

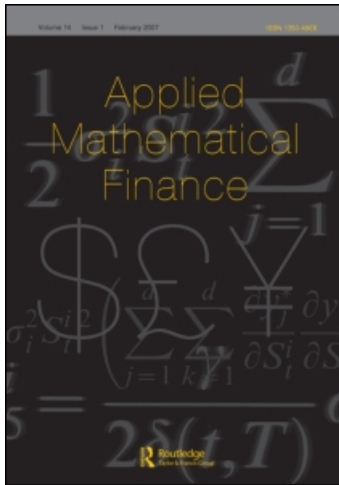
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A Numerical Method to Price Defaultable Bonds Based on the Madan and Unal Credit Risk Model

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ABSTRACT *We propose a numerical method to price corporate bonds based on the model of default risk developed by Madan and Unal. Using a perturbation approach, we derive two semi-explicit formulae that allow us to approximate the survival probability of the firm issuing the bond very efficiently. More precisely, we consider both the first- and second-order power series expansions of the survival probability in powers of the model parameter c . The zero-order coefficient of the series is evaluated using an exact analytical formula. The first- and second-order coefficients of the series are computed using an approximation algorithm based on the Laplace transform. Extensive simulation is carried out on several test cases where the parameters of the model of Madan and Unal are chosen from Grundke and Riedel, and bonds with different maturities are considered. The numerical experiments performed reveal that the numerical method proposed in this paper is accurate and computationally efficient.*

KEY WORDS: Credit risk, defaultable bonds, asymptotic expansion

1. Introduction

One of the first and most popular models for pricing defaultable bonds was developed by Madan and Unal (1998). This model is considered particularly interesting since it incorporates the most attractive features of both the reduced-form models and the structural models of credit risk. In fact, Madan and Unal (1998) relate the probability of default to the value of the equity of the firm issuing the bond. This is typical of the structural models, where the default event is linked to firm-specific variables (see, for instance, Black and Cox, 1976; Longstaff and Schwartz, 1995; Andersen and Sundaresan, 1996; Leland and Toft, 1996; Briys and de Varenne, 1997; Zhou, 2001; Hsu *et al.*, 2004; Leland, 2004, and Fouque *et al.*, 2006). Moreover, Madan and Unal (1998) model default as a random event that can occur unexpectedly at every time. Such an approach allows us to obtain high

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short-term spreads, and is peculiar to the reduced-form models (Duffee, 1999; Duffie and Singleton, 1999; Elliott *et al.*, 2000; Jarrow and Stuart, 1995; Lando, 1999). For these reasons the model of Madan and Unal (1998) is considered a middle-way approach between the reduced-form models and the structural models of default risk (for a detailed description of different models of default risk, see, for example, Duffie and Singleton, 2003).

Let $V(t, T)$ denote the price at time t of a defaultable zero-coupon bond with face value 1 maturing at time T , $T \geq t$. Using the model of Madan and Unal (1998), $V(t, T)$ is evaluated as follows:

$$V(t, T) = Q(t, T)(\Psi(t, T) + (1 - \Psi(t, T))R), \quad (1)$$

where $Q(t, T)$ is the price at time t of a Treasury (riskless) zero-coupon bond with face value 1 maturing at time T , $\Psi(t, T)$ is the probability that the firm issuing the bond survives up to time T given no default at time t , and R is the expected payoff in the case of default:

$$R = \int_0^1 yq(y)dy, \quad (2)$$

and $q(y)$ is the probability density function of the bond recovery value. Madan and Unal (1998) model $q(y)$ as the probability density function of a Beta distribution.

Let us consider the 'relativized' equity value of the firm:

$$s(t) = \frac{e(t)}{B(t)}, \quad (3)$$

where $e(t)$ is the firm's equity value and $B(t)$ is the money market account (Hull, 2003). Madan and Unal (1998) model $s(t)$ as the stochastic process

$$ds(t) = \sigma s(t)dW(t), \quad (4)$$

where σ is a volatility parameter and $W(t)$ is a standard Wiener process under the risk-neutral measure (Hull, 2003). Note that the 'relativized' equity value is modeled as a stochastic process with constant volatility, which is very common practice in mathematical finance.

Let $\phi(t)$ denote the so-called instantaneous intensity of default, that is $\phi(t)dt$ is the probability that default occurs in the time interval $[t, t+dt]$. Madan and Unal model $\phi(t)$ as a function of $s(t)$; in particular, they assume

$$\phi(t) = \frac{c}{(\log(s(t)) - \log(\delta))^2}, \quad (5)$$

where c and δ are positive constants. This choice is justified by Madan and Unal (1998). Here we only observe that, according to (5), the probability of default is measured by the distance of $s(t)$ from the critical value δ . In particular, when $s(t)$ reaches the threshold level δ , the intensity $\phi(t)$ becomes infinite, and default occurs with certainty.

As shown by Madan and Unal (1998), $\Psi(t, T)$ is a function of the time to maturity $T-t$ and $s(t)$. Then we set

$$\Psi(t, T) = P(x(t), \tau), \tag{6}$$

where

$$x(t) = \log(s(t)) - \log(\delta), \tag{7}$$

and $\tau = T - t$. It is shown by Madan and Unal (1998) that the function $P(x, \tau)$ must satisfy the following partial differential equation:

$$\frac{\partial P(x, \tau)}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 P(x, \tau)}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial P(x, \tau)}{\partial x} + \frac{cP(x, \tau)}{x^2} = 0, \tag{8}$$

with boundary conditions

$$P(0, \tau) = 0, \quad \lim_{x \rightarrow +\infty} P(x, \tau) = 1, \tag{9}$$

and initial conditions

$$P(x, 0) = 1. \tag{10}$$

Madan and Unal (1998) solved the partial differential problem (8)–(10) using certain changes of variables that reduce Equation (8) to a first-order ordinary differential equation. This ordinary differential equation can easily be approximated numerically using a finite difference scheme. However, as pointed out by Grundke and Riedel (2004), the approach of Madan and Unal lacks mathematical rigor, since, in one of the changes of variables used to transform the partial differential Equation (8), a differential term has been neglected. In particular, Grundke and Riedel (2004) show that the solutions of (8)–(10) computed using a finite difference scheme are significantly different from those obtained using the approach of Madan and Unal.

In this work we propose a perturbation approach to derive two approximate formulae that allow us to compute the solution of the partial differential problem (8)–(10) very efficiently. Let n denote a positive integer. We consider the asymptotic expansion of the solution of (8)–(10) in powers of the model parameter c with the base point in $c=0$:

$$P(x, \tau) = \sum_{j=0}^n c^j P_j(x, \tau) + o(c^n), \quad c \rightarrow 0. \tag{11}$$

Note that, in this work, we are not concerned with studying the well-posedness of problem (8)–(10). On the contrary, we simply assume that problem (8)–(10) has a unique solution for which the power series representation (11) holds true at least in the case $n=2$.

We approximate $P(x, \tau)$ using both the power series expansion (11) truncated at first order and the power series expansion (11) truncated at second order. The zero-order coefficient of the series is evaluated using an exact analytical formula. The first- and second-order coefficients are computed using a fast and accurate numerical algorithm based on the Laplace transform. More precisely, the Laplace transform of $P_1(x, \tau)$ performed with respect to the variable τ is evaluated using an exact analytical formula that contains the so-called exponential integral function. Moreover, the Laplace transform of $P_2(x, \tau)$ performed with respect to the

variable τ is accurately approximated using only elementary functions and a limited number of exponential integral functions. Finally, the Laplace transforms obtained are inverted numerically using an *ad hoc* technique based on contour integration.

An extensive numerical simulation was carried out on several test cases where the parameters of the model were chosen as per Grundke and Riedel (2004) and Madan and Unal (1998), and bonds with different maturities were considered. The numerical experiments performed reveal that the numerical method is very accurate. In fact, when the first-order power series expansion is used, the relative error obtained is always smaller than 1.0×10^{-2} , is often of order 10^{-3} or 10^{-4} , and is sometimes of order 10^{-5} or 10^{-6} . When the second-order power series expansion is used, the relative error obtained is always smaller than 9.8×10^{-4} , is often of order 10^{-5} , and is even of order 10^{-6} or 10^{-7} .

The numerical method presented in this paper is also computationally fast. In fact, the computer time necessary for the simulation is 0.028 s when the power series expansion (11) is truncated at first order, and varies from 0.24 s to 0.45 s when the power series expansion (11) is truncated at second order. Note that these execution times are obtained when the simulation is carried out on a computer with a Pentium 4 processor, 1700 MHz, 256 MB RAM, and the software programs are written using Matlab.

We remark that the numerical method presented in this paper is well suited for parallel computing, since the algorithm used to invert the Laplace transforms of $P_1(x, \tau)$ and $P_2(x, \tau)$ is fully parallelizable.

We point out that our numerical method can also be used to forecast the parameters of the model of Madan and Unal. For instance, following an approach similar to that used by Madan and Unal (1998) (we recall that the method used by Madan and Unal, 1998, to solve problem (8)–(10) is not mathematically correct), the semi-explicit formulae derived in this paper can be applied to determine the model parameters σ , c and δ by maximum likelihood fitting to the observed data. In addition, our formulae can be used to obtain the *implied* model parameters.

The paper is organized as follows. In the next section we describe the numerical method used to compute the solution of problem (8)–(10) (some mathematical details concerning the numerical method are given in Appendix A). In Section 3 we present and discuss the results obtained using the numerical algorithm developed in Section 2. Finally, conclusions are drawn in Section 4.

2. The Numerical Method

In this section we present the numerical method used to compute the solution of problem (8)–(10). For the sake of clarity, this section is divided into two subsections. In Subsection 2.1 we write down the differential problems that must be solved in order to obtain the coefficients of the power series expansion (11) and determine suitable expressions for the Laplace transforms of $P_0(x, \tau)$, $P_1(x, \tau)$ and $P_2(x, \tau)$ performed with respect to the variable τ . In Subsection 2.2 we show how to numerically invert the Laplace transforms obtained.

2.1 The Power Series Expansion Approach and the Laplace Transforms

Substituting the power series expansion (11) into Equations (8)–(10), and equating to zero the terms of the same order, we obtain for the zero-order terms:

$$\frac{\partial P_0(x, \tau)}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 P_0(x, \tau)}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial P_0(x, \tau)}{\partial x} = 0, \tag{12}$$

$$P_0(0, \tau) = 0, \quad \lim_{x \rightarrow +\infty} P_0(x, \tau) = 1, \tag{13}$$

$$P_0(x, 0) = 1, \tag{14}$$

and for the higher-order terms:

$$\frac{\partial P_j(x, \tau)}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 P_j(x, \tau)}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial P_j(x, \tau)}{\partial x} = - \frac{P_{j-1}(x, \tau)}{x^2}, \tag{15}$$

$$j = 1, 2, \dots, n,$$

$$P_j(0, \tau) = 0, \quad \lim_{x \rightarrow +\infty} P_j(x, \tau) = 0, \quad j = 1, 2, \dots, n, \tag{16}$$

$$P_j(x, 0) = 0, \quad j = 1, 2, \dots, n. \tag{17}$$

We note that the higher-order terms $P_j(x, \tau), j = 1, 2, \dots, n$, account for the presence of a non-zero default intensity in Equations (8)–(10).

The zero-order coefficient $P_0(x, \tau)$ can be obtained in closed form. In fact, Equation (12) with boundary conditions (13) and initial condition (14) constitutes a well-known parabolic problem with fixed barrier, the solution of which is given by (Black and Cox, 1976; Rich, 1994)

$$P_0(x, \tau) = N(l_1) - e^x N(l_2), \tag{18}$$

where

$$l_1 = \frac{-\sigma^2 \tau + 2x}{2\sqrt{\sigma^2 \tau}}, \tag{19}$$

$$l_2 = \frac{-\sigma^2 \tau - 2x}{2\sqrt{\sigma^2 \tau}}, \tag{20}$$

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-y^2/2} dy. \tag{21}$$

In order to evaluate higher-order coefficients in the power series expansion (11), let us consider the Laplace transform of the function $P_j(x, \tau)$ performed with respect to the variable $\tau, j = 0, 1, \dots, n$:

$$F_j(x, \omega) = \int_0^{+\infty} P_j(x, \tau) e^{-\omega\tau} d\tau, \quad j=0, 1, \dots, n, \quad (22)$$

where $\omega \in \mathbb{C}$ and $x \geq 0$.

Taking the Laplace transform of (12)–(17) with respect to the variable τ we obtain, for the zero-order term, the differential equation

$$\frac{\partial^2 F_0(x, \omega)}{\partial x^2} - \frac{\partial F_0(x, \omega)}{\partial x} - \frac{2\omega}{\sigma^2} F_0(x, \omega) = -\frac{2}{\sigma^2}, \quad (23)$$

with boundary conditions

$$F_0(0, \omega) = 0, \quad \lim_{x \rightarrow +\infty} F_0(x, \omega) = \frac{1}{\omega}, \quad (24)$$

and, for the higher-order terms, the differential equation

$$\frac{\partial^2 F_j(x, \omega)}{\partial x^2} - \frac{\partial F_j(x, \omega)}{\partial x} - \frac{2\omega}{\sigma^2} F_j(x, \omega) = \frac{2F_{j-1}(x, \omega)}{\sigma^2 x^2}, \quad (25)$$

$$j=1, 2, \dots, n,$$

with boundary conditions

$$F_j(0, \omega) = 0, \quad \lim_{x \rightarrow +\infty} F_j(x, \omega) = 0, \quad j=1, 2, \dots, n. \quad (26)$$

We approximate $P(x, \tau)$ using both the power series expansion (11) truncated at first order and the power series expansion (11) truncated at second order. That is, we compute $P(x, \tau)$ using both the first-order approximation

$$\hat{P}_1(x, \tau) = P_0(x, \tau) + cP_1(x, \tau), \quad (27)$$

and the second-order approximation

$$\hat{P}_2(x, \tau) = P_0(x, \tau) + cP_1(x, \tau) + c^2P_2(x, \tau). \quad (28)$$

In Equations (27) and (28), $P_0(x, \tau)$ is evaluated according to relations (18)–(21), whereas $P_1(x, \tau)$ and $P_2(x, \tau)$ are obtained from the Laplace transforms $F_1(x, \omega)$ and $F_2(x, \omega)$ using the numerical method described in Subsection 2.2.

Now we show how to evaluate $F_1(x, \omega)$ and $F_2(x, \omega)$. Associated with the differential Equations (23) and (25) is the algebraic characteristic equation

$$\lambda^2 - \lambda - \frac{2\omega}{\sigma^2} = 0, \quad (29)$$

the roots of which are

$$\lambda_1(\omega) = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{2\omega}{\sigma^2}}, \quad \lambda_2(\omega) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\omega}{\sigma^2}}. \quad (30)$$

The solution of the ordinary differential problem (23) and (24) is readily obtained:

$$F_0(x, \omega) = \frac{1}{\omega} (1 - e^{\lambda_1 x}). \quad (31)$$

From Equations (25) and (26) we obtain

$$F_1(x, \omega) = \frac{2}{\sigma^2(\lambda_1 - \lambda_2)} ((G_1(x, \omega) - G_2(x, \omega))e^{\lambda_1 x} + G_2(x, \omega)e^{\lambda_2 x}), \quad (32)$$

where

$$G_1(x, \omega) = \int_0^x F_0(u, \omega) \frac{e^{-\lambda_1 u} - e^{-\lambda_2 u}}{u^2} du, \quad (33)$$

$$G_2(x, \omega) = \int_x^{+\infty} F_0(u, \omega) \frac{e^{-\lambda_2 u}}{u^2} du. \quad (34)$$

Substituting (31) into (33) and (34) we obtain

$$G_1(x, \omega) = \frac{1}{\omega} \int_0^x \frac{e^{-\lambda_1 u} - e^{-\lambda_2 u} - 1 + e^{(\lambda_1 - \lambda_2)u}}{u^2} du, \quad (35)$$

$$G_2(x, \omega) = \frac{1}{\omega} \int_x^{+\infty} \frac{e^{-\lambda_2 u} - e^{(\lambda_1 - \lambda_2)u}}{u^2} du. \quad (36)$$

Note that the improper integral $G_1(x, \omega)$ is well defined. The integrals $G_1(x, \omega)$ and $G_2(x, \omega)$ can be evaluated using an explicit formula. In fact, we have (Abramowitz and Stegun, 1972)

$$\int \frac{e^{lx}}{x^2} dx = lEi(lx) - \frac{e^{lx}}{x} + K, \quad K \in \mathbb{C}, l \in \mathbb{C} \setminus \{0\}, \quad (37)$$

where Ei denotes the so-called exponential integral function, defined as

$$Ei(z) = \gamma + \log(z) + \sum_{j=1}^{+\infty} \frac{z^j}{jj!}, \quad z \in \mathbb{C} \setminus \{0\}, \quad (38)$$

where γ is the Euler–Mascheroni constant. In this paper we adopt the following extension of the logarithm function from the real positive semi-axis to the complex plane:

$$\log(z) = \begin{cases} \log|z| + \arg(z)i, & z \in \mathbb{C} \setminus \{0\}, \arg(z) \leq \pi, \\ \log|z| + \arg(z)i - 2\pi i, & z \in \mathbb{C} \setminus \{0\}, \arg(z) > \pi. \end{cases} \quad (39)$$

It is important to observe that the exponential integral function (38) can be evaluated very quickly with essentially no error using standard numerical algorithms (Amos, 1980). One of these algorithms is given in Appendix A (see relations (A9)–(A13)).

Using relation (37) to evaluate $G_1(x, \omega)$ and $G_2(x, \omega)$, we can rewrite formula (32) as

$$F_1(x, \omega) = \frac{2}{\sigma^2 \omega (\lambda_1 - \lambda_2)} [(n_1 - \lambda_1 Ei(-\lambda_1 x))e^{\lambda_1 x} + (\lambda_2 Ei(-\lambda_2 x) + (\lambda_1 - \lambda_2)Ei((\lambda_1 - \lambda_2)x) - n_2)e^{\lambda_2 x}], \quad (40)$$

where

$$n_1 = (\lambda_1 - \lambda_2)h(\lambda_1 - \lambda_2) + \lambda_2 h(-\lambda_2) - (\lambda_1 - \lambda_2)\log(\lambda_1 - \lambda_2) + \lambda_1 \log(-\lambda_1) - \lambda_2 \log(-\lambda_2), \tag{41}$$

$$n_2 = (\lambda_1 - \lambda_2)h(\lambda_1 - \lambda_2) + \lambda_2 h(-\lambda_2), \tag{42}$$

$$h(l) = \begin{cases} \pi, & l \in \mathbb{C}, \Im(l) > 0, \\ -\pi, & l \in \mathbb{C}, \Im(l) \leq 0, \end{cases} \tag{43}$$

and $\Im(l)$ denotes the imaginary part of l .

Note that in order to evaluate the improper integral (36) we have used the relation

$$\lim_{x \rightarrow +\infty} Ei(lx) = h(l), \quad l \in \mathbb{C} \setminus \{0\}, \Re(l) \leq 0, \tag{44}$$

the validity of which can easily be checked using formula (A10) reported in Appendix A.

Now we must compute the Laplace transform $F_2(x, \omega)$. From Equations (25) and (26) we have

$$F_2(x, \omega) = \frac{2}{\sigma^2(\lambda_1 - \lambda_2)} ((H_1(x, \omega) - H_2(x))e^{\lambda_1 x} + H_2(x)e^{\lambda_2 x}), \tag{45}$$

where

$$H_1(x) = \int_0^x F_1(u, \omega) \frac{e^{-\lambda_1 u} - e^{-\lambda_2 u}}{u^2} du, \tag{46}$$

$$H_2(x) = \int_x^{+\infty} F_1(u, \omega) \frac{e^{-\lambda_2 u}}{u^2} du. \tag{47}$$

It can easily be shown using relations (38) and (40) that the improper integral (46) is well defined, despite its integrand containing the function $1/u^2$, which tends to $+\infty$ as $u \rightarrow 0$.

Looking at formulae (32) and (45) we note that $F_1(x, \omega)$ and $F_2(x, \omega)$ have a similar functional form. However, contrary to what happens for the integrals (35) and (36), the integrals (46) and (47) cannot be evaluated using an exact analytical formula. Nevertheless, they can be accurately approximated using only elementary functions and a limited number of exponential integral functions. The numerical method used to evaluate the integrals (46) and (47) is shown in Appendix A.

2.2 Numerical Inversion of the Laplace Transforms

The Laplace transforms $F_1(x, \omega)$ and $F_2(x, \omega)$ cannot be inverted using an exact analytical formula. Therefore, we obtain $P_1(x, \tau)$ and $P_2(x, \tau)$ by numerical approximation of the Bromwich integrals:

$$P_j(x, \omega) = \frac{1}{2\pi i} \int_{a_j - i\infty}^{a_j + i\infty} F_j(x, \omega) e^{\omega \tau} d\omega, \quad j = 1, 2, \tag{48}$$

where $a_j \in \mathbb{R}$ is such that all the singularities of $F_j(x, \omega)$ lie in the open half-plane $\{\omega | \Re(\omega) < a_j\}$, $j=1, 2$.

It can easily be checked that both $F_1(x, \omega)$ and $F_2(x, \omega)$ are symmetric with respect to the real axis and analytic in the whole complex plan except the origin and the semi-axis $\{\omega | \Re(\omega) < -\sigma^2/8, \Im(\omega)=0\}$. Using these facts we can rewrite formula (48) as follows:

$$P_j(x, \tau) = \frac{1}{\pi i} \int_{a_j}^{a_j+i\infty} F_j(x, \omega) e^{\omega\tau} d\omega, \quad a_j \in \mathbb{R}^+, j=1, 2. \tag{49}$$

For large values of x , the quantities $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ appearing in (40) and (45), considered as functions in the variable ω , are rapidly oscillating functions when ω varies along the half-line $[a_j, a_j+i\infty)$. As a consequence, for large values of x , it is very difficult to approximate the integrals (49) accurately. Then, using an approach similar to that proposed by Talbot (1979), we change the contour integration in (49) as follows:

$$P_j(x, \tau) = \frac{1}{\pi i} \int_{a_j}^{a_j+ib_j} F_j(x, \omega) e^{\omega\tau} d\omega + \frac{1}{\pi i} \int_{a_j+ib_j}^{-\infty+ib_j} F_j(x, \omega) e^{\omega\tau} d\omega, \tag{50}$$

$$a_j, b_j \in \mathbb{R}^+, j=1, 2.$$

By Cauchy's theorem, such a contour deformation is possible since in the ω plane all the singularities of $F_j(x, \omega)$ lie on the real negative semi-axis, and $|F_j(x, \omega) e^{\omega\tau}| \rightarrow 0$ as $|\omega| \rightarrow \infty$, $j=1, 2$. Replacing (49) with (50) gives the following advantages: since in (50) the imaginary part of ω does not go to infinity, the oscillations of the functions $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are limited. Moreover, in the second integral appearing in (49) the factor $e^{\omega\tau}$ decays rapidly as $\Re(\omega) \rightarrow -\infty$ and therefore it behaves as a favourable damping term to the oscillations of $F_j(x, \omega)$, $j=1, 2$. In the numerical experiments reported in this paper we chose $a_j=0.0001$ and $b_j=0.1$, $j=1, 2$.

The integrals appearing in (50) are evaluated by numerical approximation. First, the infinite integration domain of the second integral of (50) is replaced by a bounded interval, so that formula (50) is rewritten as follows:

$$P_j(x, \tau) \cong \frac{1}{\pi i} P_{j1}(x, \tau) + \frac{1}{\pi i} P_{j2}(x, \tau), \quad j=1, 2, \tag{51}$$

where

$$P_{j1}(x, \tau) = \int_{a_j}^{a_j+ib_j} F_j(x, \omega) e^{\omega\tau} d\omega, \quad a_j, b_j, a_\infty \in \mathbb{R}^+, j=1, 2, \tag{52}$$

$$P_{j2}(x, \tau) = \int_{a_j+ib_j}^{-a_\infty+ib_j} F_j(x, \omega) e^{\omega\tau} d\omega, \quad a_j, b_j, a_\infty \in \mathbb{R}^+, j=1, 2. \tag{53}$$

In the numerical experiments presented in this paper we use $a_\infty=10/\tau$. This choice is sufficient to obtain a very accurate approximation of the integral (49).

Finally, we compute the integrals (52) and (53) using a numerical quadrature formula. In order to obtain accurate results we use Gauss–Legendre integration (Evans, 1993). Let M_{jk} denote the number of Gauss–Legendre integration nodes used to evaluate the integral $P_{jk}(x, \tau)$, $j=1, 2$, $k=1, 2$. In our numerical experience we have found that a very satisfactory level of accuracy can be obtained using $M_{11}=5$, $M_{12}=25$, $M_{21}=3$ and $M_{22}=16$.

A theoretical study of the error due to the numerical method described in Section 2 appears to be a very difficult task due to the different kinds of approximations involved (the power series expansion (11), the numerical evaluation of the integrals (46) and (47), and the numerical inversion of the Laplace transforms $F_j(x, \omega)$, $j=1, 2$). Moreover, the analysis of the error of the power-series expansion (11) is complicated by the fact that, to the best of our knowledge, a theoretical investigation of the well-posedness (existence, uniqueness and regularity) of the solution of the model of Madan and Unal is not available in the literature. Hence, the accuracy of the method proposed is tested by numerical simulation. This is done in the next section.

3. Numerical Results

The numerical simulation was carried out using a computer with a Pentium 4 processor, 1700 MHz, 256 MB RAM. The software programs were written using Matlab. The computation of the exponential integral function was performed using formula (A13).

In the first test case (Test Case 1) the parameters σ and c are chosen as follows: $\sigma=0.36633 \text{ year}^{-1/2}$ and $c=0.003419 \text{ year}^{-1}$. These values were estimated by Madan and Unal (1998) from historical series of defaultable bond prices. As far as the time to maturity τ is concerned, we consider three different test cases: $\tau=3$ months (Test Case 1.a), $\tau=1$ year (Test Case 1.b) and $\tau=10$ years (Test Case 1.c).

For each test case we computed $\hat{P}_1(x, \tau)$ and $\hat{P}_2(x, \tau)$ corresponding to different values of x . In particular, we chose values of x such that the survival probability $P(x, \tau)$ belongs to the interval $[0.1, 0.999]$. This interval contains the survival probabilities implied by the spreads of defaultable bonds that are normally observed in financial markets (Byström and Kwon Oh, 2005; Chan-Lau, 2006).

In order to test the accuracy of the results obtained, a very accurate estimation of the exact survival probability $P(x, \tau)$, denoted $P^e(x, \tau)$, is computed by numerical approximation. In particular, Equations (8)–(10) are discretized by applying the Crank–Nicholson finite difference scheme, and a very large number of collocation nodes (100,000) is used in both the x and the τ directions.

The results obtained are reported in Tables 1–3. In these tables, $RelErr_j$ denotes the relative error of the numerical approximation $\hat{P}_j(x, \tau)$, evaluated as follows:

$$RelErr_j = \frac{|\hat{P}_j(x, \tau) - P^e(x, \tau)|}{P^e(x, \tau)}, \quad j=1, 2, \quad (54)$$

and $CPUTime_j$ denotes the computer time necessary to obtain $\hat{P}_j(x, \tau)$, $j=1, 2$.

Looking at Tables 1–3, we can see that $\hat{P}_1(x, \tau)$ is a rather good approximation of $P(x, \tau)$. In fact, in all the test cases considered, $RelErr_1$ varies from 4.8×10^{-5} to

Table 1. Test Case 1.a ($\tau=1$ month).

	$x=0.1$	$x=0.3$	$x=0.5$
$P^c(x, \tau)$	0.361536	0.866920	0.986830
$RelErr_1$	5.8×10^{-3}	1.5×10^{-3}	1.4×10^{-3}
$RelErr_2$	1.2×10^{-4}	7.9×10^{-4}	8.5×10^{-4}
$CPUTime_1$ (s)	0.028	0.028	0.028
$CPUTime_2$ (s)	0.24	0.29	0.44

Table 2. Test Case 1.b ($\tau=1$ year).

	$x=0.1$	$x=0.3$	$x=1.0$
$P^c(x, \tau)$	0.160694	0.499647	0.984339
$RelErr_1$	1.0×10^{-2}	3.1×10^{-3}	4.8×10^{-5}
$RelErr_2$	9.8×10^{-4}	2.6×10^{-4}	8.2×10^{-5}
$CPUTime_1$ (s)	0.028	0.028	0.028
$CPUTime_2$ (s)	0.25	0.28	0.44

Table 3. Test Case 1.c ($\tau=10$ years).

	$x=0.6$	$x=1.0$	$x=5.0$
$P^c(x, \tau)$	0.210626	0.388445	0.998201
$RelErr_1$	6.4×10^{-3}	3.5×10^{-3}	1.5×10^{-4}
$RelErr_2$	7.1×10^{-4}	3.7×10^{-4}	7.0×10^{-5}
$CPUTime_1$ (s)	0.028	0.028	0.028
$CPUTime_2$ (s)	0.27	0.29	0.48

1.0×10^{-2} . Moreover, the computer time necessary to obtain $\hat{P}_1(x, \tau)$ is very short (0.028 s).

We also note that $\hat{P}_2(x, \tau)$ is a very accurate approximation of $P(x, \tau)$, since $RelErr_2$ varies from 7.0×10^{-5} to 9.8×10^{-4} . Moreover, the computer time necessary to obtain $\hat{P}_1(x, \tau)$ is rather short; in fact, $CPUTime_2$ varies from 0.24 s to 0.48 s.

It is worth noting that the largest relative errors of the approximations $\hat{P}_1(x, \tau)$ and $\hat{P}_2(x, \tau)$ are obtained in Test Case 1.b when $x=0.1$ ($RelErr_1=1.0 \times 10^{-2}$, $RelErr_2=9.8 \times 10^{-4}$). However, in this test case, $P^c(x, \tau)$ is rather small ($P^c(x, \tau)=0.160694$), so the bond considered has a high probability of default. Pricing near-default bonds very accurately is not usually a matter of concern for financial researchers and practitioners.

Note that, in Test Case 1.b, when $x=1$ we obtain $RelErr_1=4.8 \times 10^{-5}$ and $RelErr_2=8.2 \times 10^{-5}$, that is $\hat{P}_1(x, \tau)$ is a more accurate approximation of $P_1(x, \tau)$ than $\hat{P}_2(x, \tau)$. This has a clear explanation: in Test Case 1.b, when $x=1$, the error due to the numerical approximation of the integrals appearing in (50) is larger than the error due to the truncation of the series (27). In other words, $\hat{P}_2(x, \tau)$ can become

more accurate than $\hat{P}_1(x, \tau)$ only if the numerical approximation of the integrals appearing in (50) is performed more accurately. In fact, let us try to evaluate these integrals using a larger number of quadrature nodes: we set $M_{11}=10$, $M_{12}=32$, $M_{21}=6$ and $M_{22}=32$. We obtain $RelErr_1=7.0 \times 10^{-5}$ and $RelErr_2=5.2 \times 10^{-6}$, that is $\hat{P}_2(x, \tau)$ is still a more accurate approximation of $P(x, \tau)$ than $\hat{P}_1(x, \tau)$. In this new experiment we obtain $CPUTime_1=0.047$ s and $CPUTime_2=0.89$ s, that is the computer times necessary to obtain $\hat{P}_1(x, \tau)$ and $\hat{P}_2(x, \tau)$ are about two times longer than those required in the previous simulation. Note, however, that the level of accuracy of the approximation $\hat{P}_2(x, \tau)$ obtained in the new experiment is considerably higher than the level of accuracy that is usually required in everyday business practice.

In the following test case (Test Case 2) we chose $\sigma=0.5199$ year $^{-1/2}$ and $c=0.0017$ year $^{-1}$. These parameters were used by Grundke and Riedel (2004) and are the average values of those estimated by Madan and Unal (1998). The results obtained are reported in Tables 4–6.

In Tables 4–6 we note that both $\hat{P}_1(x, \tau)$ and $\hat{P}_2(x, \tau)$ are very accurate approximations of $P(x, \tau)$. In fact, in Test Case 2, $RelErr_1$ varies from 5.9×10^{-6} to 8.7×10^{-4} , and $RelErr_2$ varies from 6.7×10^{-7} to 1.3×10^{-4} .

The computer times necessary to calculate $\hat{P}_1(x, \tau)$ and $\hat{P}_2(x, \tau)$ are rather short. In fact, $CPUTime_1$ is always 0.028 s, and $CPUTime_2$ varies from 0.24 s to 0.45 s.

4. Conclusions

We have proposed a numerical method of pricing defaultable bonds using the model of default risk developed by Madan and Unal (1998). In particular, the survival probability $P(x, \tau)$ of the firm issuing the bond is approximated using both a

Table 4. Test Case 2.a ($\tau=1$ month).

	$x=0.1$	$x=0.3$	$x=0.5$
$P^e(x, \tau)$	0.259738	0.707014	0.927519
$RelErr_1$	5.6×10^{-4}	2.0×10^{-4}	1.7×10^{-4}
$RelErr_2$	6.1×10^{-5}	8.1×10^{-5}	1.3×10^{-4}
$CPUTime_1$ (s)	0.028	0.028	0.028
$CPUTime_2$ (s)	0.25	0.29	0.34

Table 5. Test Case 2.b ($\tau=1$ year).

	$x=0.1$	$x=0.3$	$x=1.0$
$P^e(x, \tau)$	0.109931	0.346882	0.909232
$RelErr_1$	8.7×10^{-4}	3.4×10^{-4}	4.4×10^{-5}
$RelErr_2$	4.9×10^{-5}	2.7×10^{-5}	1.9×10^{-5}
$CPUTime_1$ (s)	0.028	0.028	0.028
$CPUTime_2$ (s)	0.24	0.27	0.34

Table 6. Test Case 2.c ($\tau=10$ years).

	$x=0.6$	$x=1.0$	$x=5.0$
$P^c(x, \tau)$	0.106825	0.204120	0.976987
$RelErr_1$	6.0×10^{-4}	3.8×10^{-4}	5.9×10^{-6}
$RelErr_2$	3.2×10^{-6}	5.0×10^{-6}	6.7×10^{-7}
$CPUTime_1$ (s)	0.028	0.028	0.028
$CPUTime_2$ (s)	0.25	0.27	0.45

first- and a second-order power series expansion in the parameter c that measures the default intensity.

The zero-order coefficient of the series is obtained using an exact analytical formula. Moreover, the first- and second-order coefficients of the series are approximated very efficiently using a numerical method based on the Laplace transform.

An extensive numerical simulation was performed on several test cases where the parameters of the model were chosen as per Grundke and Riedel (2004) and Madan and Unal (1998), and bonds with different maturities were considered. These experiments reveal that the numerical method is accurate. In fact, when the first-order power series expansion is used, the relative error obtained is always smaller than 1.0×10^{-2} , is often of order 10^{-3} or 10^{-4} , and is even of order 10^{-5} or 10^{-6} . When the second-order power series expansion is used, the relative error obtained is always smaller than 9.8×10^{-4} , is often of order 10^{-4} or 10^{-5} , and is even of order 10^{-6} or 10^{-7} .

The numerical method proposed in this paper is computationally very fast. In fact, using the first-order power series approximation the survival probability can be evaluated in 0.028 s on a modest personal computer. Moreover, when the second-order power series approximation is used the computer times vary from 0.24 s to 0.45 s. Note that the numerical experiments are performed using Matlab, and the vectorial capabilities of this programming environment have not been exploited at all. The computer times necessary for the simulation could be significantly reduced if the software programs were rewritten using a compiled programming language, such as, for instance, FORTRAN, C or C++. In this regard, note also that our numerical method is well suited for parallel computing, since the algorithm used to evaluate the integrals appearing in (50) is fully parallelizable.

In addition, we observe that the survival probability $P_2(x, \tau)$ is evaluated using the semi-explicit formulae (27) and (28). Therefore, the numerical method proposed in this paper has the advantage that the sensitivity of defaultable bond prices to the variables x and τ , or to the model parameters σ , c and δ , can be computed by direct differentiation using formulae (27) and (28).

In addition, we remark that the numerical method presented in this paper can also be used to forecast the parameters of the model of Madan and Unal. For instance, the model parameters σ , c and δ can be estimated following the approach used by Madan and Unal (1998). That is, the parameters σ , c and δ can be determined by maximum likelihood fitting of the approximate formulae (27) and (28) to realized survival probabilities (obtained from historical bond prices using relation (1)).

In addition, formulae (27) and (28) can be applied directly to obtain *implied* model parameters.

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Appendix A

We describe the numerical method used to compute the integrals (46) and (47). This algorithm exploits the special functional form of the integrand functions of (46) and (47), and the fact that the exponential integral function can be approximated with essentially no error using few elementary functions. As a result, the integrals (46) and (47) are computed with excellent accuracy (relative error of order 10^{-8} or less) using only a limited number of elementary functions and exponential integral functions. Hence, from the computational standpoint, the quadrature scheme proposed is more efficient than methods based on standard interpolation (e.g. Lagrange and Gaussian interpolation), which require a significantly large number of integrand function evaluations in order to give very accurate results.

We note that the integrals (46) and (47) are evaluated using the same algorithm, since both the integrand function in (46) and the integrand function in (47) have the same functional form. We consider here only the integral (47).

Substituting (40) into (47) we obtain

$$H_2(x) = \frac{2}{\sigma^2 \omega (\lambda_1 - \lambda_2)} (H_{21}(x) + H_{22}(x) + H_{23}(x) + H_{24}(x)), \quad (A1)$$

where

$$H_{21}(x) = - \int_x^{+\infty} \frac{n_2}{u^2} du, \quad (A2)$$

$$H_{22}(x) = \int_x^{+\infty} n_1 \frac{e^{(\lambda_1 - \lambda_2)u}}{u^2} du, \quad (A3)$$

$$H_{23}(x) = \int_x^{+\infty} \frac{\lambda_2 Ei(-\lambda_2 u) + (\lambda_1 - \lambda_2) Ei((\lambda_1 - \lambda_2)u)}{u^2} du, \quad (A4)$$

$$H_{24}(x) = \int_x^{+\infty} - \frac{\lambda_1 Ei(-\lambda_1 u) e^{(\lambda_1 - \lambda_2)u}}{u^2} du. \quad (A5)$$

The integral (A2) is an elementary integral and is readily obtained. The integral (A3) can be evaluated analytically using formula (37). Therefore, $H_{22}(x)$ can be evaluated using the exponential integral function. The integral (A4) can be evaluated analytically using the relation (Abramowitz and Stegun, 1972)

$$\int \frac{Ei(lx)}{x^2} = lEi(lx) - \frac{Ei(lx) + e^{lx}}{x} + K, \quad K \in \mathbb{C}, l \in \mathbb{C} \setminus \{0\}. \quad (A6)$$

Therefore, $H_{23}(x)$ can be obtained using the exponential integral function. Finally, let us consider the integral (A5). This integral cannot be evaluated using an explicit formula. However, it can be approximated very efficiently. Using the change of variable $z = -\lambda_1 u$, let us rewrite (A5) as

$$H_{24}(x) = \lambda_1^2 \int_{-\lambda_1 x}^{-\lambda_1 \infty} \frac{Ei(z)e^{mz}}{z^2} dz, \tag{A7}$$

where

$$m = \frac{\lambda_2}{\lambda_1} - 1. \tag{A8}$$

Approximation of $Ei(z)$

We want to approximate the exponential integral function $Ei(z)$ in (A7) using elementary functions. Let M denote a positive integer; we consider the power series expansion (38) truncated at the the M th term:

$$Ei_M(z) = \gamma + \log(z) + \sum_{j=1}^M \frac{z^j}{jj!}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{A9}$$

When $|z|$ is large the truncated series (A9) converges very slowly to $Ei(z)$ as $M \rightarrow \infty$ (Amos, 1980). In this case, however, $Ei(z)$ can be efficiently approximated as follows. Let L denote a positive integer; we consider the continued fraction expansion (Abramowitz and Stegun, 1972)

$$\overline{Ei}_L(z) = -e^z \left(\frac{1}{-z+1} \frac{1}{1+} \frac{1}{-z+1} \frac{2}{1+} \frac{2}{-z+1} \frac{3}{1+} \frac{3}{-z+} \dots \frac{L}{1+} \frac{L}{-z} \right) + h(z). \tag{A10}$$

The summation appearing in (A10) inside the parentheses can be understood as follows: each term must be added to the denominator of the preceding term; that is, the summation is equal to $1/-z$ when $L=1$, to

$$\frac{1}{-z + \frac{1}{1 + \frac{1}{-z}}}$$

when $L=2$, to

$$\frac{1}{-z + \frac{1}{1 + \frac{1}{-z + \frac{2}{1 + \frac{2}{-z}}}}}$$

when $L=3$, etc.

It can be shown that, as $L \rightarrow \infty$, $\overline{Ei}_L(z)$ converges to the exponential integral function $Ei(z) \forall z \in \mathbb{C} \setminus \{0\}$. Moreover, when $|z|$ is large, say $|z| \gg 1$, the continued

fraction expansion (A10) is a very accurate approximation of $Ei(z)$ also for small values of L . Moreover, the highest rates of convergence of $\overline{Ei}_L(z)$ to $Ei(z)$ as $L \rightarrow \infty$ are achieved when $\Re(z) < 0$. Therefore, let us define

$$\Gamma_1 = \{z \in \mathbb{C}, \Re(z) \geq 0, |z| \leq 30, z \neq 0\} \cup \{z \in \mathbb{C}, \Re(z) < 0, |z| \leq 12\}, \tag{A11}$$

$$\Gamma_2 = \{z \in \mathbb{C}, \Re(z) \geq 0, |z| > 30\} \cup \{z \in \mathbb{C}, \Re(z) < 0, |z| > 12\}. \tag{A12}$$

In (7) we approximate $Ei(z)$ as

$$Ei(z) \cong \begin{cases} Ei_{100}(z), & z \in \Gamma_1, \\ \overline{Ei}_4(z), & z \in \Gamma_2. \end{cases} \tag{A13}$$

The accuracy of the approximation (A13) can be tested by numerical simulation. We have found that, when $z \in \mathbb{C} \setminus \{0\}$, $Ei(z)$ is computed using (A13) with a relative error smaller than 1.0×10^{-6} . Moreover, when $|z| \leq 5, z \neq 0$, the relative error of the approximation (A13) is always smaller than 1.0×10^{-12} .

We can rewrite $\overline{Ei}_4(z)$ as follows:

$$\overline{Ei}_4(z) = \frac{z^4 - 19z^3 + 102z^2 - 154z + 24}{z^5 - 20z^4 + 120z^3 - 240z^2 + 120z} e^z + h(z). \tag{A14}$$

Let z_1, z_2, \dots, z_5 denote the roots of the polynomial appearing in the denominator of (A14). Clearly, one of these roots, say z_1 , is equal to zero. Moreover, it can be shown that z_2, z_3, z_4 and z_5 are real numbers. Therefore, $\overline{Ei}_4(z)$ can be rewritten as

$$\overline{Ei}_4(z) = e^z \left(\frac{w_1}{z} + \sum_{j=2}^5 \frac{w_j}{z - z_j} \right) + h(z). \tag{A15}$$

It can easily be shown that $w_1 = 24/120$. Numerical approximations of z_j and w_j accurate to the tenth significant digit have been obtained using Matlab, $j = 2, 3, 4, 5$. These values are reported in Table A1.

Table A1. Numerical approximations of z_j and w_j .

$z_1 = 0$	$w_1 = 24/120$
$z_2 = 0.7432919279$	$w_2 = 0.6012046901$
$z_3 = 2.571635007$	$w_3 = 0.1857323340$
$z_4 = 5.731178751$	$w_4 = 0.01294284962$
$z_5 = 10.95389431$	$w_5 = 0.0001201261988$

Evaluation of $H_{24}(x)$

For the sake of brevity we show how to compute the integral (A7) only in the case where $-\lambda_1 x \in \Gamma_1$. If $-\lambda_1 x \in \Gamma_1$ the two functional laws appearing in (A13) are both used to approximate $Ei(z)$ inside the integral (A7). If $-\lambda_1 x \in \Gamma_2$, only the second of relations (A13) is used to approximate $Ei(z)$ in (A7). Then the case $-\lambda_1 x \in \Gamma_2$ does not differ substantially from the case $-\lambda_1 x \in \Gamma_1$, and is left to the reader.

The integral (A7) can be rewritten as

$$H_{24}(x) = \lambda_1^2 (I_1(x) + I_2), \quad (\text{A16})$$

where

$$I_1(x) = \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{Ei(z) e^{mz}}{z^2} dz, \quad (\text{A17})$$

$$I_2 = \int_{-\lambda_1 \bar{x}}^{-\lambda_1 \infty} \frac{Ei(z) e^{mz}}{z^2} dz, \quad (\text{A18})$$

and $\bar{x} = 30/|\lambda_1|$. Substituting $Ei(z)$ in (A17) with the approximate formula (A13) we obtain

$$I_1(x) \cong I_{11}(x) + I_{12}(x) + I_{13}(x) + I_{14}(x), \quad (\text{A19})$$

where

$$I_{11}(x) = \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{\gamma e^{mz}}{z^2} dz, \quad (\text{A20})$$

$$I_{12}(x) = \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{\log(z) e^{mz}}{z^2} dz, \quad (\text{A21})$$

$$I_{13}(x) = \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{e^{mz}}{z} dz, \quad (\text{A22})$$

$$I_{14}(x) = \sum_{j=1}^{99} \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{z^{j-1} e^{mz}}{(j+1)(j+1)!} dz. \quad (\text{A23})$$

The integral (A20) can be evaluated analytically using relation (37). Therefore, $I_{11}(x)$ can be conveniently expressed in terms of exponential integral functions.

The integral (A21) cannot be evaluated using an exact analytical formula; nevertheless, it can be approximated efficiently as follows. Let S denote a positive integer, we substitute the function e^{mz} in (A21) by its Taylor series truncated at the S th order. We obtain

$$\begin{aligned} I_{12}(x) \cong & \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{\log(z)}{z^2} dz + m \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{\log(z)}{z} dz \\ & + \sum_{j=1}^{S-1} m^{j+1} \int_{-\lambda_1 x}^{-\lambda_1 \bar{x}} \frac{\log(z) z^{j-1}}{(j+1)!} dz. \end{aligned} \quad (\text{A24})$$

All the integrals appearing in (A24) are elementary integrals and can be expressed in terms of elementary functions using well-known integration formulae (Abramowitz and Stegun, 1972).

By numerical experiments we found that the choice $S=60$ is always sufficient to evaluate $I_{12}(x)$ with 10 exact significant digits.

The integral (A22) can be evaluated analytically using the following relation (Abramowitz and Stegun, 1972):

$$I_{13}(x) = \int \frac{e^{mx}}{x} dx = Ei(mx) + K, \quad K \in \mathbb{C}, m \in \mathbb{C} \setminus \{0\}. \tag{A25}$$

Therefore, $I_{13}(x)$ can be expressed conveniently in terms of exponential integral functions.

Finally, we note that $I_{14}(x)$ is a finite sum of elementary integrals. These integrals can be evaluated using integration formulae that contain only elementary functions (Abramowitz and Stegun, 1972).

Now let us describe how to compute $I_2(x)$. Substituting $Ei(z)$ in (A18) with the approximate formula (A13) we obtain

$$I_2 \cong I_{21} + I_{22} + I_{23}, \tag{A26}$$

where

$$I_{21} = w_1 \int_{-\lambda_1 \bar{x}}^{-\lambda_1 \infty} \frac{e^{(m+1)z}}{z^3} dz, \tag{A27}$$

$$I_{22} = \sum_{j=2}^5 w_j \int_{-\lambda_1 \bar{x}}^{-\lambda_1 \infty} \frac{e^{(m+1)z}}{z^2(z-z_j)} dz, \tag{A28}$$

$$I_{23} = \int_{-\lambda_1 \bar{x}}^{-\lambda_1 \infty} h(z) \frac{e^{mz}}{z^2} dz. \tag{A29}$$

The integral (A27) can be calculated analytically using the following relation (Abramowitz and Stegun, 1972):

$$\int \frac{e^{(m+1)x}}{x^3} dx = \frac{1}{2}(m+1)^2 Ei((m+1)x) - \frac{e^{(m+1)x}(mx+x+1)}{2x^2} + K, \tag{A30}$$

$K \in \mathbb{C}, m \in \mathbb{C} \setminus \{-1\}.$

Therefore, I_{21} can be evaluated explicitly using a few exponential integral functions. The integral (A28) is the sum of four integrals that can be evaluated using the following relation (Abramowitz and Stegun, 1972):

$$\int \frac{e^{(m+1)x}}{x^2(x-\alpha)} dx = K + \frac{e^{(m+1)x}}{\alpha x} + \frac{-(\alpha+1+\alpha m)Ei((m+1)x) + e^{\alpha+zm}Ei((m+1)(x-\alpha))}{\alpha^2}, \tag{A31}$$

$\alpha \in \mathbb{C} \setminus \{0\}, K \in \mathbb{C}, m \in \mathbb{C} \setminus \{-1\}.$

Hence, I_{22} can be expressed conveniently in terms of exponential integral functions.

Finally, we note that, according to definition (43), the term $h(z)$ inside the integral (A29) is actually a constant. Thus, formula (37) can be applied and I_{23} can be evaluated using exponential integral functions.

Remark A1. According to the numerical method described above the integrals (46) and (47) are computed using only a finite sum of elementary functions and a small number of exponential integral functions.