

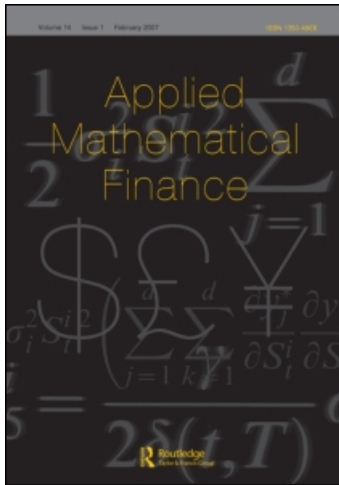
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Boundary Values and Finite Difference Methods for the Single Factor Term Structure Equation

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ABSTRACT *We study the classical single factor term structure equation for models that predict non-negative interest rates. For these models we develop a fast and accurate finite difference method (FD) using the appropriate boundary conditions at zero.*

KEY WORDS: Term structure equation, degenerate parabolic equations, stochastic representation, finite difference method

1. Introduction

When determining option prices using the Black–Scholes equation with finite difference methods (FDs), boundary conditions need to be imposed both for vanishing asset values and for large asset values. In the case of one underlying asset, the appropriate value at zero for European options is simply the discounted value of the pay-off function at that point. This is the case since the boundary is absorbing corresponding to the underlying asset going bankrupt. The question of appropriate boundary values for several underlying assets is investigated by Janson and Tysk (2006). One should perhaps note that, for several models, for instance geometric Brownian motion, the stock process reaches the boundary with probability zero and the boundary conditions are thus redundant to specify from a mathematical point of view. However, using FDs, boundary conditions are needed, being mathematically redundant or not. Let us note that the conditions discussed above are valid for models predicting positive asset values as well as those that allow bankruptcy with positive probability.

The present note deals with FDs for the classical term structure equation in single factor models. Using this equation, bond prices and bond option prices can be determined. We consider models that predict non-negative interest rates. In most interest rate models the boundary is not absorbing since the short rate typically would not stay zero if the value zero is reached. Moreover, the diffusion coefficient

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tends to zero and the drift is non-negative at the boundary for models predicting non-negative rates. Consequently, it is not clear what boundary conditions should be specified for the term structure equation. Modelling the short rate $X(t)$ directly under the pricing measure as

$$dX(t) = \beta(X(t), t)dt + \sigma(X(t), t)dW,$$

the bond option price u corresponding to a pay-off function g is given, using risk-neutral valuation, by

$$u(x, t) = E_{x,t}[e^{-\int_t^T X(s)ds} g(X(T))].$$

As indicated above, we assume that $\sigma(0, \cdot) = 0$ and $\beta(0, \cdot) \geq 0$. For the precise conditions on σ and β , see Ekström and Tysk (2008). One important example is the Cox–Ingersoll–Ross (CIR) model, for which $\beta(x, t) = a(b - x)$ and $\sigma(x, t) = c\sqrt{x}$, where a , b and c are positive constants. We note that if the pay-off $g \equiv 1$, then bond prices are obtained. The function u satisfies the term structure equation

$$u_t(x, t) + \frac{1}{2}\sigma^2(x, t)u_{xx}(x, t) + \beta(x, t)u_x(x, t) = xu(x, t), \quad (1)$$

with terminal condition $u(x, T) = g(x)$. The term structure equation holds at all interior points $(x, t) \in (0, \infty) \times [0, T)$. Oleinik and Radkevich (1973) discuss in detail the issue of when boundary conditions are needed at $x = 0$. No boundary condition is needed if the so-called Fichera function, which in a one-dimensional time-homogeneous setting is $\beta(x) - \frac{1}{2}(\partial\sigma^2/\partial x)(x)$, satisfies

$$\lim_{x \searrow 0} \left(\beta(x) - \frac{1}{2} \frac{\partial\sigma^2}{\partial x}(x) \right) \geq 0.$$

In the example of the CIR model, this condition reduces to $ab - \frac{1}{2}c^2 \geq 0$. This is of course consistent with the usual Feller condition that states that zero is not attainable for the process X . However, to use FDs it is necessary to know the behaviour of the solution close to the boundary, even though a boundary condition might be redundant from a mathematical perspective.

Only recently has the question of appropriate boundary behaviour for Equation (1) been treated mathematically. The main result of Ekström and Tysk (2008) states that the bond option price u is the unique classical solution to the term structure equation satisfying the boundary condition

$$u_t(0, t) + \beta(0, t)u_x(0, t) = 0. \quad (2)$$

Observe that this boundary condition is obtained by formally plugging $x = 0$ into the equation. Alternatively, to obtain an intuitive explanation of Equation (2), assume that u is sufficiently regular and use Ito's formula to compute

$$d(u(X(t), t)) = \left(u_t + \beta u_x + \frac{1}{2}\sigma^2 u_{xx} \right) (X(t), t) dt + (\sigma u_x)(X(t), t) dW.$$

Standard arbitrage theory says that the local rate of return should equal the short rate X . At the boundary we therefore obtain Equation (2) since σ vanishes there.

Remark 1. It is perhaps misleading to always refer to Equation (2) as a boundary condition. When the boundary is not attainable for the stochastic process X , Equation (2) should perhaps rather be referred to as the boundary behaviour of u . For simplicity, however, we will refer to Equation (2) as a boundary condition.

The recent book by Duffy (2006) has a section entitled ‘The thorny issue of boundary conditions’, treating the term structure equation. Also, other references in this area, such as d’Halluin *et al.* (2001), deal with this question. In these references, boundary conditions are only specified for certain models and for parameter values when the boundary is reached with positive probability, and the general case is avoided. In Example 1.1 of Heston *et al.* (2007), the authors encounter several solutions to the pricing partial differential equation (PDE) when not considering the boundary behaviour, and they discuss these solutions as different possible prices. Our point of view is that only the solution that satisfies appropriate boundary conditions represents the price as given by the risk-neutral expected value.

In the present note we develop a fast and accurate FD using Equation (2). The advantage of FDs compared with Monte Carlo methods for Equation (1) is the accuracy and the efficiency for low-dimensional problems (see, e.g. Lötstedt *et al.*, 2007). Our method requires no tailoring for the specific model in question, but is instead valid for *all* models that predict non-negative rates. We quote from Duffy (2006, p. 280): ‘Much of the literature is very Spartan in the author’s opinion when it comes to defining boundary conditions, and their assembly into the discrete system of equations.’ This note is one step towards filling this gap.

The paper is organized as follows. The term structure Equation (1) with the boundary condition Equation (2) is discretized by a FD of second-order accuracy in Section 2. The FD is applied to the CIR model (Brigo and Mercurio, 2001; Cox *et al.*, 1985) and a model with a diffusion proportional to $x^{3/4}$ in Section 3. Finally, some conclusions are drawn.

2. Numerical Method

The term structure Equation (1) is solved by a FD on the grid $x_n = nh$, $n = 0, \dots, N$. The upper boundary of the computational domain is x_{\max} and the step size h is x_{\max}/N . The constant time step is $\Delta t = T/M$ between the discrete time points $t^m = m\Delta t$, $m = 0, \dots, M$. The numerical solution at (x_n, t^m) is denoted by u_n^m and the spatial derivatives there are approximated by

$$u_x \approx \frac{1}{2}h^{-1}(u_{n+1}^m - u_{n-1}^m), \quad u_{xx} \approx h^{-2}(u_{n+1}^m - 2u_n^m + u_{n-1}^m). \quad (3)$$

At the lower boundary, $x_0 = 0$,

$$u_x \approx -h^{-1}\left(\frac{3}{2}u_0^m - 2u_1^m + \frac{1}{2}u_2^m\right) \quad (4)$$

in Equation (2) and at $x_N = x_{\max}$,

$$\begin{aligned} u_x &\approx h^{-1} \left(\frac{3}{2} u_N^m - 2u_{N-1}^m + \frac{1}{2} u_{N-2}^m \right), \\ u_{xx} &\approx h^{-2} (2u_N^m - 5u_{N-1}^m + 4u_{N-2}^m - u_{N-3}^m). \end{aligned} \tag{5}$$

In this way, only values of the solution between x_0 and x_N appear in the approximations. The second formula of Equation (5) is a linear extrapolation of the difference approximations of u_{xx} at x_{N-1} and x_{N-2} . All approximations are second-order accurate.

Let \mathbf{u}^m denote the solution vector at t_m with the components u_n^m . The time derivative is approximated in the same manner as the space derivative in Equation (4). Then the complete integration scheme backward in time for Equation (1) is

$$\left(\frac{3}{2} I - \Delta t A \right) \mathbf{u}^{m-1} = 2\mathbf{u}^m - \frac{1}{2} \mathbf{u}^{m+1}, \quad m = M - 1, M - 2, \dots, 1, \tag{6}$$

where the constant matrix A represents the space discretizations in Equations (3), (4) and (5). The implicit time integration method is the backward differentiation formula of order two (BDF2). The first step is taken with a first-order method, the Euler backward method or BDF1 of order one,

$$(I - \Delta t A) \mathbf{u}^{M-1} = \mathbf{u}^M, \quad u_n^M = g(x_n). \tag{7}$$

Both methods are stable if all eigenvalues $\lambda(A)$ of A satisfy $\Re \lambda(A) \leq 0$ (Hairer *et al.*, 1993). The error in the solution after the first time step is of $\mathcal{O}(\Delta t^2)$ and the truncation error is of order two in both time and space at all points (m, n) with $m < M$. The matrices in Equations (6) and (7) are almost tridiagonal and the systems of equations are both solved easily in a number of operations proportional to N . If σ and β are time-independent, then A is constant and a LU -factorization is first computed for the system matrices in Equations (6) and (7) (Dahlquist and Björck, 1974, Ch. 5.4). This factorization is then used in every time step with a cost of about $5N$ operations to obtain \mathbf{u}^{m-1} .

3. Numerical Results

We solve Equation (1) using the scheme in Section 2 for two different models: the CIR model (Brigo and Mercurio, 2001; Cox *et al.*, 1985) and a model with $\sigma \sim x^{3/4}$. An exact solution is known for the CIR equation and the convergence properties of our method can then be investigated. The second model is chosen to demonstrate the flexibility of the FD in a case without an analytical solution.

Let

$$\beta(x) = a(b - x), \quad a = 0.55, \quad b = 0.035, \quad \sigma(x) = 0.39\sqrt{x} \tag{8}$$

in Equation (1) with similar parameters as in the CIR model of d'Halluin *et al.* (2001) and let $g(x) = 1$. The end points in space and time are $x_{\max} = 0.1$ and $T = 1$. The

analytical solution v at grid and time points $v_n^m = v(x_n, t^m)$ were found by Brigo and Mercurio (2001, p. 58).

Remark 1. The analytical solution given by Brigo and Mercurio (2001) is only specified for parameter values such that the Fichera function is strictly positive at $x = 0$, and the parameter specification Equation (8) does not fulfil this condition. However, the same formula is also correct in the case when the Fichera function is negative at $x = 0$. To see this, it suffices to check that the correct boundary condition Equation (2) is satisfied.

The difference \mathbf{d}^m between \mathbf{u}^m and \mathbf{v}^m at all time points is measured in the norm defined by

$$\|\mathbf{d}\|^2 = \sum_{m=0}^M \sum_{n=0}^N h\Delta t |u_n^m - v_n^m|^2.$$

In Figure 1, the second-order convergence rate is confirmed with our choice of T and x_{\max} . For long integration times, the solution error has its maximum at $x = x_{\max}, t = 0$.

The eigenvalues of A are all real except for two and all have a negative real part for $N = 10, 20, 40$ and 80 , implying a stable integration in Equation (6) in those cases. The minimum and maximum modulus of $|\lambda(A)|$ are found in Table 1. The minimal value is

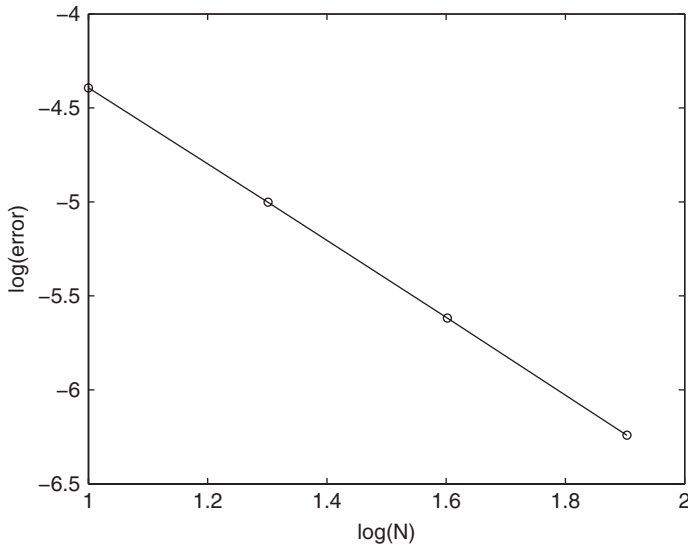


Figure 1. The difference $\log_{10} \|\mathbf{d}\|$ between the exact solution and the solution computed with the FD in Section 2 versus $\log_{10} N$ for the same number time steps M and space steps N .

Table 1. The minimum and maximum modulus of the eigenvalues of A .

N	10	20	40	80
$\min \lambda(A) $	0.0294	0.0294	0.0294	0.0294
$\max \lambda(A) $	0.0186×10^4	0.0926×10^4	0.4152×10^4	1.7711×10^4

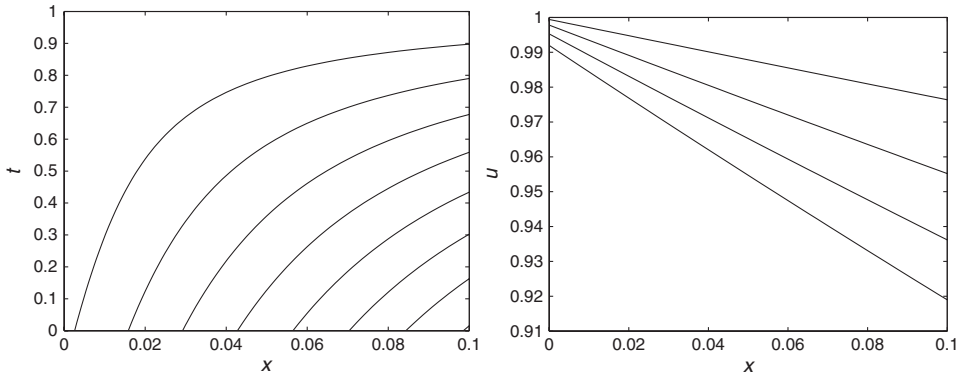


Figure 2. The solution of Equation (1) with diffusion $\sim x^{3/4}$ and $M = 21$ and $N = 21$: isolines from $u = 0.92$ to 1.0 with step 0.01 (left) and u at $t = 0.75, 0.5, 0.25$ and 0 , from top to bottom (right).

associated with an eigenvector which is almost constant in n . This is the mode with the slowest decay. With an explicit method the time step is restricted by

$$\Delta t \sim \frac{1}{\max |\lambda(A)|}$$

for a stable integration. From the table, it follows that with $N = 80$ we have an upper bound on an explicit time step $\Delta t \sim 0.56 \times 10^{-4}$. With the implicit method in Section 2 $\Delta t = 1/M = 0.0125$, about 200 times longer. The work per time step for the implicit method is less than two times the work for the simplest explicit method and the error in the solution is dominated by the spatial error in both cases.

The term structure equation is solved with $g(x) = 1$ and the same drift term as in Equation (8), but with $\sigma(x) = 0.39x^{3/4}$. Only a minor change in the code is necessary. The solution is displayed in Figure 2.

4. Conclusions

We implement a general boundary condition at $x = 0$ for the term structure equation and propose a finite difference method based on this boundary condition. In this way, we partly resolve ‘The thorny issue of boundary conditions’, which as mentioned before is the title of Section 25.6 treating the term structure equation in Duffy (2006). The numerical method is implicit in time and second-order accurate. The flexibility of a finite difference method makes it easy to change the drift and diffusion terms in the model.

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