MA5233 Homework 3

(Due date: 10:00pm, November 7, 2016 (Monday))

1. Given the initial-value problem (IVP)

$$\begin{aligned} &y'(t) = -2y(t) + 1, \qquad 0 < t \le 1, \\ &y(0) = 1. \end{aligned}$$

The analytical solution is

$$y(t) = \frac{1}{2}(1 + e^{-2t}), \qquad t \ge 0.$$

Find the explicit formula for $\{y^n\}_{n=0}^N$ with the choice of time step $k = \frac{1}{N}$ and the corresponding stability condition for the time step k by using the following numerical methods for the IVP.

(a) (Forward) Euler method.
(b) Backward Euler method.
(c) Trapezoidal method.
(d) The midpoint method

$$\begin{aligned} &y^{n+1} = y^n + k\,f\left(t_n + \frac{k}{2}, y^n + \frac{k}{2} f(t_n, y^n)\right), \qquad n = 0, 1, 2, \ldots, N-1, \\ &y^0 = y(0) = 1. \end{aligned}$$

(e) The fourth-order Runge-Kutta method.
(f) Compare the numerical solution and the analytical solution, what conclusion can you get? why?

2. Determine b_0, b_1 and b_2 such that the difference method

$$y^{n+1} = y^n + k[b_0\,f(t_{n+1}, y^{n+1}) + b_1\,f(t_n, y^n) + b_2\,f(t_{n-1}, y^{n-1})], \qquad n = 1, 2, \ldots,$$

is a third-order method for the differential equation

$$y'(t) = f(t, y).$$

Apply the above difference method to construct the difference equation for the following initial value problem

$$\begin{aligned} &y'(t) = \frac{2t+1}{1 + y(t)^2}, \qquad 0 < t \le 1, \\ &y(0) = 1. \end{aligned}$$

3. Find the stability condition of the midpoint method

$$\begin{aligned} &y^{n+1} = y^n + k\,f(t_n + k/2, y^n + k\,f(t_n, y^n)/2), \qquad n = 0, 1, 2, \ldots, N-1, \\ &y^0 = y(0) = \alpha. \end{aligned}$$

Suppose that $f(t, y)$ is Lipschitz continuous with respect to y, prove convergence of the midpoint method.

4. (a) Determine α and β such that the difference method

$$y^{n+1} = y^n + k\, f(t_n + \alpha k, y^n + \beta k\, f(t_n, y^n)), \qquad n = 0, 1, 2, \ldots,$$

for the differential equation

$$y'(t) = f(t, y), \qquad t > 0,$$

with order of accuracy as high as possible. Find the leading term in the local truncation error. What is the order of accuracy? Find the stability condition.

(b) Apply the above difference method to construct the difference equation for the following second-order differential equation

$$\begin{aligned} &y''(t) - 2y'(t) + y(t)^2 = t\, e^t - t, \qquad 0 < t \leq 3, \\ &y(0) = 0, \qquad y'(0) = 1. \end{aligned}$$

What is the stability condition for the time step k for this problem?

(c) Write a code to implement the method for solving the above problem. Plot the numerical solution $y(t)$ and $y'(t)$ as functions of time t. Plot the phase portrait, i.e. $(y(t), y'(t))$ in the plane.

5. Consider the initial value problem (IVP)

$$\begin{aligned} &y'(t) = t + y, \qquad 0 < t \leq 1, \\ &y(0) = 1, \end{aligned}$$

with exact solution

$$y(t) = -t - 1 + 2e^t, \qquad 0 \leq t \leq 1.$$

Write codes to implement the following methods, use them to simulate the above problem and compare the numerical solutions with the exact solution. Find the order of accuracy for the methods numerically.

(a) (Forward) Euler method.
(b) Trapezoidal method.
(c) 4th-order Runge-Kutta method.

What conclusion do you get from your numerical experiments?

6. Develop a code to implement the 4th-order Runge-Kutta method to solve

$$\begin{aligned} &x'(t) = f(t, x, y), \qquad a < t < b, \\ &y'(t) = g(t, x, y), \qquad a < t < b, \\ &x(a) = \alpha, \qquad y(a) = \beta. \end{aligned}$$

Apply your code to simulate the system of initial value problem describing an electric circuit

$$
\begin{aligned}
&I_1'(t) = f_1(t, I_1, I_2) = -4I_1 + 3I_2 + 6\sin(t^2), \qquad 0 < t < 4,\\
&I_2'(t) = f_2(t, I_1, I_2) = -2.4I_1 + 1.6I_2 + 3.6\cos(t^2), \qquad 0 < t < 4,\\
&I_1(0) = 1, \qquad I_2(0) = 0.
\end{aligned}
$$

7. Consider the two-dimensional (2D) Poisson equation on a unit square:

$$
-\Delta u(x, y) + \alpha\, u(x, y) = -\partial_{xx} u(x, y) - \partial_{yy} u(x, y) + \alpha\, u(x, y) = f(x, y), \qquad 0 < x, y < 1,
$$

with Dirichlet boundary conditions

$$
\begin{aligned}
&u(0, y) = f_1(y), \qquad u(1, y) = f_2(y), \qquad 0 \le y \le 1,\\
&u(x, 0) = g_1(x), \qquad u(x, 1) = g_2(x), \qquad 0 \le x \le 1,
\end{aligned}
$$

where $\alpha \ge 0$ is a constant.

(a) Write down a second order centered finite difference scheme for the problem on a uniform mesh $h = \frac{1}{N}$.

(b) Express the difference equations in linear system form.

(c) Design a fast direct Poisson solver for solving the linear system.

(d) Design a 4th-order compact scheme for the problem.

(e) Write codes to implement the methods in (a) and (d), respectively. Design a numerical example to test the order of accuracy of the methods in (a) and (d). Plot the numerical solutions (contour plots or surface plots) with different mesh sizes.