# MA5233 Homework 2

(Due date: 10:00pm, October 24, 2016 (Monday))

1. Consider the iteration: $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$ for solving the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{d}_k$ is a vector called the direction of search, and $\alpha_k$ is a scalar. It is assumed throughout that $\mathbf{d}_k$ is a nonzero vector. Consider a method which determine $\mathbf{x}_{k+1}$ so that the residual $\|\mathbf{r}_{k+1}\|_2$ with $\mathbf{r}_{k+1} = \mathbf{b} - A\mathbf{x}_{k+1}$ is the smallest possible.

(a) Determine $\alpha_k$ so that $\|\mathbf{r}_{k+1}\|_2$ is minimal.

(b) Show that the residual vector $\mathbf{r}_{k+1}$ obtained in this manner is orthogonal to $A\mathbf{d}_k$.

(c) Show that the residual vectors satisfy the relation

$$\|\mathbf{r}_{k+1}\|_2 = \|\mathbf{r}_k\|_2 \,\sin(\theta_k),$$

where the angle $\theta_k \in [0, \pi]$ is defined by

$$\cos(\theta_k) = \frac{(\mathbf{r}_k, A\mathbf{d}_k)}{\|\mathbf{r}_k\|_2 \|A\mathbf{d}_k\|_2}.$$

(d) Assume that at each step $k$, we have $(\mathbf{r}_k, A\mathbf{d}_k) \neq 0$. Will the method always converge?

(e) Now assume that $A$ is positive definite and select at each step $\mathbf{d}_k \equiv \mathbf{r}_k$. Prove that the method will converge for any initial guess $\mathbf{x}_0$.

2. Show how the GMRES method converge by going through (by hand or by Matlab) a few number of iterations of this method for the linear system $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

and $\mathbf{x}_0 = \mathbf{0}$.

3. Given the linear system

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8, \\ 2x_1 + 5x_2 + 2x_3 &= 3, \\ x_1 + 2x_2 + 4x_3 &= 11, \\ 4x_1 + x_2 + 2x_3 &= 9, \\ 2x_1 + 4x_2 - x_3 &= -5, \\ x_1 + x_2 - 3x_3 &= -9. \end{aligned}$$

(a) Find the least square solution of the above problem by the normal equation (NE) method.

(b) Find the least square solution of the above problem by the singular-value-decomposion (SVD) method.

4. Consider different numerical methods for computing $\sqrt{a}$ with $a > 0$.

(a) Construct the Newton's method based on the nonlinear equation $f(x) = x^2 - a = 0$. Prove that, for $k = 1, 2, \ldots$, $x_k \geq \sqrt{a}$ and the sequence $x_1$, $x_2$, ... is decreasing. Find

$$\lim_{k\to\infty} \frac{|\sqrt{a} - x_{k+1}|}{|\sqrt{a} - x_k|^2}, \qquad \lim_{k\to\infty} \frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|^2}.$$

What is the order of convergence of this method?

(b) Construct the Newton's method based on the nonlinear equation $f(x) = 1 - \frac{a}{x^2} = 0$. Find

$$\lim_{k\to\infty} \frac{|\sqrt{a} - x_{k+1}|}{|\sqrt{a} - x_k|^2}, \qquad \lim_{k\to\infty} \frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|^2}.$$

What is the order of convergence of this method?

(c) Show that the iterative method

$$x_{k+1} = \frac{x_k(x_k^2 + 3a)}{3x_k^2 + a}, \qquad k = 0, 1, 2, \ldots; \qquad x_0 > 0,$$

is a third-order method for computing $\sqrt{a}$ by finding

$$\lim_{k\to\infty} \frac{|\sqrt{a} - x_{k+1}|}{|\sqrt{a} - x_k|^3}.$$

(d) Write codes to implement the above three methods. Apply them to compute numerically $\sqrt{115}$ and $\sqrt{0.111}$ with different initial data $x_0 > 0$. Compare the convergence rates and the choices of different initial data for different numerical methods. What conclusion can you obtain?