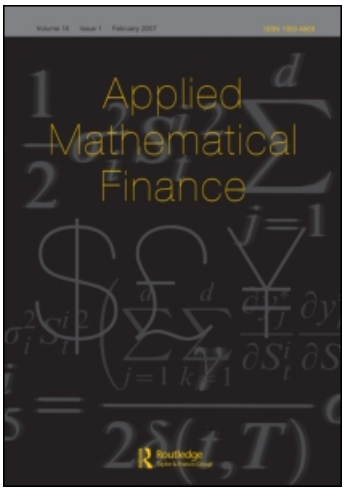

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An EZI Method to Reduce the Rank of a Correlation Matrix in Financial Modelling

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ABSTRACT *Reducing the number of factors in a model by reducing the rank of a correlation matrix is a problem that often arises in finance, for instance in pricing interest rate derivatives with Libor market models. A simple iterative algorithm for correlation rank reduction is introduced, the eigenvalue zeroing by iteration, EZI, algorithm. Its convergence is investigated and extension presented with particular optimality properties. The performance of EZI is compared with those of other common methods. Different data sets are considered including empirical data from the interest rate market, different possible market cases and criteria, and a calibration case. The EZI algorithm is extremely fast even in computationally complex situations, and achieves a very high level of precision. From these results, the EZI algorithm for financial application has superior performance to the main methods in current use.*

KEY WORDS: Correlation matrix, rank reduction, market models

Introduction

The value of financial derivatives often depends on more than one underlying market variable. Hence models for pricing and hedging can involve many state variables, each modelled as a stochastic process. When explicit solutions for derivatives prices are not available, numerical methods are used, for instance Monte Carlo simulation. Numerical methods become computationally burdensome when a high number of independent stochastic driving factors are used. For instance in market models for the term structure of interest rates there may be as many state variables as relevant forward Libor rates, perhaps more than thirty or forty. Hence the serious problem

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arises of reducing the number of independent factors to a number small enough for the numerical method to cope with.

When the interdependency among the instantaneous stochastic shocks driving the different variables is Gaussian, with some correlation matrix ρ, then reducing the number of independent stochastic factors is equivalent to reducing the rank of the correlation matrix. This is the case for Libor Market Models.

The problem of reducing the rank of an exogenous correlation matrix in financial modelling is the focus of this work. Rebonato and Jäckel (2000) investigate a related problem, discussing the use of an eigenvalue zeroing method (EZN) and a method based on an angles parameterization (OAP). Brigo (2002) extended their work, specifically examining the same problem as this paper. Zhang and Wu (2003) use Lagrange multiplier techniques, and Grubisic and Pietersz (2003) use geometric optimization.

We introduce an iterative algorithm for correlation rank reduction called eigenvalue zeroing by iteration (EZI). This algorithm is intuitive and simple to implement. Empirical results demonstrate the superiority in application of our algorithm over the two alternative methods commonly used in finance. It is much more accurate than EZN, achieving an accuracy comparable to the high precision OAP method in a fraction of the time. This allows one to increase remarkably computational speed in relevant financial applications, such as the calibration of multi-factor models.

In the second section the role of correlation in multi-dimensional financial models is discussed. In the third section we formally state the correlation rank reduction problem, present some properties of correlation matrixes, and describe two existing methods for solving the correlation rank reduction problem. In the fourth section we introduce and describe the EZI algorithm. We analyse convergence and discuss the optimality issue, relating the algorithm to alternating projections and introducing an extension with particular optimality properties. In the fifth section we give empirical results for EZI on various data sets including financial data from the interest rate market, making a precise comparison of its performance with the two methods common in finance. In the sixth section the methods are applied to a case of calibration of a Libor Market Model, showing how they affect computational efficiency and the quality of results. The final section concludes.

Correlation in Multivariate Financial Models

We present below a common use of correlation rank reduction in finance, with reference to multidimensional models as used in pricing applications or in risk management. In particular we consider, as the main example, a Libor Market Model for pricing interest rate derivatives. Consider a multivariate model involving M state variables $F_k(t)$, $k=1, 2, \ldots M$, where the vector $F(t)=(F_1(t), \ldots F_M(t))'$ has the dynamics

$$\mathrm{d}F(t)=\Psi(F, t)\mathrm{d}t+\Gamma^i\Upsilon(t)\mathrm{d}Y(t),\ i=1, 2 \tag{1}$$

where, for $r \leqslant M$, $Y(t)$ is a standard r-dimensional uncorrelated Wiener process under

a measure Q, $\Psi(F, t)$ is an $M \times 1$ vector, $\Upsilon(t)$ is an $M \times r$ matrix function of time only, and Γ^i is an $M \times M$ diagonal matrix with $\Gamma^1 = I_M$, the identity matrix, and $\Gamma^2 = diag(F(t))$.

When $i=1$ the diffusion part is modelled as an arithmetic Brownian Motion with time-varying diffusion coefficient. When $i=2$ the diffusion part is modelled as a geometric Brownian motion with time-varying covariance term $\Upsilon(t)$.

If $F(t)$ is a vector of discretely-compound forward rates and $i=2$, then for a suitable $\Psi(F, t)$ we have a Libor Market Model.

In (1) $\Upsilon(t)$ represents the covariance structure, describing explicitly the functional relationship between individual shocks. If (1) is used as a starting point to simulate F, the number r of independent stochastic factors heavily affects the computational burden, so that r is often chosen to be much smaller than M, the number of state variables.

Parameters in the functional form for $\Upsilon(t)$ can be found by calibration, minimizing some loss function representing the distance between model and market prices. However, in practice the most liquid products, such as caps and swaptions in the interest rate market, may bear imprecise or little, if any, information about an implied correlation matrix ρ. This leads to calibrated correlations often irregular, unreliable and not significant. Therefore often in the market volatility and correlation components are separated. Correlation may be computed exogenously, for instance estimated via econometric analysis of historical market data as in Rebonato (2002), and volatilities are instead the main parameters used to calibrate to current market prices.

In this context one can represent the model dynamics as

$$dF(t) = \Psi(F, t)dt + \Gamma^i \Sigma(t) dZ(t), \; i = 1, 2 \tag{2}$$

where $Z(t)$ is a standard M-dimensional Wiener process under Q with instantaneous correlation matrix ρ, and $\Sigma(t) = diag\{\sigma_i(t)\}_{i=1, \ldots, M}$ is a matrix of volatilities.

For (1) and (2) to be equivalent one needs

$$\Upsilon(t)\Upsilon(t)' = \Sigma(t)\rho\Sigma(t) \tag{3}$$

which implies $rank(\rho) = r$, the desired number of independent stochastic factors.

In general an exogenously given correlation matrix ρ is a full M-rank matrix. We need to replace ρ with a matrix $\hat{\rho}$ approximating ρ such that $rank(\hat{\rho}) = r$.

If the model being considered is a Libor Market Model, the correlation ρ represents the instantaneous forward rates correlation. M is the number of forward rates required by the financial products being priced and hedged, while r is the number of factors that the trader wishes to use in the numerical implementation of the model. r is usually much lower than M, since a low r enhances the speed of computation. Also empirical testing shows that a lower factor model can efficiently account for the prices of most derivatives. The matrix ρ can typically be obtained via an econometric analysis of fixed income time series, while $\Sigma(t)$ is fixed to match market prices of reference products, usually caps or swaptions.

Having introduced the correlation rank reduction problem in financial modelling, we present some facts about correlation matrixes and describe two common existing solution methods.

Correlation Rank Reduction

Define the following sets

$$S = \{ Y \in \mathbb{R}^{M \times M} | Y = Y' \}$$
$$P = \{ Y \in \mathbb{R}^{M \times M} | x' Yx \geq 0, x \in \mathbb{R}^M \}$$
$$D = \{ Y \in \mathbb{R}^{M \times M} | y_{ij} = 0, i \neq j \}$$
$$U = \{ Y \in \mathbb{R}^{M \times M} | y_{ii} \leq 1, i = 1, \ldots, M \}$$
$$U_x = \{ Y \in \mathbb{R}^{M \times M} | y_{ii} = x, i = 1, \ldots, M \}$$

S is the set of $M \times M$ symmetric matrixes, P the set of $M \times M$ positive semidefinite matrixes, D the set of diagonal matrixes, and U_1 the set of $M \times M$ matrixes with unit diagonal. S and U_0 are vector subspaces and P is a cone. U_x is an affine space. S, P, and U_1 are closed and convex.

A correlation matrix ρ is characterized by three properties: (1) symmetry; (2) positive semidefiniteness; (3) unit diagonal. These three properties imply, via Cauchy–Schwartz inequality, also the following property: (4) normalized entries, namely $|\rho_{ij}| \leqslant 1$. Thus the set C of $M \times M$ correlation matrixes is

$$C = S \cap P \cap U_1$$

which is closed and convex.

We define B_r to be the set of all $M \times M$ matrixes of rank r

$$B_r = \{ Y \in \mathbb{R}^{M \times M} | rank(Y) = r \}$$

and K_r to be the set of all $M \times M$ matrixes of rank less than or equal to r

$$K_r = \{ Y \in \mathbb{R}^{M \times M} | rank(Y) \leq r \}$$

K_r is not convex. However, it is always closed, unlike B_r.

The problem of correlation rank reduction can be formally stated as follows. Given a correlation matrix $\rho \in C$, the problem is to find a matrix $\hat{\rho} \in C \cap K_r$ such that

$$\hat{\rho} = \underset{Y \in C \cap K_r}{\arg\min} \{ \| Y - \rho \| \} \tag{4}$$

where $\| \cdot \|$ is some chosen metric.

Notice that the problem of correlation rank reduction in financial practice often aims at minimizing over $C \cap B_r$ rather than $C \cap K_r$. However, if Y is constrained to belong to $C \cap B_r$ the problem may have no solution. We discuss this issue in a later section.

Following Rebonato and Jäckel (2000) and Brigo (2002) we consider first the metric

$$SE = \| A - B \|^2 = \sum_{i,j=1}^{M} | A_{ij} - B_{ij} |^2 \tag{5}$$

induced by the Frobenius norm, used extensively in approximation and financial literature. Later we consider also robustness of the results when using different metrics.

In a market application context there are additional issues to consider in solving (4).

First, in practical applications accuracy must be traded off against computation time. In particular, in financial applications computational efficiency is extremely important, while accuracy is often not crucial since the correlation matrix to be approximated is a parametric form or an historical estimation, not obtained only from current prices. The broad structure of the correlation matrix is often the important aspect. In fact, the volatility matrix $\Sigma(t)$ is used to match current market prices, rather than the correlation matrix.

Secondly, in numerical methods for finance one exploits $\hat{\rho}=JJ'$, and the characteristics of J can affect computation, for instance in calibration (an example is given later).

We briefly present some results that will be used later in our discussion of correlation rank reduction methods.

The angles parameterization of a correlation matrix. (Rebonato and Jäckel, 2000). A parametric form for $A \in C \cap K_r$ of rank r is

$$A = BB'$$

where $B=\{b_{i,\,k}\}_{i=1,\,\ldots,\,M,\;k=1,\,\ldots,\,r}$ is an $M \times r$ matrix whose i^{th} row is given by

$$\begin{aligned} b_{i,\,1} &= \cos\theta_{i,\,1} \\ b_{i,\,k} &= \sin\theta_{i,\,1} \ldots \sin\theta_{i,\,k-1}\cos\theta_{i,\,k},\ 1<k<r \\ b_{i,\,r} &= \sin\theta_{i,\,1} \ldots \sin\theta_{i,\,r-1} \end{aligned} \tag{6}$$

Set $\theta=\{\theta_{i,\,k}\}_{i=1,\,\ldots,\,M,\;k=1,\,\ldots,\,r}$. We write $A(\theta)$ for the angles parameterization of A.

Spectral decomposition of a correlation matrix. $\rho \in C$ is symmetric positive semidefinite so it admits M linearly independent eigenvectors forming an orthonormal basis with corresponding eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_M \geqslant 0$. Let $\Delta=diag\{\lambda_i\}_{i=1,\,\ldots,\,M}$ and form the eigenvectors into a matrix X so that

$$\rho X = X\Delta$$

and $\rho=X\Delta X^{-1}=X\Delta X'$. We can write $\Delta=\Lambda\Lambda'$, where $\Lambda=\Lambda'$ is diagonal and $\Lambda_{ii}=\sqrt{\lambda_i}$, leading to

$$\rho = X\Lambda\Lambda' X' = (X\Lambda)(X\Lambda)' \tag{7}$$

We have the following theorem for symmetric matrixes.

Theorem 1. (Optimal rank reduction) Let $A \in S \cap K_k$ be of rank $k \geqslant r$ and let λ_1, λ_2, …, λ_k with $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_k|$, be its non-zero eigenvalues. Consider the spectral decomposition $A=X\Delta X'$ where $\Delta=diag(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$. Then for any

matrix $B \in K_r$

$$\|B-A\|^2 = \sum_{i,j=1}^{M} \left(B_{ij}-A_{ij}\right)^2 \geq \lambda_{r+1}^2 + \cdots + \lambda_k^2$$

and equality is attained when

$$B = X\widetilde{\Delta}X', \text{ with } \widetilde{\Delta} = diag(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$$

(See Harville, 1999).

Rank Reduction Methods

Descriptions of two techniques for correlation rank reduction are given by Brigo (2002) and Rebonato and Jäckel (2000).[1] We now present these techniques.

Eigenvalue zeroing with normalization (EZN). EZN is based on the spectral decomposition (7) of a symmetric positive semidefinite matrix, $\rho = (X\Lambda)(X\Lambda)'$. Let $\widetilde{\Lambda}$ be the matrix obtained by setting to zero the $M-r$ last entries of Λ, corresponding to the smallest $M-r$ eigenvalues of ρ. Set $L_r = S \cap P \cap U \cap K_r$ and define

$$f_1 : C \to L_r \tag{8}$$

$$f_1 : \rho \mapsto \widetilde{\rho} = \left(X\widetilde{\Lambda}\right)\left(X\widetilde{\Lambda}\right)' \tag{9}$$

$f_1(\rho)$ is symmetric and its eigenvalues are non-negative so it is positive semidefinite, but the diagonal values $\widetilde{\rho}$ of are not necessarily equal to 1. Since

$$\rho_{ii} = \sum_{j=1}^{M} X_{ij}^2 \lambda_j = 1, \; i = 1, \ldots, M$$

setting to zero the $M-r$ smallest eigenvalues implies

$$0 \leq \widetilde{\rho}_{ii} = \sum_{j=1}^{r} X_{ij}^2 \lambda_j \leq 1, \; i = 1, \ldots, M$$

so from the Cauchy–Schwartz inequality we have property 4, but property 3, namely unit diagonal, can be lost.

Brigo (2002) and Rebonato and Jäckel (2000) propose a rescaling. Set $B = X\widetilde{\Lambda}$ and write B_i for the ith row[2] of B. Set $\hat{\rho} = \left\{\hat{\rho}_{ij}\right\}_{i,j=1,\ldots,M}$ to be

$$\hat{\rho}_{ij} = \frac{\widetilde{\rho}_{ij}}{\sqrt{\widetilde{\rho}_{ii}\widetilde{\rho}_{jj}}} = \frac{(BB')_{ij}}{\sqrt{(BB')_{ii}(BB')_{jj}}} = \left(\frac{B_i}{\sqrt{B_i(B_i)'}}\right)\left(\frac{(B_j)'}{\sqrt{B_j(B_j)'}}\right) \tag{10}$$

This defines a map $\langle\cdot\rangle : S \cap P \cap K_r \to C$. $\langle\widetilde{\rho}\rangle = \hat{\rho}$ is the EZN r-rank correlation matrix. $<\cdot>$ is well defined only if $\widetilde{\rho}_{ii} > 0$ for all i.

EZN appears to be a method commonly used by practitioners. As we will see later, it is very fast but also inaccurate.

Optimization of the angles parameterization (OAP). The OAP method exploits the angles parameterization of a correlation matrix. The problem (4) is reduced to the unconstrained optimization

$$\hat{\rho} = \arg\min_{\theta} \|\rho - A(\theta)\|$$

where $A(\theta)$ is the angles parameterization of A. The result $\hat{\rho}$ of this optimization is the OAP reduced rank correlation matrix.

The OAP method is much more accurate than EZN, as will be shown in empirical tests. However, it relies on a complex optimization which can be very slow, in particular when M and r are large.

The Eigenvalue Zeroing by Iteration Algorithm, EZI

The OAP method requires the use of general optimization techniques to solve problem (4). The only specific structure it imposes is the use of angles parameterization. On the other hand, EZN has a strong theoretical justification, based on Theorem 1; setting to zero the $M-r$ smallest eigenvalues is always the optimal procedure to reduce the rank of a symmetric matrix.[3] Since correlation matrices are symmetric, it induces the map f_1 for reducing the rank of a correlation matrix ρ. $\tilde{\rho}=f_1(\rho)$ is the optimal matrix satisfying properties 1, 2 and 4 but $\tilde{\rho}$ is not guaranteed to be in C.

The second step (10) in the EZN algorithm modifies $\tilde{\rho}$ to obtain unit diagonal, but the resulting matrix is no longer optimal.

It is easy to see that when $A \in S \cap P$ with $a_{ii} \leqslant 1$, $i=1, \ldots, M$, then for any $B \in S \cap P$ with $b_{ii}=1$, $i=1, \ldots, M$, the quantity $\|B-A\|^2$ is minimized when $B=A+H$ where

$$H = diag(1-a_{11}, 1-a_{22}, \ldots, 1-a_{MM})$$

Hence we can define a map

$$f_2 : S \cap P \cap U \to C$$

$$f_2 : \tilde{\rho} \mapsto \rho = \tilde{\rho} + diag(1-\tilde{\rho}_{11}, \ldots, 1-\tilde{\rho}_{MM})$$

We also denote by f_2 the induced map $f_2: L_r = S \cap P \cap U \cap K_r \to C$. f_2 yields the correlation matrix ρ closest to $\tilde{\rho}$, but there is now no guarantee that $f_2(\tilde{\rho}) \in K_r$.

This suggests the possibility of iterating the two maps f_1 and f_2. Successive application of the map $g=f_1 \circ f_2$: $C \to C$ may give an improvement in accuracy compared to EZN while, unlike OAP, exploiting the efficiency of the eigenvalues zeroing procedure.

We formally state the algorithm. We start from an $M \times M$ correlation matrix ρ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$, whose rank is to be reduced to r.

Algorithm 2. Eigenvalue zeroing by iteration (EZI)

1. Set the iteration number $s=1$, $\rho^s=\rho$ and $a_s=0$.
2. Reduce the rank: Set $\tilde{\rho}^s=f_1(\rho^s)$ and $a_s=\|\rho^s-\tilde{\rho}^s\|$

3. If a stopping condition is true, stop. Return $\hat{\rho} = \langle \tilde{\rho}^s \rangle$ where $\langle \tilde{\rho}^s \rangle$ is defined by (10).
4. Recover a unit diagonal: Set $\rho^{s+1} = f_2(\tilde{\rho}^s)$. Set $s = s+1$ and go to step (2).

There are two stopping conditions.

1. For a tolerance level ε_1, stop when $\|\tilde{\rho}^s - \langle \tilde{\rho}^s \rangle\| < \varepsilon_1$.
2. Stop when $|a_s - a_{s-1}| < \varepsilon_2$.

In the first case $\langle \tilde{\rho}^s \rangle$ is surely a viable correlation of the required rank, and it is not distinguishable from $\tilde{\rho}^s$ to the tolerance ε_1.

In the second case the algorithm can proceed no further, and convergence to $\hat{\rho} \in C \cap K_r$ has not been achieved.

We now investigate convergence of the algorithm and identify structures in ρ^s that can prevent the algorithm from converging to a rank-reduced correlation matrix.

Convergence

We analyse the convergence of the sequence $a_s = \|\rho^s - \tilde{\rho}^s\|$. We use s-superscripts to denote matrices at the sth iteration, so that $\rho^s = X^s \Delta^s X^{s\prime}$ for instance.

Let $\overline{\Delta}^s = diag\{0, \ldots, 0, \lambda^s_{r+1}, \ldots \lambda^s_M\} \in D$ and define

$$Z^s = X^s \overline{\Delta}^s X^{s\prime} = \sum_{j=r+1}^{M} \lambda_j^s (X^s)^j (X^s)^{j\prime} = \rho^s - \tilde{\rho}^s$$

$$D^s = diag\{1 - \tilde{\rho}^s_{ii}\} = diag\left\{ \sum_{j=r+1}^{M} \lambda_j^s \left(X^s_{ij}\right)^2 \right\} = diag\{Z^s_{ii}\} = \rho^{s+1} - \tilde{\rho}^s$$

By Theorem 1, $\left\|\rho^{s+1} - \tilde{\rho}^{s+1}\right\| \le \left\|\rho^{s+1} - \tilde{\rho}^s\right\|$, so $\left\|Z^{s+1}\right\| \le \|D^s\|$ and

$$0 \le \left\|Z^{s+1}\right\| \le \|D^s\| \le \|Z^s\| \tag{11}$$

so the sequence $a_s = \|\rho^s - \tilde{\rho}^s\| = \|Z^s\|$ converges uniformly to $a \geqslant 0$.

If $\left\|Z^{s+1}\right\| = \|Z^s\|$ for some s then $\|D^s\| = \|Z^s\|$ so $Z^s = D^s$ is diagonal and $\|Z^t\| = \|Z^s\|$ for all $t \geqslant s$. Conversely if $Z^s = D^s$ is diagonal for some s then $\|Z^t\| = \|Z^s\|$ for all $t \geqslant s$.

Since $a \le \|D^s\| \le \|Z^s\|$ we have $\|Z^s\|^2 - \|D^s\|^2 = \|Z^s - D^s\|^2 \to 0$, that is, the non-diagonal elements of Z^s go to zero. Since $D^s_{ii} \ge 0$ for all i we conclude $D^s \to D^\infty \in D$ and $Z^s \to D^\infty \in D$ also. Note that $a_{s+1} < a_s$ if and only if $Z^s \notin D$.

Since $\rho^{s+1} - \rho^s = D^s - Z^s$ we have $\left\|\rho^{s+1} - \rho^s\right\| \to 0$, achieving the limit if $Z^s = D^s$ for some s. Similarly, $\tilde{\rho}^{s+1} - \tilde{\rho}^s = D^s - Z^{s+1}$ and $\left\|\tilde{\rho}^{s+1} - \tilde{\rho}^s\right\| \to 0$.

We also have $\left\|\rho^{s+1} - \tilde{\rho}^s\right\| = \|D^s\| \le \|Z^s\| = \|\rho^s - \tilde{\rho}^s\|$ and

$$\|Z^s - D^s\|^2 + \left\|Z^{s+1}\right\|^2 = \|Z^s\|^2 - \|D^s\|^2 + \left\|Z^{s+1}\right\|^2.$$

If $a_s \to 0$ then there exists $\rho^\infty = \langle \tilde{\rho}^\infty \rangle \in C \cap K_r$ such that $\rho^s \to \rho^\infty$. The EZI algorithm converges to a matrix ρ^∞, which is a correlation matrix of the required rank r.

If $\|Z^s\| = 0$ for some s we have convergence to zero so $\langle \widetilde{\rho}^s \rangle = \widetilde{\rho}^s = \rho^s = \rho^\infty$ and the algorithm has converged to a correlation matrix of the desired rank. If $0 \neq Z^s \in D$ for some s then $a = \|Z^s\|$ and the algorithm stops. If $0 \neq Z^s \notin D$ is never diagonal then we have seen that Z^s converges to a diagonal matrix D^∞ and stopping rule 2 will apply.

EZI can allow a remarkable improvement in accuracy compared to EZN (as we see in the empirical tests of the fifth section). This is true even if $a_s \nrightarrow 0$. EZI returns $\langle \widetilde{\rho}^s \rangle$ for some $s \geqslant 1$. EZN returns $\langle \widetilde{\rho}^1 \rangle$. Since $\|\rho^s - \widetilde{\rho}^s\| \leq \left\|\rho^{s-1} - \widetilde{\rho}^{s-1}\right\|$ EZI returns a matrix rescaled from one at least as close to a correlation matrix as that returned by EZN.

Eigenvector Structure

Suppose that for some s, $0 \neq Z^s \in D$ so that $a = \|Z^s\| \neq 0$. We investigate implications for the structure of eigenvectors of X^s and of ρ^s.

Since

$$Z^s = X^s \overline{\Delta}^s X^{s\prime} = X^s \overline{\Delta}^s (X^s)^{-1} \tag{12}$$

the matrixes Z^s and $\overline{\Delta}^s$ are similar and have the same set of eigenvalues. Thus if $0 \neq Z^s \in D$ then Z^s and $\overline{\Delta}^s$ have the same elements, possibly in a different order. Furthermore (12) implies

$$Z^s (X^s)^i = \overline{\Delta}^s_{ii} (X^s)^i, \quad i = 1, \ldots, M \tag{13}$$

where $(X^s)^i$ is the ith column of X^s.

Let $\mathbb{M} = \{1, 2, \ldots, M\}$ and define

$$\alpha^Z = \left\{ j \in \mathbb{M} \middle| Z^s_{jj} = 0 \right\} \tag{14}$$

$$\alpha^\Delta = \left\{ i \in \mathbb{M} \middle| \overline{\Delta}^s_{ii} = 0 \right\} \tag{15}$$

to be the set of all indices for Z^s and $\overline{\Delta}^s$ corresponding to null diagonal elements. α^Z and α^Δ have the same number of elements. Write $\overline{\alpha}^Z = \mathbb{M} \backslash \alpha^Z$ and $\overline{\alpha}^\Delta = \mathbb{M} \backslash \alpha^\Delta$ for the set of indices not in α^Z or α^Δ respectively.

From (13),

$$Z^s (X^s)^i = \begin{cases} 0, & i \in \alpha^\Delta \\ \overline{\Delta}^s_{ii} (X^s)^i, \ \overline{\Delta}^s_{ii} > 0, & i \notin \alpha^\Delta \end{cases}$$

This implies the following complementarity conditions:

$$\left[1_{\alpha^\Delta}(i) 1_{\overline{\alpha}^Z}(j) + 1_{\overline{\alpha}^\Delta}(i) 1_{\alpha^Z}(j) \right] X^s_{ij} = 0 \tag{16}$$

where $1_A(i)$ is the indicator function for i belonging to the set A. We conclude that when $0 \neq Z^s \in D$, the columns $(X^s)^i$ are divided in two sets: those with index in α^Δ have zeros in every position not in α^Z, while those with indexes not in α^Δ have zeros in every position in α^Z.

The structure of ρ^s. We analyse now the structure of ρ^s when $0 \neq Z^s \in D$. Recall that

$$\rho^s_{i,j} = \left(X^s\sqrt{\Delta^s}\right)_i \left(X^s\sqrt{\Delta^s}\right)'_j$$

and the rows (and its columns) of X^s form an orthonormal set. From (16)

$$\left[1_{\alpha^Z}(i)1_{\overline{\alpha}^Z}(j) + 1_{\overline{\alpha}^Z}(i)1_{\alpha^Z}(j)\right]\rho^s_{ij} = 0$$

Furthermore, when $i, j \in \overline{\alpha}^Z$,

$$\rho^s_{ij} = \sum_{k=1}^{M} \lambda^s_k X^s_{ik} X^s_{jk} = \sum_{k \notin \alpha^\Delta} \lambda^s_k X^s_{ik} X^s_{jk} = Z^s_{ij}$$

So when $Z^s \in D$, if $i \neq j$ and one of i or j is in $\overline{\alpha}^Z$, then $\rho^s_{ij} = 0$. We have

$$\rho^s_i = \left((\rho^s)^i\right)' = e_i,\ i \in \overline{\alpha}^Z \tag{17}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the ith position.

Definition 3. For an $M \times M$ correlation matrix $\rho \in C$, if there exists $\phi \neq H \subseteq \mathbb{M}$ such that $i \in H \Rightarrow \rho_i = e_i$, then we say that ρ has block elementary structure for indices $i \in H$, and for each index $i \in H$, ρ has an uncorrelated factor.

We have shown that Z^s diagonal and non-null always implies a block elementary structure with uncorrelated factors for all $i \in \overline{\alpha}^Z$. We now determine conditions for an uncorrelated factor to have index in $\overline{\alpha}^Z$.

Given a matrix A with block elementary structure, there exists a permutation matrix Q such that QAQ' is block-diagonal. QAQ' is the Gantmacher Normal Form (GNF) of matrix A (De Giuli and Magnani, 1998).

The spectral properties of a matrix are invariant when moving to its GNF, since QAQ' is similar to A. λ is an eigenvalue of A if and only if λ is an eigenvalue of one of the diagonal blocks in the GNF of A. If ρ^s has block elementary structure the diagonal block represented by a diagonal value ρ^s_{ii} associated with an uncorrelated factor has a unit eigenvalue. Consequently for $0 \neq Z^s \in D$ if $i \in \overline{\alpha}^Z$ then the associated eigenvector $\lambda = 1$ is one of the $M - r$ smallest eigenvalues.

Definition 4. When there exist uncorrelated factors in a block elementary structure that belong to the set of $M - r$ smallest eigenvalues we call this a relevant block elementary structure (RBE structure) and call these uncorrelated factors relevant uncorrelated factors.[4]

We have shown that

$$0 \neq Z^s \in D \Rightarrow \rho^s \text{ has REB structure} \tag{18}$$

A stronger statement is possible. Suppose ρ^s has χ relevant uncorrelated factors. When $0 \neq Z^s \in D$, $\chi = \left|\overline{\alpha}^Z\right|$. From the definition of $\overline{\alpha}^Z$, these are the only non-zero eigenvalues in the set of the $M - r$ smallest eigenvalues. Therefore

$$rank(\rho^s) = r + \chi$$

with r eigenvalues $\geqslant 1$.

Definition 5. When a correlation matrix with χ relevant uncorrelated factors has rank $r+\chi$, we say that it has fixed RBE (FRBE) structure.

So

$$0 \neq Z^s \in D \Rightarrow \rho^s \text{ has FRBE structure} \tag{19}$$

It follows from the discussion below that matrixes with FRBE structure are fixed points of the EZI algorithm. When $\chi=0$, a matrix with FRBE structure is already in $C \cap K_r$.

RBE structure and convergence. We have seen that if $\|\rho^s - \tilde{\rho}^s\| \to a = \|Z^s\|$ for some s then ρ^s has RBE structure for some s. We now investigate the converse.

Suppose ρ^s has RBE structure with $\chi=|H|$ relevant uncorrelated factors for indices $i \in H \subseteq \mathbb{M}$. Let Q be the permutation matrix moving rows with indices in H to the first χ positions. Then $Q\rho^s Q' = QX^s D^s X^{s\prime} Q'$ is block diagonal,

$$Q\rho^s Q' = \begin{pmatrix} I & 0 \\ 0 & \rho^s_{SE} \end{pmatrix}$$

where ρ^s_{SE} is $(M-\chi)\times(M-\chi)$. Set to zero the $M-r$ smallest eigenvalues. These include the unit eigenvalues in the top left block. We find again a block diagonal matrix, where the top left block is null,

$$Q\tilde{\rho}^s Q' = QX^s \tilde{D}^s X^{s\prime} Q' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\rho}^s_{SE} \end{pmatrix}$$

where $\tilde{\rho}^s_{SE}$ is $(M-\chi)\times(M-\chi)$. Then $Q'QX^s \tilde{D}^s X^{s\prime} Q' Q = X^s \tilde{D}^s X^{s\prime} = \tilde{\rho}^s$ has

$$\tilde{\rho}^s_i = \left((\tilde{\rho}^s)^i \right)' = 0,\ i \in H$$

Correct the diagonal to recovery a correlation matrix ρ^{s+1}. Then

$$\rho^{s+1}_i = \left((\rho^{s+1})^i \right)' = e_i,\ i \in H$$

Since in EZI the r biggest eigenvalues never decrease in size,[5] if a unit eigenvalue belongs to the set of the $M-r$ smallest eigenvalues at iteration s, then it belongs to this set for any $t \geqslant s$ so ρ^s and $\tilde{\rho}^s$ will always differ by 1s in the ith diagonal values, $i \in H$. Hence

$$\|\rho^s - \tilde{\rho}^s\| \geq \chi \not\to 0 \text{ as } s \to \infty$$

and we conclude

Proposition 6. If ρ^s has RBE structure then $a_s \to a > 0$.

This implies that if the initial matrix ρ has RBE structure, the EZI algorithm does not ultimately converge to a correlation matrix of rank r. Instead, stopping rule 2 will apply at some point.

We have identified a set of correlation matrixes ρ for which the EZI algorithm will not converge to a rank-r correlation matrix. Even so, EZI will always improve on the initial correlation matrix ρ, unless ρ already has FRBE structure.

An extreme case is when $\rho=I$, the identity matrix. In this case there is no correlation at all among the underlying variables. Here the problem of reducing model dimension via reducing the rank of the correlation matrix is not determined.

Note that if ρ has RBE structure, then the map $<\cdot>$ (10) is undefined for $\widetilde{\rho}$ (since some denominators would be zero). Thus neither in EZN nor in EZI can it be used.

If $a_s \to a \neq 0$ and $a \neq \|Z^s\|$ for any s, we know that Z^s converges to a diagonal matrix D^∞. The analysis above goes through 'module ε'.

When a matrix ρ has RBE structure, a different approach can be considered. Assume ρ_{ii} is a diagonal block in the GNF affected by rank reduction. Blocks like this, together with adjacent null blocks, can be cut out from ρ. The other eigenvalues of such a matrix are not affected by this operation and the matrix consisting of the remaining blocks is still a correlation matrix. Its rank can be reduced and then the ρ_{ii} block can be re-inserted in the resulting matrix. The re-insertion will increase the rank, but no other eigenvalues will be altered.

In the next section we show how EZI relates to the theory of alternating projections and to approximation algorithms in the literature, and present an extension of the EZI algorithm with OAP optimality properties.

EZI and Alternating Projections

K_r is always closed, unlike B_r. In the previous section we mentioned the fact that the correlation rank reduction problem in financial practice is often ill-posed, by requiring a minimization over $C \cap B_r$ rather than over $C \cap K_r$. See Chu, Funderlich and Plemmons (2003) for a discussion of similar issues in a general context. In the financial literature usually $C \cap K_r$ is considered, as in (4). An advantage of the algorithm EZI is that it can automatically solve also the problem with the minimization over $C \cap B_r$ when this is possible.

The EZI algorithm can be formulated as an alternating projections scheme. Suppose H is a Hilbert space and suppose $T_k \subseteq H$, $k=1, \ldots, K$ are subsets of H. A map $\Pi_k: H \to T_k$ is an orthogonal projection if $\Pi_k^2 = \Pi_k$ and $<a-\Pi_k(h), h-\Pi_k(h)>=0$ for all $a \in T_k$, $h \in H$. The point $\Pi_k(h) \in T_k$ is the point in T_k closest to $h \in H$. Let $T = \bigcap_{k=1}^{K} T_k$ and suppose $T \neq \phi$, and let Π_T be the orthogonal projection onto T.

Von Neumann (1950) showed that if the T_k are closed subspaces of H then for $h \in H$

$$\lim_{n \to \infty} \|(\Pi_K \Pi_{K-1} \ldots \Pi_1)^n(h) - \Pi_T(h)\| = 0 \tag{20}$$

so that the limit of alternating projections onto each T_k converges to the projection onto T. The same results applies when the T_k are affine sets[6] rather than subspaces.

This theory has been extended to sets T_k which are closed and convex. For closed and convex sets the optimal algorithm involves a correction to the projections. It

reduces to the Von Neumann case when the T_k are closed subspaces or affine sets. See Boyle and Dykstra (1985).

Maps in EZI are orthogonal projections. f_1 is an orthogonal projection into $L_r = S \cap P \cap U \cap K_r$ and f_2 is an orthogonal projection into C so EZI consists of alternating projections.[7] In addition the projection f_1 into L_r coincides both with projection into K_r and into B_r.

However we have seen that the correlation rank reduction problem presents some anomalies, so that the precise result (20) on optimality does not extend to this case. In the following we first investigate the magnitude of such anomalies, and then we present an algorithm for exploiting EZI efficiency together with the optimality properties of different methods. In order to investigate the magnitude of the anomalies in the sets relevant to correlation rank reduction, we see now some results for matrixes in L_r.

For $Y \in \mathbb{R}^{M \times M}$ write $\lambda_i(Y)$ for its ith largest eigenvalue.

Theorem 7. (See Schott, 1996) Let $A \in S$ and $B \in P$. Then for $i=1, \ldots, M$, we have

$$\lambda_i(A+B) \geq \lambda_i(A).$$

When both $A, B \in S \cap D$ then $\lambda_i(A+B) \geqslant \max(\lambda_i(A), \lambda_i(B)) \geqslant 0$.

EZI starts from ρ. The rank reduction step does not affect the r highest eigenvalues, and Theorem 7 can be applied to the map f_2, so for any rank reduced matrix $\tilde{\rho}^s = f_1(\rho^s)$ generated by EZI we have $\lambda_i(\tilde{\rho}^s) \geq \lambda_i(\rho)$, $i=1, \ldots, r$.

Define the set L_r^ρ as

$$L_r^\rho = \left\{ Y \in L_r \mid \lambda_i(Y) \geq \frac{1}{2}\lambda_i(\rho),\ i=1, \ldots, r \right\}$$

(The reason for the factor $\frac{1}{2}$ will become clear shortly.) Then f_1 is the projection $\Pi_r^\rho : C \to L_r^\rho$ giving the closest matrix in L_r^ρ to matrixes in C.

We can investigate the distance of L_r^ρ from convexity. Suppose $A, B \in L_r^\rho$. From Theorem 7, for all $0 \leqslant \alpha \leqslant 1$, $i=1, \ldots, r$,

$$\begin{aligned} \lambda_i(\alpha A + (1-\alpha)B) &\geq \max(\alpha\lambda_i(A), (1-\alpha)\lambda_i(B)) \\ &\geq \max(\alpha\lambda_i(\rho), (1-\alpha)\lambda_i(\rho)) \\ &\geq \frac{1}{2}\lambda_i(\rho) \end{aligned}$$

Since for any $Y \in L_r \subset U$ we have $\sum_{i=1}^M \lambda_i(Y) = tr(Y) \leq M$, it follows that for $0 \leqslant \alpha \leqslant 1$,

$$\sum_{i=1}^{M} \lambda_i(\alpha A + (1-\alpha)B) \leq M$$

$$\sum_{i=r+1}^{M} \lambda_i(\alpha A + (1-\alpha)B) \leq M - \sum_{i=1}^{r} \lambda_i(\alpha A + (1-\alpha)B) \leq M - \sum_{i=1}^{r} \frac{1}{2}\lambda_i(\rho)$$

We define the distance of a matrix X from L_r^ρ as $\left\|X, L_r^\rho\right\| = \min_{Y \in L_r^\rho} \|X - Y\|^2$. Since the minimum is achieved for a matrix Y which is also the minimum in K_r,

$$\left\|\alpha A + (1-\alpha)B, L_r^\rho\right\| \leq \left(M - \sum_{i=1}^{r} \frac{1}{2}\lambda_i(\rho)\right)^2$$

This bound is related to the anomaly of the rank reduction problem in the setting of alternating projections, namely to the distance from convexity of the set L_r^ρ. It depends on the spectral properties of the initial matrix ρ, on M and on r. The bound is larger when the eigenvalues of ρ are of similar magnitude and when $M-r$ is high. It is narrower when (1) some eigenvalues dominate the others; or when (2) $M-r$ is smaller.

The first case occurs when the correlation matrix has a pronounced factor structure, as is the situation for many financial examples, such as term structure data. The next section shows that the EZI algorithm performs very well, in terms of both accuracy and speed, on this kind of financial matrix. This is particularly noticeable when the second case also applies, namely when $M-r$ is smaller.

Remark. The above analysis suggests that it may be relevant to incorporate the projection correction given by Dykstra (1983) and Boyle and Dykstra (1985) for closed and convex sets. But a natural implementation of this correction into the EZI algorithm fails since it may lead to diagonal values greater than unity for $\widetilde{\rho}^s$ and f_2 may not project onto C. These problems were confirmed by empirical tests.

The structure of the EZI algorithm and the anomalies of the sets involved closely recall the composite property mapping algorithm of Cadzow (1988), and the extension for Toeplitz structures of Chu, Funderlich and Plemmons (2003). These alternating projection algorithms for non-convex sets have been successfully applied in signal and image enhancement, speech encoding and filter design. We will see, in our tests in the next section, that also in finance they allow a remarkable advantage in efficiency compared to traditional methods.

Although empirical testing shows that the accuracy of these methods is very high, one may desire to recover some optimality properties typical of general optimisation methods, for example of OAP which involves the use of a general optimisation method. A similar issue is considered in Chu, Funderlich and Plemmons (2003). They use an alternating projection algorithm in order to render a general optimization method, with desired properties, more tractable and efficient in dimension reduction problems.

In our context a natural implementation is the algorithm below, that in the following we call the EZI+ algorithm.

Algorithm 8.

1. With $\rho^1 = \rho$, apply the EZI algorithm until a stopping condition, such as $\|f_1(\rho^s) - \langle f_1(\rho^s)\rangle\| < \varepsilon_1$, is true
2. Set $\theta_0 = A^{-1}$ ($<f_1(\rho^s)>$)

3. Starting from θ_0, solve

$$\hat{\rho} = \arg\min_{\theta} \|\rho - A(\theta)\|$$

The solution $\hat{\rho}$ satisfies the optimality properties typical of OAP, however the algorithm exploits the high efficiency and accuracy of EZI. We will see in the next section that this algorithm allows us to replicate OAP results with noticeable improvement in efficiency, although not with the same speed as EZI alone.

Numerical Results on Financial Data

We give numerical results to compare the performance of EZI with that of EZN and OAP. Four test matrixes are used including parametric forms and market correlations. Our test matrixes represent features commonly found in money market applications, for instance. We describe the test matrixes and then present numerical results.

Example: ρ_1. This matrix is used by Rebonato and Jäckel (2000) in a risk management example:

$$\rho_1 = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.4 \\ 0.7 & 0.4 & 1 \end{pmatrix}$$

ρ_1 is not a correlation matrix,[8] since it has a negative eigenvalue. However the negative eigenvalue is also the smallest one in absolute value, so setting it to zero one simultaneously recovers a viable correlation matrix and reduces the rank. Hence EZI, OAP and EZN can be applied as usual.

Example: ρ_2. The test matrix is a full rank 10×10 matrix $\rho_2 = \{\rho_{2,\,ij}\}_{i,\,j=1,\,\ldots,\,10}$ with parametric form

$$\rho_{2,\,ij} = 0.5 + (1 - 0.5)\exp(-0.05|i-j|) \tag{21}$$

This form was introduced for forward Libor rates in Rebonato (1999). Brigo (2002) uses ρ_2 to give comparisons between EZN and OAP. Eigenvalues of ρ_2 are given in Table 1 which also shows the cumulative percentage variation accounted for by the leading eigenvalues.

Example: ρ_3. This 10×10 matrix $\rho_3 = \{\rho_{3,\,ij}\}_{i,\,j=1,\,\ldots,\,10}$ is given by the modified parametric form

$$\rho_{3,\,ij} = \exp(-|i-j|) \tag{22}$$

also used by Brigo (2002). As $|i-j|$ increases there is a sharp decrease in correlations. Eigenvalues of ρ_3 are given in Table 1.

Table 1. Eigenvalues for test matrices

	Example ρ_2		Example ρ_3		Example ρ_4				
	Eig.	%Cum.	Eig.	%Cum.	Eig.	%Cum.		Eig.	%Cum.
1	9.271	92.7	2.04	20.4	11.7	61.6	11	0.20	96.2
2	0.421	96.9	1.74	37.8	2.15	72.9	12	0.17	97.1
3	0.127	98.2	1.39	51.7	1.18	79.1	13	0.16	97.9
4	0.059	98.8	1.10	62.7	0.72	82.9	14	0.15	98.7
5	0.036	99.1	0.88	71.5	0.64	86.2	15	0.09	99.2
6	0.025	99.4	0.72	78.8	0.43	88.5	16	0.06	99.5
7	0.019	99.6	0.62	84.9	0.39	90.5	17	0.05	99.7
8	0.016	99.7	0.54	90.3	0.34	92.3	18	0.03	99.9
9	0.014	99.9	0.50	95.3	0.28	93.8	19	0.02	100
10	0.013	100	0.47	100	0.25	95.1			

Example: ρ_4. The final test matrix ρ_4 is a 19×19 correlation matrix for annual discretely compounded forward rates, given in Table 2. The matrix is estimated from market quoted Euro data during the period from 1 February 2001 to 1 February 2002. The eigenvalues of ρ_4 are given in Table 1. Reducing the rank of matrixes such as these is an essential step in calibrating Libor Market Models of interest rates. For example, ρ_4 is the correlation matrix used in the calibration to market swaption data of 1 February 2002, shown in the following section.

Table 2. Market forward rate correlation matrix

$$\rho_4 = \begin{pmatrix}
1.00 & 0.82 & 0.69 & 0.65 & 0.58 & 0.47 & 0.29 & 0.23 & 0.43 & 0.47 & 0.33 & 0.43 & 0.29 & 0.23 & 0.26 & 0.21 & 0.23 & 0.29 & 0.25 \\
0.82 & 1.00 & 0.80 & 0.73 & 0.68 & 0.55 & 0.45 & 0.40 & 0.53 & 0.57 & 0.42 & 0.45 & 0.48 & 0.34 & 0.35 & 0.32 & 0.32 & 0.31 & 0.32 \\
0.69 & 0.80 & 1.00 & 0.76 & 0.72 & 0.63 & 0.47 & 0.56 & 0.67 & 0.61 & 0.48 & 0.52 & 0.48 & 0.54 & 0.46 & 0.42 & 0.45 & 0.42 & 0.35 \\
0.65 & 0.73 & 0.76 & 1.00 & 0.78 & 0.67 & 0.58 & 0.56 & 0.68 & 0.70 & 0.56 & 0.59 & 0.58 & 0.50 & 0.50 & 0.48 & 0.49 & 0.44 & 0.35 \\
0.58 & 0.68 & 0.72 & 0.78 & 1.00 & 0.84 & 0.66 & 0.67 & 0.71 & 0.73 & 0.70 & 0.67 & 0.64 & 0.59 & 0.58 & 0.65 & 0.65 & 0.53 & 0.42 \\
0.47 & 0.55 & 0.63 & 0.67 & 0.84 & 1.00 & 0.77 & 0.68 & 0.73 & 0.69 & 0.77 & 0.69 & 0.66 & 0.63 & 0.61 & 0.68 & 0.70 & 0.57 & 0.45 \\
0.29 & 0.45 & 0.47 & 0.58 & 0.66 & 0.77 & 1.00 & 0.72 & 0.71 & 0.65 & 0.65 & 0.62 & 0.71 & 0.62 & 0.63 & 0.66 & 0.64 & 0.52 & 0.38 \\
0.23 & 0.40 & 0.56 & 0.56 & 0.67 & 0.68 & 0.72 & 1.00 & 0.73 & 0.66 & 0.64 & 0.56 & 0.61 & 0.72 & 0.59 & 0.64 & 0.64 & 0.49 & 0.46 \\
0.43 & 0.53 & 0.67 & 0.68 & 0.71 & 0.73 & 0.71 & 0.73 & 1.00 & 0.75 & 0.59 & 0.66 & 0.69 & 0.69 & 0.69 & 0.63 & 0.64 & 0.52 & 0.40 \\
0.47 & 0.57 & 0.61 & 0.70 & 0.73 & 0.69 & 0.65 & 0.66 & 0.75 & 1.00 & 0.63 & 0.68 & 0.70 & 0.63 & 0.64 & 0.65 & 0.62 & 0.52 & 0.40 \\
0.33 & 0.42 & 0.48 & 0.56 & 0.70 & 0.77 & 0.65 & 0.64 & 0.59 & 0.63 & 1.00 & 0.83 & 0.72 & 0.64 & 0.58 & 0.68 & 0.73 & 0.57 & 0.45 \\
0.43 & 0.45 & 0.52 & 0.59 & 0.67 & 0.69 & 0.62 & 0.56 & 0.66 & 0.68 & 0.83 & 1.00 & 0.82 & 0.69 & 0.67 & 0.70 & 0.69 & 0.65 & 0.43 \\
0.29 & 0.48 & 0.48 & 0.58 & 0.64 & 0.66 & 0.71 & 0.61 & 0.69 & 0.70 & 0.72 & 0.82 & 1.00 & 0.79 & 0.78 & 0.79 & 0.72 & 0.59 & 0.42 \\
0.23 & 0.34 & 0.54 & 0.50 & 0.59 & 0.63 & 0.62 & 0.72 & 0.69 & 0.63 & 0.64 & 0.69 & 0.79 & 1.00 & 0.82 & 0.83 & 0.79 & 0.60 & 0.45 \\
0.26 & 0.35 & 0.46 & 0.50 & 0.58 & 0.61 & 0.63 & 0.59 & 0.69 & 0.64 & 0.58 & 0.67 & 0.78 & 0.82 & 1.00 & 0.90 & 0.80 & 0.50 & 0.22 \\
0.21 & 0.32 & 0.42 & 0.48 & 0.65 & 0.68 & 0.66 & 0.64 & 0.63 & 0.65 & 0.68 & 0.70 & 0.79 & 0.83 & 0.90 & 1.00 & 0.94 & 0.71 & 0.46 \\
0.23 & 0.32 & 0.45 & 0.49 & 0.65 & 0.70 & 0.64 & 0.64 & 0.64 & 0.62 & 0.73 & 0.69 & 0.72 & 0.79 & 0.80 & 0.94 & 1.00 & 0.82 & 0.66 \\
0.29 & 0.31 & 0.42 & 0.44 & 0.53 & 0.57 & 0.52 & 0.49 & 0.52 & 0.52 & 0.57 & 0.65 & 0.59 & 0.60 & 0.50 & 0.71 & 0.82 & 1.00 & 0.84 \\
0.25 & 0.32 & 0.35 & 0.35 & 0.42 & 0.45 & 0.38 & 0.46 & 0.40 & 0.40 & 0.45 & 0.43 & 0.42 & 0.45 & 0.22 & 0.46 & 0.66 & 0.84 & 1.00
\end{pmatrix}$$

Comparison of Results

The EZN, OAP and EZI methods were implemented in Matlab and run on a 1 Ghz Pentium III PC. First we aim at assessing the performance of each algorithm in terms of accuracy, with slight restraints on computational effort. EZN is not an iterative algorithm thus no stopping criteria need to be specified. As for EZI, stopping condition 1 is used, with tolerance level taken to be $\varepsilon_1 = 10^{-9}$. Since results with this implementation appear satisfactory both in terms of accuracy and computational effort, we keep this implementation also in the succeeding comparison.

OAP is implemented via the Matlab fminsearch algorithm, as in Brigo (2002). We maintain Matlab default termination tolerances (10^{-4} on both the objective function value and the argument), and default maximum number of iterations ($200 \times$ number of variables). The maximum level of function evaluations allowed is set to 10^7.

In our tests, these criteria allow OAP to exploit fully its potential in terms of accuracy, with very long computational times, confirming indications in previous literature. OAP appears to be very accurate. According to error (5), it achieves the same accuracy as EZI does with the above ε_1, and often it is even slightly more accurate. However, this requires computational times which are much longer than those of EZI. This happens in particular for the realistic correlation matrix ρ_4. For example, at rank 8 more than 10 minutes are required by OAP to achieve 0.5972, while EZI achieves 0.6128 in 0.28 seconds. At rank 12, 0.0983 is reached by OAP in 2203 seconds, while EZI achieves 0.1008 in 0.27 seconds. At rank 14, 0.0216 is achieved by both methods, but OAP takes 2523 seconds, compared with 0.24 seconds for EZI.

In order to make comparison of the methods as clear as possible, we modify the OAP criteria to make OAP achieve in all tests an error at least as low as EZI, to an accuracy of 4 dp as above, but in the shortest time possible. This was obtained by using as stopping criterion a termination tolerance of 10^{-8} on the objective function value and 10^{-1} on the argument, with Matlab default bound on iterations and 10^8 bound on function evaluations to avoid cases of explosion of computational time. A tolerance level of higher order brings about some errors for OAP which are higher than errors for EZI. Thus these results give maximum efficiency possible for OAP without becoming less precise than EZI to 4 dp.

Table 3 gives results for ρ_1, ρ_2 and ρ_3, Table 4 for ρ_4. For each test matrix results are shown for reducing the rank down to a target rank r. For each method the sum of square errors (5) is shown and the time taken in seconds (in round brackets). The number of iterations used by EZI is given in square brackets.

Compare first the accuracy of the standard methods OAP and EZN. For the very simple matrix ρ_1 EZN performs almost as well as OAP. But as matrixes become more realistic, and r increases, EZN performs progressively worse than OAP. For ρ_4 EZN is much less accurate than OAP, usually achieving an error twice as great as the error achieved by OAP.

EZI accuracy is very close to that of OAP. In the examples in Table 3 it is often within 0.1% of OAP. For the more realistic case of ρ_4 given in Table 4 its error is always within a few percent of that of OAP, often much closer, and they go to coincide as r increases.

Table 3. Comparison of rank reduction methods

Comparison of methods: errors and times						
Matrix:	ρ_1	ρ_2			ρ_3	
Target rank, r:	2	2	4	7	4	7
EZN:	1.004e-4 (0.02)	0.1134 (0.02)	0.0163 (0.05)	2.32e-3 (0.02)	6.14 (0.07)	1.20 (0.06)
OAP:	0.947e-4 (0.08)	0.0764 (0.34)	0.0069 (3.1)	0.919e-3 (15.2)	5.95 (12.7)	1.12 (27.6)
EZI:	0.946e-4 (0.02) [8]	0.0765 (0.29) [50]	0.0070 (0.08) [19]	0.918e-3 (0.05) [8]	5.96 (0.14) [23]	1.13 (0.10) [10]

Note the computation times. EZN is always very fast, and EZI is of comparable speed. By contrast, even after optimizing computational times, OAP is considerably slower than either EZN or EZI. Already for the example ρ_3 it takes almost one hundred times longer at rank 4 than EZI to achieve only slightly greater accuracy. For the empirical example ρ_4 it takes more than eight minutes for $r=10$, more than 2000 times longer than EZI with the same accuracy. In some cases it is even more time consuming. Computation time for OAP tends to increases as r increases, in particular up to $r=12$, whereas for EZI the number of iterations, and hence the computation time, decreases.

Different metrics. In the tests above, consistent with Rebonato and Jäckel (2000) and Brigo (2002), we used the sum of square errors (5) as a reference metric. One may be interested in seeing how the methods perform when also other criteria, less common, but possibly at times relevant, are considered. Therefore we compare performances under the following three financially reasonable criteria.

$$SE\% = \sum_{i,j} \left(\frac{a_{ij} - b_{ij}}{a_{ij}} \right)^2; \; AE = \sum_{i,j} |a_{ij} - b_{ij}|; \; AE\% = \sum_{i,j} \frac{|a_{ij} - b_{ij}|}{|a_{ij}|}$$

Table 4. Comparison of rank reduction methods

Comparison of methods for the empirical matrix: errors and times									
r:	2	4	6	8	10	12	14	16	18
EZN:	27.04 (0.03)	9.00 (0.04)	3.67 (0.04)	1.51 (0.03)	0.56 (0.03)	0.24 (0.03)	0.046 (0.03)	0.011 (0.03)	0.0007 (0.03)
OAP:	19.11 (9)	4.54 (51)	1.51 (160)	0.60 (433)	0.23 (505)	0.098 (1318)	0.022 (583)	0.006 (846)	0.0004 (504)
EZI:	19.59 (1.1) [100]	4.84 (0.59) [59]	1.57 (0.34) [34]	0.61 (0.23) [23]	0.23 (0.18) [19]	0.101 (0.16) [17]	0.022 (0.13) [13]	0.006 (0.09) [12]	0.0004 (0.06) [8]

Table 5. Comparison of methods: errors with different metrics

Matrix:		ρ_2		ρ_3		ρ_4			
Target rank, r:		2	7	4	7	4	8	12	16
OAP	SE%	0.089	0.001	671e4	802e3	13.70	1.80	0.34	0.027
	AE	2.20	0.229	18.70	8.14	32.03	10.34	4.36	0.946
	AE%	2.42	0.243	6010	2491	56.90	17.81	7.93	1.819
EZI	SE%	0.089	0.001	495e4	353e3	13.05	1.81	0.35	0.027
	AE	2.24	0.226	18.42	8.02	31.90	10.37	4.49	0.909
	AE%	2.38	0.240	5241	1855	53.84	17.77	8.13	1.745

Notice, in particular, that percentage criteria SE% and AE% can be particularly relevant in this context. In fact, due to the differences in magnitude of the correlation entries, the relative importance of the discrepancies can be more informative than their absolute value, although at times relative errors can be very high. In Table 5 we present results for EZI and OAP.[9]

We see that again errors are very close, apart from some pathological cases for percentage errors. These further tests confirm that the two methods have similar accuracy.

High dimension matrixes. These tests on high dimension matrixes are of particular relevance for financial application. Often a 6-month tenor, rather than a 1-year tenor, is needed for forward rates, usually for consistency with the typical tenor structure underlying caplets in the Euro market. To cover the same 20-year period as the matrix ρ_4, a trader must consider 39 semiannual forward rates, and a 39×39 correlation matrix. Such a number of variables, or even higher, is common with interest rate derivatives. Obviously, in this case it is even more important that the dimension of the model gets reduced for efficient implementation, so correlation rank reduction is particularly relevant. We test correlation rank reduction when the correlation matrix is given by functional forms (21) and (22), but with M set to 39. In particular we consider reduction to rank 7, a number of factors more tractable but still retaining flexibility.

For the (21) matrix, with the same implementation details as before, both methods reach SE=0.271. OAP takes 1348 seconds, but EZI takes only 2.3 seconds.

For (22) matrix OAP gives an error SE=119.78 in 1671 seconds, while EZI gives an error SE=115.06 in 2.1 seconds.

This confirms that in relevant and realistic tests EZI is very efficient, beside being highly accurate. In fact OAP, in order to reach (almost) the same accuracy as EZI, takes hundreds of times longer than EZI.

Previous tests have shown that both OAP and EZI have a clear advantage in accuracy over the most commonly used method EZN. Considering a range of different errors and different matrixes, the levels of accuracy allowed by these two methods appear similar. However EZI reaches such accuracy in a much shorter time. We conclude that, for most financial applications, where rapid computation is

relevant, EZI may be the method of choice. An example of such application is shown in the following section.

EZI+. We now test EZI+ of Algorithm 8 on ρ_4. EZI+ has the same optimality properties as OAP but exploits the efficiency of EZI. Our target is to recover the same level of accuracy achieved by OAP, as shown in the empirical results of Table 4. Table 6 compares computation times for OAP (taken from Table 4) with those for EZI+ to achieve the same level of accuracy.

Apart from the rank 2 case, which was already quite fast, in all other cases computational times are cut at least by a half compared to OAP, and often much more. The most burdensome case, rank 12, is reduced to about a third of the OAP time, while many of the others are reduced to about a fifth.

OAP is expensive to run. EZI+ speeds up the general OAP optimization by using alternating projections and is much faster than OAP. However, for a very slight difference from the error achieved by EZI, the general optimization step in EZI+ requires a computational time which is often hundreds of times longer than EZI alone.

In spite of the general optimization step, EZI+ is noticeably more efficient than OAP, with the same accuracy. Consequently we recommend the use of EZI+ as in Algorithm 8 when the optimality properties of optimization methods such as OAP are required, while we suggest that for most financial applications EZI of Algorithm 2 remains preferable.

LMM Calibration

One important application of rank reduction methods in finance is the calibration of multi-factor models, such as the Libor Market Model for interest rate derivatives. When an exogenous correlation matrix is given, for instance via econometric analysis, one must ensure that the correlation is reduced to a rank tractable for subsequent pricing, so rank reduction is an important part of the calibration procedure. This is underlined for instance by Rebonato (2002). Since one starts from a correlation matrix not obtained solely from current prices of tradable assets, extreme closeness to such econometric correlation is not the main goal. Because very frequent recalibration is now a well-established standard, computational speed is instead a crucial issue.

In the following we compare the performances of EZI and OAP as parts of a Libor Market Model calibration procedure. We also assess, since it is relevant from a financial point of view, how the use of a particular rank reduction method can affect the results.

Table 6. Computational times for Algorithm 8

r:	2	4	6	8	10	12	14	16	18
OAP:	(9)	(51)	(160)	(433)	(505)	(1318)	(583)	(846)	(504)
EZI+:	(8)	(18)	(72)	(116)	(92)	(431)	(111)	(102)	(148)

The calibration methodology we apply is based on the swaption volatility formula of Rebonato (1998). We follow Rebonato (2002), section 10.5, which shows for a set of co-terminal swaptions that, thanks to this swaption volatility formula, calibration can be performed by simple matrix manipulation. We consider in particular the case of a general piecewise constant parameterization of volatility, as in Brigo and Mercurio (2001, 2002) and Brigo *et al.* (2005). In this case the matrix relationships allow calibrating to various sets of co-terminal swaptions via an efficient cascade algorithm inverting the formula of Rebonato (1998).[10]

As part of the calibration, an exogenous correlation matrix ρ must be reduced to a rank r correlation matrix $\hat{\rho} = JJ'$. The calibrated volatility and the matrix J can then be used, for instance, in pricing exotic products with r independent stochastic factors. Both OAP and EZI methods can be readily used in this context. OAP returns a matrix $\hat{\rho} = BB'$ where B is given by (6). As for EZI output correlation matrix $\langle \widetilde{\rho}^s \rangle$, it is easily decomposed as $\langle \widetilde{\rho}^s \rangle = JJ'$. With notation as in (10), set $J_i = \frac{B_i^s}{\sqrt{B_i^s \left(B_i^s\right)'}}$ where $B^s = X^s \sqrt{\widetilde{D}^s}$. Then

$$\langle \widetilde{\rho}^s \rangle_{ij} = \left(\frac{B_i^s}{\sqrt{B_i^s \left(B_i^s\right)'}} \right) \left(\frac{\left(B_j^s\right)'}{\sqrt{B_j^s \left(B_j^s\right)'}} \right) = J_i J_j'$$

Rebonato (2002) in section 9.1.2 gives a number of relevant calibration cases. The third case regards calibration to swaptions with an exogenous forward rate correlation matrix. We calibrate a model for 19 forward rates to a 10×10 swaption matrix of 1 February 2002. The forward rate estimated correlation matrix consistent with this trading day is ρ_4. We test calibration of models of all ranks from 19 to 2.

Both with OAP and EZI exact recovery of market prices is achieved, so calibration error is zero in both cases. Looking at the parameters obtained, both methods usually give calibrations that are robust and significant, but when OAP is used the rank 4 calibration encounters numerical problems, returning a few negative parameters. This typically happens when reduced rank correlation matrixes are less smooth and regular. When EZI was used no numerical problems were found with any rank of the estimated correlation matrix.

We now give some examples of computational times for the whole calibration procedure. For calibrating with two factors, with EZI the procedure takes about 5 seconds, with OAP about 13 seconds. When the number of factors increases, the efficiency of EZI relative to the other method is more pronounced. With four factors we have almost 1 minute for OAP versus 5 seconds with EZI, with six factors almost 3 minutes for OAP while EZI reduces to about 4.5 seconds. For higher rank, computational time is dominated by the rank reduction step: calibrating with OAP the computational time increases at about the same rate as seen in the above tables, for example increasing to over 1300 seconds at rank 12, while with EZI calibration time is stable around 4 seconds.

Computational times reflect the differential already seen in the rank reduction tests. Hence OAP can be even more remarkably inefficient in the corresponding cap calibration, the type of calibration considered in Chapter 9 of Rebonato (2002). In

fact, in this case calibration can be based on semi-annual forward rates and on large correlation matrixes such as the 39×39 matrix considered earlier.

OAP appears the most burdensome step when it is used in a calibration routine. With EZI the entire calibration procedure is very fast.

Conclusions

In this paper we introduce an iterative algorithm, EZI, for correlation matrix rank reduction. The algorithm is intuitive and simple to implement.

We provide an analysis of conditions affecting its convergence to a viable correlation matrix of desired rank. We describe the relationship of EZI with alternating projection theory, and with similar algorithms in signal enhancement. We also present an extension of EZI, called EZI+, with the optimality properties of particular optimization methods, such as the common alternative method OAP.

We present empirical tests using different matrixes, both parametric and historically estimated on forward rate market data. We compare EZI with the two methods commonly used in finance, the fast but inaccurate EZN method and the slow but accurate OAP method. Results show that EZI is much more accurate than EZN, comparable in accuracy to OAP, and can be more accurate than OAP if a range of relevant error metrics is considered. However the speed of EZI is comparable to EZN, making it much faster than OAP, particularly when the required rank is large. OAP is even more time consuming on large matrixes typical in interest rate applications, taking hundreds of times longer than EZI in order to approach the accuracy of EZI. We also test the EZI+ algorithm. It returns the same results as OAP in a fraction of the time.

The high computational efficiency of EZI can be crucial in relevant financial applications such as the calibration of the Libor Market Model. Using EZI, swaption calibration yields regular results in a remarkably shorter time than with OAP.

The EZI algorithm appears to provide a good balance of accuracy and speed, and is recommended for financial applications.

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Notes

[1] Rebonato and Jäckel investigate a problem related to ours, that of finding some correlation matrix close to a general matrix.

[2] Given a matrix A, we write A_i for its ith row and A^j for its jth column.
[3] With respect to the metric $\|\cdot\|$ defined in (5).
[4] When $\lambda_r=\lambda_{r+1}=1$ there is no unique choice of eigenvalue to set to zero. However, the algorithm always chooses the same eigenvalue to set to zero.
[5] See the following subsection.
[6] Here, an affine set is defined to be a translation of a subspace.
[7] We thank Igor Grubisic for recently signalling an application of alternating projections to correlation rank reduction in his master thesis 'Interest rate theory. BGM model', Leiden University, 2002.
[8] Rebonato and Jäckel's goal is to recover a viable correlation matrix starting from ρ_1.
[9] We no longer consider the grossly inaccurate EZN method.
[10] In particular we use the version described in Brigo and Morini (2004), including endogenous interpolation to recover values of missing data. This algorithm avoids or reduces numerical difficulties that might otherwise occur, such as negative volatility parameters.

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