration with equal *weight* with respect to the underlying normal distribution. By using equally weighted points we concentrate the computation where they have a greater importance and so increase the precision for a given number of points in the numerical integration.

All the models used in the section have been implemented using the same language (Matlab[3]) and the computation time was measured at the same moment running all of them in a loop, without operator intervention.

European Swaption Speed

In this section we briefly study the pricing speed of several Hull–White model implementation for European swaptions. It may seem strange to study the European swaption computation speed in a paper on Bermudan swaptions. But a 2-Bermudan option is equal to an expected value involving European swaption or composed European options. So when you compute the external expectation you have the choice of computing the internal one independently. In this section we show that the explicit method on which this paper relies is more efficient when a good precision is required.

Even if we show this it does not mean that it necessarily has to be used for Bermudan options. Efficient use of the second step computations can lead to significant time saving. The recombining property of the Hull–white tree is one of them. Most of the points of the second expiry date are used to compute several points at the first date.

But this section will show that there is no hope to use a method for the first expiry independently (without efficient use of previous computations) of the method used for the second expiry without using the explicit approach for the second step.

For this we use a numerical integration technique with *equi-probable* points (not equi-distant). The Hull–White trinomial tree implementation is a standard one (as described in Brigo and Mercurio (2001) with long-term discount factors recovered from the one-step one described in Hull (2000, Section 21.9)). The explicit solution is the one described in Henrard (2003).

Figure 1 represents the computation time for $n=10$, 20, 50, 100, 200 and 500 steps for the Hull–White tree and numerical integration (for the numerical integration we also added $n=1000$, 2000 and 5000) and the constant time for the explicit solution. By step we mean the Hull–White tree equivalent. In a Hull–White trinomial tree, for n steps there are (approximatively) $2n+1$ final points. To have a correct comparison we also use $2n+1$ points in the numerical integration procedure.

As it can be seen from the graph, the tree approach is a lot slower when a lot of points are used (50 or above). The numerical integration is faster up-to 200 steps, but slower for more points. For the option used, one need more than 200 steps to have a price that is within 0.1% of the correct one.

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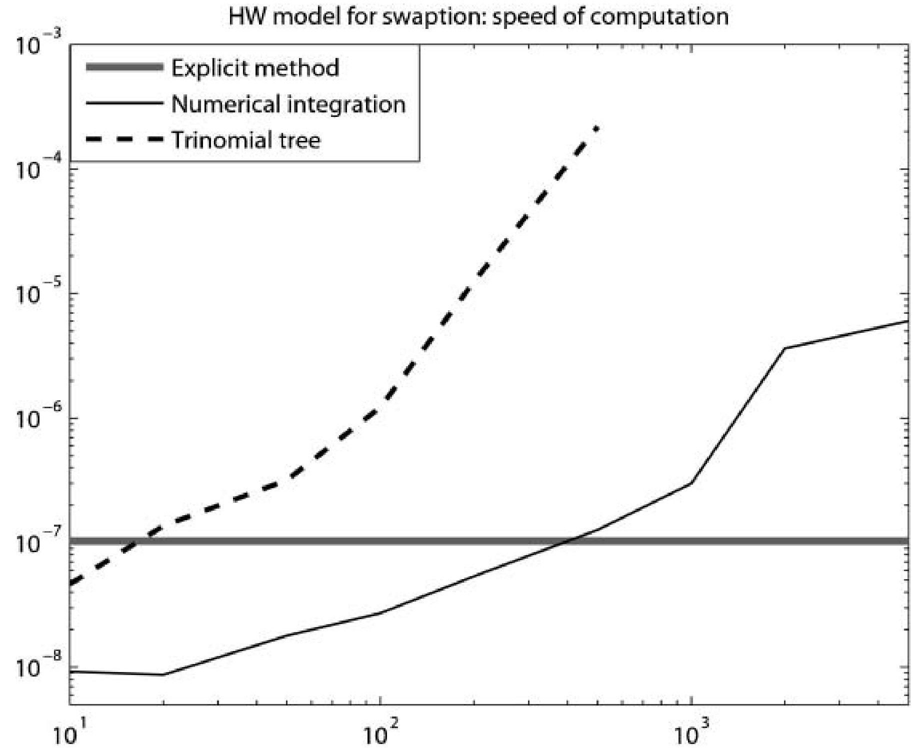

Figure 1. Computation time for European swaptions with the semi-explicit method, the numerical integral method and Hull–White tree

At this stage there is no clear evidence of the speed superiority of the explicit solution to the numerical integration when high precision is not required. But as the next section evidences, an intelligent use of intermediary computations, as described in Theorem 4 will prove that method largely superior.

Tree and Numerical Integration Speed

We now come back to our main subject, 2-Bermudan swaptions, and compare the speed of different implementations. On one hand we use the same classical Hull–White tree implementation and on the other hand for the numerical integration we use three different implementations: the *brute* implementation using directly Theorem 3, the *fast* implementation described in Theorem 4 with equiprobable points and the *semi-analytical* implementation of Theorem 5. The brute implementation recomputes the full price (and in particular the κ) of the European swaption at each point of the numerical integration. The two others compute only once the Λ of Theorem 4 and use it to deduce the κ for each point. We could have used also a double numerical integration. But as the explicit method as a speed of the same order of magnitude (or faster) and the brute explicit method will be proved a lot slower than the recombining approach, the extra comparison would not bring more information.

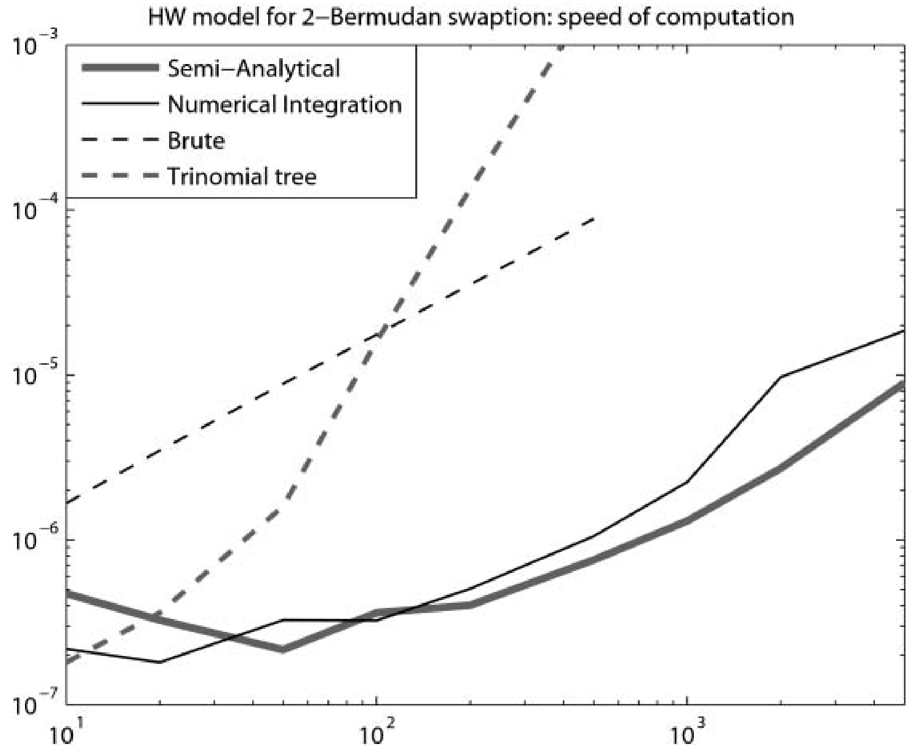

Figure 2. Computation time for 2-Bermudan swaptions with brute, fast and semi-explicit methods, and Hull–White trinomial tree

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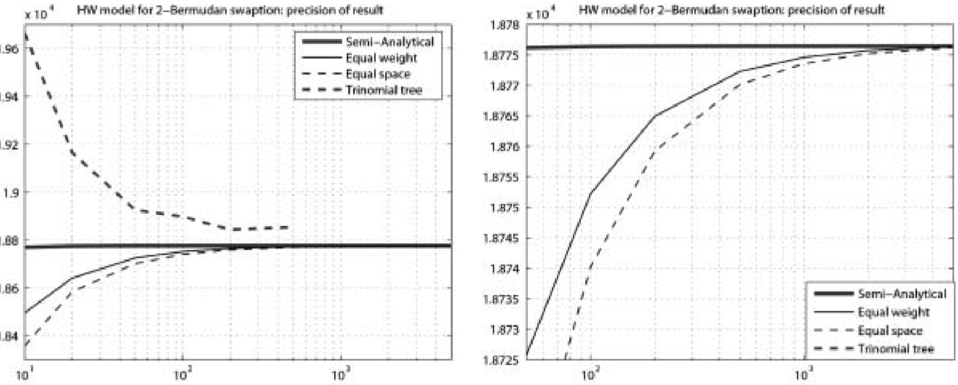
The examples are all on a 1y × 5y and 1.5y × 4.5y receiver swaption. The strike is close to the at-the-money rate of the first option. The yield curve used is the one of 28 October 2004[4]. We measure the speed for $n=10$, 20, 50, 100, 200 and 500 steps. For the semi-explicit and numerical integration we do it also for $n=1000$, 2000 and 5000. As in the previous section for n steps there are $2n+1$ points at first date in the tree and we use the same number of points in the numerical integration. The tree is extended up to the second expiry date while the numerical integration stops at the first date. The same number of steps is used for each of the two periods. So what we call a n step computation means $2n$ total steps and $4n+1$[5] final points in the tree and $2n+1$ points in the numerical integration. For the semi-explicit approach only the points corresponding to the non-explicit part are computed. So for an option with a 50% probability of exercise at the first date, only one half of the points are computed.

The results are graphically represented in Figure 2. The graph is on a log-log scale. So lines represent *exponent* laws. A regression of the log-steps with the log-speed gives a slope (exponent) of 2.1 for the tree and 1.0 for the brute force. This is what was expected from the number of computations. For the other methods the numbers are 0.6 and 0.5. There the situation is more complex as a large part of the computation (Λ and μ) is done before starting the points computation and is independent of the number of points.



(a) (b)

Figure 3. Convergence of the results for 2-Bermudan swaptions with brute, fast and semi-explicit methods, and Hull–White trinomial tree

It can be seen that using 5000 points with one of the efficient numerical integration approaches take still less time than for 100 points in the tree or brute approaches. Without discussing the convergence, it is clear that the proposed approaches are significantly more efficient than the tree of brute approaches.

Also the semi-explicit approach is faster for large number of points than the full numerical integration. Even if an extra equation is solved to find μ, the number of points is reduced in proportion of the probability of exercise at the first date.

Convergence of the Results

We compare the convergence of the results for several implementations. We still use the tree implementation but now for the numerical integration we use two versions of Theorem 4, one with *equi-distant* and one with *equi-probable* points, and the semi-explicit implementation of Theorem 5.

The results for $n=10$, 20, 50, 100, 200 and 500 steps for the tree and also $n=1000$, 2000 and 5000 for the integration approaches are given in Figure 3.

The graph clearly indicates that the semi-explicit is the best approach in terms of convergence. This can be explained by the way a Bermudan swaption behaves. The most valuable part is the first option. This part is valued explicitly and so is as precise as the double precision of the computer. The rest, which is small, is computed in the integral and converges to its true value. The second best is the implementation with equally probable points. This is also not surprising as the computations are concentrated on the more relevant points. Finally come the equally spaced point and the trinomial tree approaches. The tree has also equally spaced points at each step, it is therefore not surprising that they perform in a similar way.

The poor convergence/time ratio of the tree is not surprising as the number of grid points increases as n^2 while the precision, measured by the distance between final points, is in $\sqrt{n}$.

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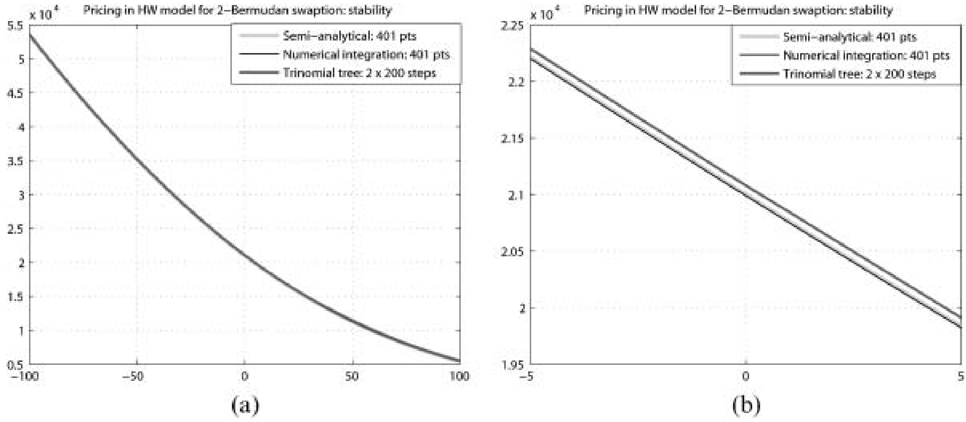

Figure 4. Price of 2-Bermudan swaption with numerical integration, semi-explicit methods, and Hull–White trinomial tree

Delta and Gamma

The results on price convergence can be extended even more successfully to delta and in an unrivalled way to Gamma.

The tree approaches are notoriously unstable for *greeks* computations and gamma numbers are dominated by numerical noises (see Henrard (2004) for some computation examples in the case of European swaptions).

We take the 2-Bermudan swaption of the previous section and compute its yield curve delta and gamma. By this we mean that we try to assess the first- and second-order price change coming from a parallel movement of one basis point of all market rates that compose the curve.

We compute those numbers for the initial yield curve but also for other yield curves resulting from shifts by half a basis point increment up to 100 basis points away from the current curve. The similar experience for European swaptions done in Henrard (2004) indicates that there is very little hope to obtain a correct result through the tree approach. We show the results for 2-Bermudan swaptions in Figures 4, 5 and 6. Those results are obtained with 200 steps (with the meaning of step described above)[6].

The price seems acceptable for all the methods if one doesn't look from too close. When one goes to the first order sensitivity, the delta, the results are bad for the tree method but there is still some hope to improve it by increasing the number of steps.

In the case of the gamma, the tree numbers are meaningless. Curing the problem would require more points in the tree than is possible for the memory of the computer. The two numerical integration approaches give acceptable results. They exhibit small oscillations, but still acceptable ones.

One of the tree problems is that even if a lot of points are used, the extreme ones are almost useless. In the example we study the *value* (as defined by the Q function in Brigo and Mercurio, 2001) of the 200 first points and 200 last points at the second expiry date have an average value of 4×10^{-95}. This is to be compared with an average of 0.0095 for the 101 central points. The central points of the tree are the only ones that bring value.

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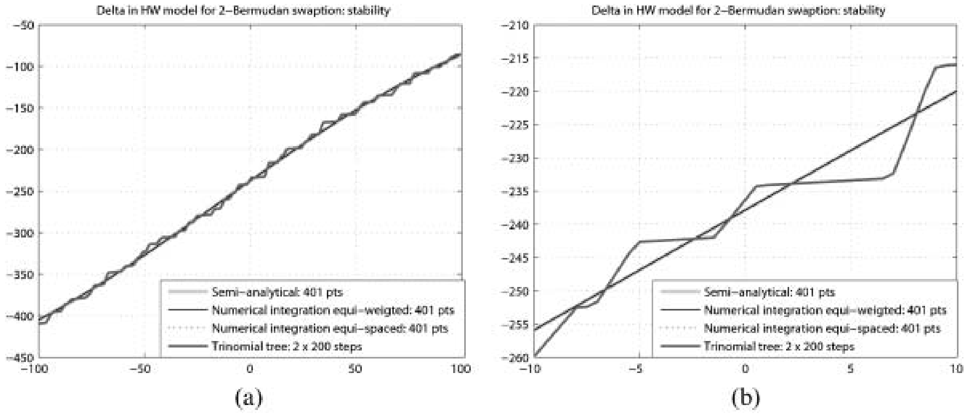

Figure 5. Delta of 2-Bermudan swaption with numerical integration, semi-explicit methods, and Hull–White trinomial tree

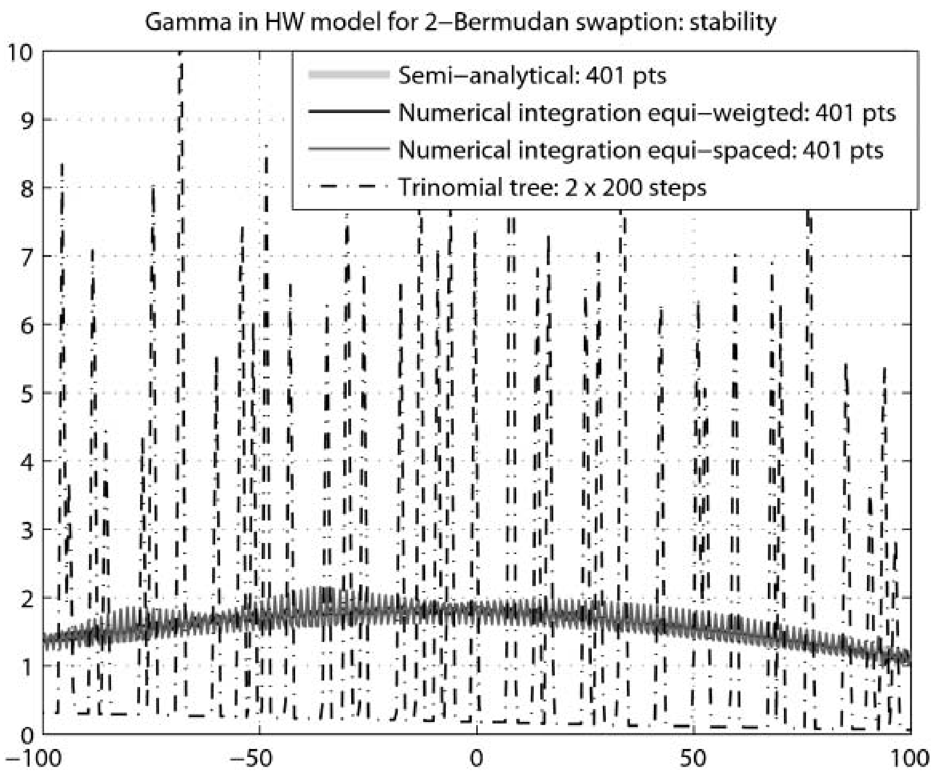

Figure 6. Gamma of 2-Bermudan swaption with numerical integration, semi-explicit methods, and Hull–White trinomial tree

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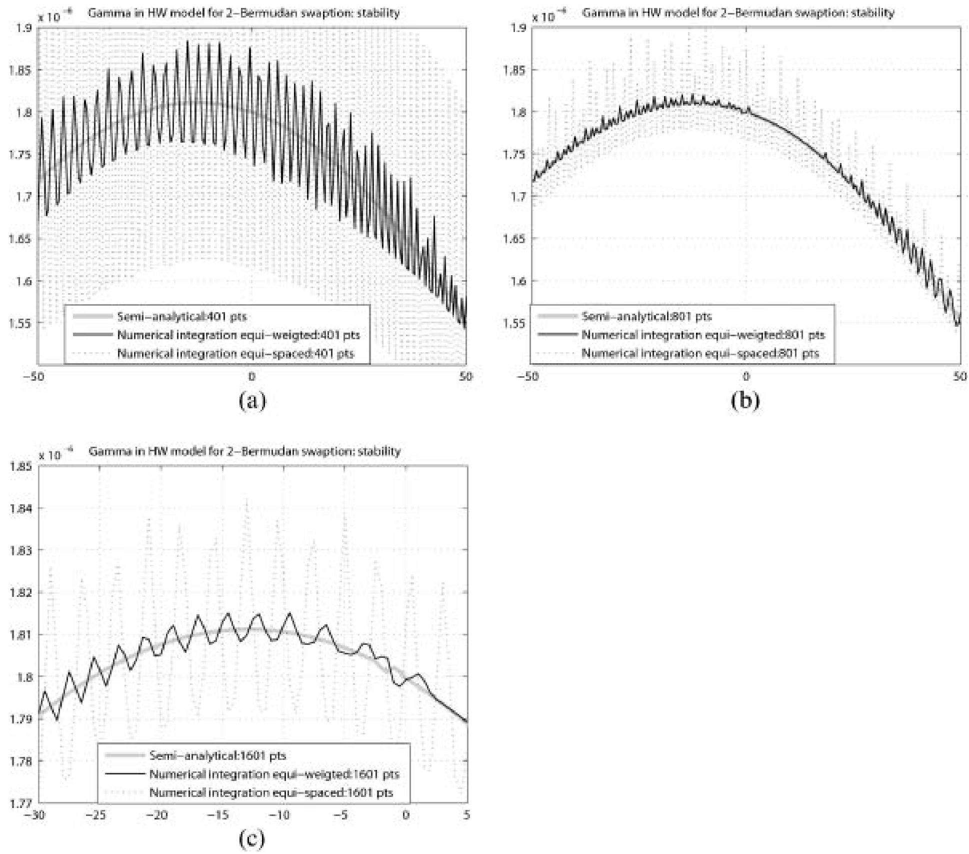

Figure 7. Gamma of 2-Bermudan swaption with numerical integration and semi-explicit methods

With 200 steps (401 points), the numerical integration gamma still lacks precision. But thanks to the speed efficiency we can increase the number of points without problems. Figure 7 gives the gamma profile for 401, 801 and 1601 points.

At initial scale only one line is visible. This is why we increase the scale with the number of points. The increased number of points gives results that for all practical purposes are smooth and precise enough. This is done with a computation time below the one for 50 steps in the trinomial tree approach!!!

Extension to General Bermudan Swaption

This approach will not work directly in practice for n-Bermudan swaption ($n \geq 3$) as $n-1$ iterated integrations would be required for a total of points of the order of p^{n-1} where a Hull–White tree has a number of final points in pn (total of the order of $(pn)^2$).

Nevertheless some extra analytical manipulation and selection of the points for the integration can bring the number of computation for an integration-like formula to

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pn^2. Part of the semi-explicit result can also be extended. This will be developed in a forthcoming article (in preparation). This approach is related to the one of Gandhi and Hunt (1997) who also suggest a recombining and numerical integration implementation.

Conclusions

Both in terms of speed and convergence the semi-explicit approach proposed in Theorem 5 performs better than more simple methods described in this paper. The improvements, both in terms of speed and precision, are even more impressive with respect to a standard Hull–White trinomial tree. The scope of the improvement is currently limited to 2-Bermudan swaptions but part of the method can be extended efficiently to more general swaptions.

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Notes

[1] Bounded is too strong for the proof we use, some L^1 and L^2 conditions are enough, but as all the examples we present are bounded, we use this condition for simplicity.

[2] See Hunt and Kennedy (2000) for the definition of a numeraire pair. Note that here we require that the bonds of *all* maturities are martingales for the numeraire pair (N, N).

[3] Matlab code available from the author.

[4] There is nothing special about that date, except it is my sister's birthday!

[5] As the second step is shorter (6m), the distance between points is also smaller and more than $4n+1$ final points are used.

[6] It took around four hours on my computer to run the (non-optimized) code to compute 3×401 yield curves and the prices for the 4 implementations using 200 steps precisions. As can be inferred from Figure 2, most of the time was devoted to the tree computations.

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