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Two Central Limit Problems for Dependent Random Variables

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0. Introduction

In this paper, we consider the central limit problem for dependent random variables in two different settings. In the first setting, we deal with certain dependent random variables indexed by an arbitrary set. Our main interest is in a special case called finitely dependent random variables which is a generalization of m-dependent random variables. Because the index set is arbitrary, finite dependence is applicable to multi-parameter processes. On the other hand, since the nature of dependence of a single-parameter process is in general altered by a re-ordering of the index set, many mixing sequences and non-mixing sequences are in fact finitely dependent sequences of random variables (see Kesten and O'Brien [16]). In Section 2, we shall give a detailed account of this dependence.

In the second setting, we consider a combinatorial problem which arises in the theory of nonparametric inference and dates back to Wald and Wolfowitz [29]. Special cases of this problem and their refinements have been considered by many authors (see [1, 5, 10, 11, 13, 18, 19, 23 and 29]).

Although the two central limit problems are unrelated, they are solved by taking the same approach which combines the method of characteristic functions with the techniques associated with Stein's method (see [4, 5, 9, 11 and 28]). In the first problem, it is found necessary to extend weak convergence of probability measures to that of finite signed measures.

The main results in this paper consist of establishing a necessary and sufficient condition for the sequence of sums of random variables in each setting to converge in distribution to any specified infinitely divisible distribution. Although a number of authors (see, for example, [3, 7 and 21]) have considered the central limit problem for dependent random variables, their dependence assumptions depend crucially on the linear ordering of the index set and, in most cases, only sufficient conditions are established. The present settings differ from those considered by most authors in that the dependence among the random variables is not formulated in terms of any linear ordering (in fact any ordering in the first problem) of the index set.

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Among the known results on the central limit problem, the more notable and perhaps most general ones are those of Dvoretzky [7]. In Section 3, we shall construct two examples of triangular arrays which arise from finitely dependent random variables, whose row sums converge in distribution to a normal distribution, but which do not satisfy Dvoretzky's conditions.

1. A Convergence Criterion

We adopt the following natural extension of weak convergence of probability measures to that of finite signed measures. We shall denote the class of Borel sets in $\mathbb{R}^k$, $k \geqq 1$, by $\mathscr{B}(\mathbb{R}^k)$.

Definition 1.1. Let $\{\nu_n\}$ be a sequence of finite signed measures on $\mathscr{B}(\mathbb{R}^k)$, $k \geqq 1$. It is said to converge weakly to another finite signed measure ν on $\mathscr{B}(\mathbb{R}^k)$, if for every continuous real-valued function f defined on $\mathbb{R}^k$ and vanishing at infinity, $\lim_{n \to \infty} \int f \, d\nu_n = \int f \, d\nu$.

We note that if ν_n and ν are probability measures then this definition of weak convergence is equivalent to that of Billingsley [2]. Since the space $C_0(\mathbb{R}^k)$ of all continuous real-valued functions defined on $\mathbb{R}^k$ and vanishing at infinity with the sup norm is separable, any subset of the dual space $C_0^*(\mathbb{R}^k)$ which is weak * compact is metrizable in the weak* topology. Combining this fact with the Banach-Alaoglu theorem and the Riesz representation theorem, we have the following important property: every sequence of finite signed measures $\{\nu_n\}$ on $\mathscr{B}(\mathbb{R}^k)$ such that $\sup_n \|\nu_n\| < \infty$, where $\|\cdot\|$ denotes total mass, has a subsequence which converges weakly to a finite signed measure ν on $\mathscr{B}(\mathbb{R}^k)$. Thus, in order to prove that ν_n converges weakly to ν, we need only to show that every weakly convergent subsequent of $\{\nu_n\}$ has ν as its limit. Clearly, if $\{\nu_n\}$ is a tight sequence of probability measures, then ν must necessarily be a probability measure.

We first state the following easy lemma without proof.

Lemma 1.1. *Let $\{\nu_n\}$ be a sequence of finite signed measures on $\mathscr{B}(\mathbb{R})$ such that $\sup_n \|\nu_n\| < \infty$. Then it converges weakly to a finite signed measure ν on $\mathscr{B}(\mathbb{R})$ if and only if*

$$\lim_{n \to \infty} \int (e^{itx} - 1)\, x^{-1} \, d\nu_n(x) = \int (e^{itx} - 1\, x^{-1} \, d\nu(x)$$

for every real number t, where $(e^{itx} - 1)\, x^{-1} = it$ if $x = 0$.

The next lemma is the main result in this section. It will be used in place of the usual characteristic function technique. An advantage of using this lemma is that it enables us to apply the techniques associated with Stein's method (see [4, 5, 9, 11 and 28]). This is particularly apparent in the case of the combinatorial problem considered in Section 5.

Lemma 1.2. *Let $\{Z_n\}$ be a uniformly integrable sequence of random variables, $\{\beta_n\}$ a sequence of real numbers, and $\{\nu_n\}$ a sequence of finite signed measures on $\mathscr{B}(\mathbb{R})$ such that $\sup_n \|\nu_n\| < \infty$. Suppose for every real number t,*

$$\lim_{n\to\infty} \{EZ_n e^{itZ_n} - [\beta_n + \int (e^{itx}-1)x^{-1}\, d\nu_n(x)]\, E e^{itZ_n}\} = 0 \tag{1.1}$$

where $(e^{itx}-1)x^{-1} = it$ if $x=0$. Then $\mathscr{L}(Z_n)$ converges weakly to the infinitely divisible distribution with characteristic function ϕ given by

$$\phi(t) = \exp\{it\beta + \int (e^{itx}-1-itx)x^{-2}\, d\nu(x)\} \tag{1.2}$$

where ν is a finite measure, if and only if

$$\beta_n \to \beta \tag{1.3}$$

and

$$\nu_n \text{ converges weakly to } \nu. \tag{1.4}$$

Suppose, in addition, the ν_n's are (positive) measures. Then any weak limit of $\mathscr{L}(Z_n)$ must necessarily be infinitely divisible with finite variance.

Proof. We first note that the uniform integrability of $\{Z_n\}$ implies the tightness of $\{\mathscr{L}(Z_n)\}$. It also implies $\lim_{k\to\infty} EZ_{n_k} e^{itZ_{n_k}} = -i\tilde{\phi}'(t)$ for every real number t, whenever the subsequence $\{\mathscr{L}(Z_{nk})\}$ converges weakly to a probability measure with characteristic function $\tilde{\phi}$. Thus if (1.3) and (1.4) hold, then by (1.1) and Lemma 1.1, the characteristic function $\tilde{\phi}$ of the limit of any weakly convergent subsequence of $\{\mathscr{L}(Z_n)\}$ must satisfy the differential equation

$$-i\tilde{\phi}'(t) = [\beta + \int (e^{itx}-1)x^{-1}\, d\nu(x)]\, \tilde{\phi}(t) \tag{1.5}$$

with $\tilde{\phi}(0)=1$. As the solution of (1.5) is unique and given by (1.2), it follows that $\mathscr{L}(Z_n)$ converges weakly to an infinitely divisible distribution with characteristic function ϕ given by (1.2). For the converse, we have $\lim_{n\to\infty} EZ_n = -i\phi'(0) = \beta$. Let $\tilde{\nu}$ be the limit of any weakly convergent subsequence of $\{\nu_n\}$. By (1.1) and Lemma 1.1 again, we have

$$-i\phi'(t) = [\beta + \int (e^{itx}-1)x^{-1}\, d\tilde{\nu}(x)]\, \phi(t) \tag{1.6}$$

for every real number t. Differentiating (1.2), comparing the result with (1.6) and using the nonvanishing of ϕ, we obtain $\tilde{\nu} = \nu$. This proves (1.3) and (1.4). The second part of the lemma follows from the fact that the limit of any weakly convergent subsequence of $\{\nu_n\}$ must be a finite measure.

We give an example to show that if the ν_n's in Lemma 1.2 are not (positive) measures, then $\mathscr{L}(Z_n)$ may converge to a probability measure which is not infinitely divisible.

Example 1.1. Let $\mathscr{L}(Z_n) = (1-p)\,\delta + p\,\mu$ where δ is the Dirac measure at zero, μ a probability measure on $\mathscr{B}(\mathbb{R})$ with finite second moment, and $0<p<\frac{1}{2}$. Trivially

$\mathscr{L}(Z_n)$ converges weakly to $(1-p)\,\delta+p\,\mu$. Now let γ be the finite signed measure given by

$$\gamma=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{p}{1-p}\right)^k\mu^k$$

where μ^k is the k-fold convolution of μ. It is not difficult to see that (1.1) holds with $\beta_n=\int x\,d\gamma(x)$ and ν_n defined by $d\nu_n(x)=x^2\,d\gamma(x)$. But if μ has bounded support, then $(1-p)\,\delta+p\,\mu$ is not infinitely divisible.

2. Finitely Dependent Random Variables

In this section we define and give a few examples of finitely dependent random variables.

Definition 2.1. A nonempty family of random variables $\{X_\alpha\colon \alpha\in J\}$ is said to be finitely dependent if for every nonempty finite subset A of J there exists another finite subset $B=B(A)$ (including A) such that $\{X_\alpha\colon \alpha\in A\}$ is independent of $\{X_\alpha\colon \alpha\in B^c\}$ and such that $\sup_A\inf_B|B|/|A|<\infty$, where $|\cdot|$ denotes the order of a set. The order of dependence of the family is defined to be the smallest integer not less than $\sup_A\inf_B|B|/|A|$.

Definition 2.2. Let $\{X_\alpha\colon \alpha\in J\}$ be a finitely dependent family of random variables and let A and B be nonempty finite subsets of J. The set $\{X_\alpha\colon \alpha\in B\}$ is said to be a dependent set of $\{X_\alpha\colon \alpha\in A\}$ if the latter is independent of $\{X_\alpha\colon \alpha\in B^c\}$ and $|B|/|A|$ does not exceed the order of dependence.

We note that every finite set of random variables is finitely dependent, since in this case Definition 2.1 is vacuously satisfied by taking B to be the whole index set. However, for a triangular array, we define finite dependence as follows.

Definition 2.3. A triangular array of random variables $X_{n1},\ldots,X_{nr_n}$, $n\geqq 1$, is said to be finitely dependent if the sequence $\{d_n\}$ is bounded where d_n is the order of dependence of $\{X_{n1},\ldots,X_{nr_n}\}$. It is said to be a finitely dependent uniformly infinitesimal array if it is finitely dependent and X_{nj} converges in probability to zero uniformly in j for $1\leqq j\leqq r_n$ as $n\to\infty$.

Here and throughout the rest of this paper, a triangular array $X_{n1},\ldots,X_{nr_n}$, $n\geqq 1$, is assumed to be such that $r_n\to\infty$ as $n\to\infty$. Note that if $\{X_n\}$ is a finitely dependent sequence of random variables, then the triangular array $X_{n1},\ldots,X_{nr_n}$, $n\geqq 1$, defined by $X_{nj}=X_j/b_n$, where $\{b_n\}$ is a sequence of nonzero numbers, is a finitely dependent array.

It is easy to see that an m-dependent sequence of random variables is finitely dependent. Another example of finite dependence is as follows. Let J be an r-dimensional lattice set in which the coordinates of each point are integers. For each two points $x=(x_1,\ldots,x_r)$ and $y=(y_1,\ldots,y_r)$ in J, define $\|x-y\|=\sum_{j=1}^{r}|x_j-y_j|$. Let $\{Y_\alpha\colon \alpha\in J\}$ be a set of independent random variables and let $\{X_\alpha\colon \alpha\in J\}$

be another set of random variables such that for every $\alpha \in J$, X_α is a function of
Y_β $\{\cancel{X_\beta}: \|\alpha-\beta\| \leqq d\}$ where $d \geqq 1$. Then $\{X_\alpha: \alpha \in J\}$ is a finitely dependent family of random variables. If $r \geqq 2$, then $\{X_\alpha: \alpha \in J\}$ cannot be reduced to an m-dependent sequence no matter how it is enumerated. To see this, we consider for simplicity the special case where $J=\{(j,k): j,k=1,2,\ldots\}$ and $d=1$. (Our argument can easily be extended to the general case.) Suppose that $\{X_n\}$ is an enumeration of $\{X_\alpha: \alpha \in J\}$ and were m-dependent for some $m \geqq 0$. Then for every $j \geqq 1$, $\{X_1, \ldots, X_j\}$ would be independent of $\{X_{j+m+1}, X_{j+m+2}, \ldots\}$. Now take $j=k^2$. Then by the definition of $\{X_\alpha: \alpha \in J\}$, every minimal dependent set of $\{X_1, \ldots, X_{k^2}\}$ must contain at least k^2+2k random variables. Consequently $\{X_1, \ldots, X_{k^2}\}$ cannot be independent of $\{X_{k^2+m+1}, X_{k^2+m+2}, \ldots\}$ if k is sufficiently large. This is a contradiction.

The above examples show that finite dependence includes an m-dependent analogue for random fields. This application of finite dependence to random fields was also suggested by Kai Lai Chung who in addition suggested that it might also be interesting to consider finitely dependent random variables indexed by a partially ordered set (see, for example, [27]).

Further examples of finitely dependent random variables can be found in Kesten and O'Brien [16] who showed that corresponding to each type of mixing (strong, ϕ- or ψ-mixing) and each mixing rate, there exists a finitely dependent sequence of random variables with order of dependence equal to 2, which is also a mixing sequence of the given type and with the given mixing rate. They also showed that a finitely dependent sequence of random variables need not be a mixing sequence.

3. Two Examples

The aim of this section is to show that some of the very general central limit theorems are not applicable to finitely dependent random variables.

First, we observe that many central limit theorems for mixing sequences of random variables impose some conditions on the mixing rate (see, for example, [8, 15, 21 and 25]). The results of Kesten and O'Brien [16] show that there are indeed many finitely dependent sequences of random variables to which these theorems are not applicable.

Second, a number of authors have considered the central limit problem for dependent random variables (see, for example, [3, 7 and 21]), and many more have considered the special case where the limiting distribution is normal (see, for example, [6, 8, 14, 15, 17, 20, 24, 25 and 26]). Among these results, the more notable and perhaps most general ones are those of Dvoretzky (see [7] and [8]). However, we shall construct two finitely dependent arrays of random variables, which do not satisfy Dvoretzky's conditions, yet whose row sums have asymptotically the normal distribution.

Let $X_{n1}, \ldots, X_{nr_n}$, $n \geqq 1$, be a triangular array of random variables, and let $S_{n0}=0$ and $S_{nj}=\sum_{k=1}^{j} X_{nk}$ for $1 \leqq j \leqq r_n$. We shall show that our examples do not satisfy the following condition which is the first of the three sufficient conditions of Theorem 1 in [7] (or Theorem 2.2 in [8]):

$$\sum_{j=1}^{r_n} E(X_{nj}|S_{n,j-1}) \xrightarrow{P} 0 \quad \text{as} \quad n\to\infty, \tag{3.1}$$

where $\xrightarrow{P}$ denotes convergence in probability.

We need a simple lemma which we state without proof.

Lemma 3.1. *Let X and Y be independent and normally distributed random variables with zero means and variances σ^2 and τ^2 respectively. Then*

$$E(X|X+Y)=\sigma^2(X+Y)/(\sigma^2+\tau^2).$$

Example 3.1. Let $\{Y_n\}$ be a sequence of independent random variables each having the standard normal distribution, and let $X_{n1},\dots,X_{nr_n}$, $n\geqq 1$, be a triangular array such that $(r_n^{\frac{1}{2}}X_{n1},\dots,r_n^{\frac{1}{2}}X_{nr_n})$ is a permutation of $(Y_1,\dots,Y_1,\dots,Y_{k_n},\dots,Y_{k_n})$ for some k_n where each Y_j occurs m_{nj} times, $2\leqq m_{nj}\leqq m$ and m is fixed integer, and such that $r_n^{-1}\sum_{j=1}^{k_n} m_{nj}^2$ converges to a positive number τ^2 as $n\to\infty$. Clearly the triangular array is finitely dependent and

$$\mathscr{L}\left(\sum_{j=1}^{r_n} X_{nj}\right)=\mathscr{L}\left(r_n^{-\frac{1}{2}}\sum_{j=1}^{k_n} m_{nj}Y_j\right)$$

converges weakly to $N(0,\tau^2)$. We assert that the triangular array does not satisfy the condition (3.1).

Proof. For each $n\geqq 1$, let $Y_{nj}=r_n^{\frac{1}{2}}X_{nj}$, $1\leqq j\leqq r_n$. Also let $S_{n0}=0$ and $S_{nj}=\sum_{k=1}^{j} X_{nk}$ for $1\leqq j\leqq r_n$. Further let $b_{n0}^2=1$ and $b_{nj}^2=\operatorname{var}\left(\sum_{k=1}^{j} Y_{nk}\right)$ for $1\leqq j\leqq r_n$. If none of the $X_{n1},\dots,X_{n,j-1}$ equals X_{nj}, then $E(X_{nj}|S_{n,j-1})=0$. Otherwise, by Lemma 3.1,

$$\begin{aligned} E(X_{nj}|S_{n,j-1}) &= \left[\frac{q_{nj}\operatorname{var}(X_{nj})}{\operatorname{var}(S_{n,j-1})}\right]S_{n,j-1} \\ &= r_n^{-\frac{1}{2}}b_{n,j-1}^{-2}q_{nj}\sum_{k=1}^{j-1} Y_{nk} \end{aligned} \tag{3.2}$$

where q_{nj} is the number of times X_{nj} occurs in $X_{n1},\dots,X_{n,j-1}$. Let $U_n=\sum_{j=1}^{r_n} E(X_{nj}|S_{n,j-1})$ and J_n be the set of j for which (3.2) holds (with $q_{nj}\geqq 1$). Then

$$U_n=\sum_{j\in J_n} r_n^{-\frac{1}{2}}b_{n,j-1}^{-2}q_{nj}\sum_{k=1}^{j-1} Y_{nk}=\sum_{k=1}^{r_n} c_{nk}Y_{nk}$$

where

$$c_{nk}=r_n^{-\frac{1}{2}}\sum_{j\in J_n,\, j\geqq k+1} b_{n,j-1}^{-2}q_{nj}.$$

Therefore for every real number t, $\phi_n(t)=\exp\{-\frac{1}{2}\sigma_n^2 t^2\}$ where ϕ_n is the characteristic function of U_n and $\sigma_n^2=\mathrm{var}(U_n)$. Since some of the Y_{nk}'s are equal to one another, $\sigma_n^2 \geqq \sum_{k=1}^{r_n} c_{nk}^2$. Now for all $n \geqq 1$ we have $|J_n| \geqq k_n$, $r_n \leqq m\,k_n$, and for $j \in J_n$ we have $q_{nj} \geqq 1$ and $b_{n,j-1}^2 \leqq j\,m \leqq m r_n$. Therefore for $k \leqq [\frac{1}{2}k_n]$, we have

$$|J_n \cap \{k+1, k+2, \ldots, r_n\}| \geqq k_n - [\tfrac{1}{2}k_n] \geqq \tfrac{1}{2}k_n$$

so that $c_{nk} \geqq k_n/2m\,r_n^{\frac{3}{2}} \geqq 1/2m^2\,r_n^{\frac{1}{2}}$, where $[x]$ denotes the integral part of the real number x. It follows that

$$\begin{aligned}\liminf_{n\to\infty} \sigma_n^2 &\geqq \liminf_{n\to\infty} \sum_{k=1}^{[\frac{1}{2}k_n]} c_{nk}^2 \\ &\geqq \liminf_{n\to\infty} [\tfrac{1}{2}k_n]/4m^4 r_n \\ &= 1/8m^5 > 0. \end{aligned} \tag{3.3}$$

If $\sigma_n^2 \to \infty$ as $n \to \infty$, then clearly U_n does not converge in probability to any random variable. On the other hand, if a subsequence $\{\sigma_{n'}^2\}$ converges to some σ^2, then by (3.3), σ^2 must be positive and $\mathscr{L}(U_{n'})$ converges weakly to $N(0,\sigma^2)$. An application of Kolmogorov's zero-one law then implies that U_n does not converge in probability to any random variable. This proves our assertion.

The above example may appear a little artificial but the next example which arises more naturally can be reduced to essentially a special case of it.

Example 3.2. Let $\{Y_{jk}\colon j,k=1,2,\ldots\}$ be a set of independent random variables each having the standard normal distribution and let $\{Z_{jk}\colon j,k=1,2,\ldots\}$ be the finitely dependent family of random variables defined by

$$Z_{jk} = \sum_{|l-j|+|m-k| \leqq 1} Y_{lm} \qquad \text{for } j,k=1,2,\ldots.$$

Let $\{X_n\}$ be an enumeration of $\{Z_{jk}\colon j,k=1,2,\ldots\}$ such that $n^{-1}\,\mathrm{var}\left(\sum_{j=1}^n X_j\right) \to \sigma^2$. We assert that $0<\sigma^2<\infty$ and $\mathscr{L}\left(n^{-\frac{1}{2}} \sum_{j=1}^n X_j\right)$ converges weakly to $N(0,\sigma^2)$. We also assert that there exist such an enumeration for which $n^{-\frac{1}{2}} \sum_{j=1}^n E(X_j | S_{j-1})$ does not converge in probability to any random variable, where $S_0=0$ and for $j \geqq 2$, $S_{j-1} = \sum_{k=1}^{j-1} X_k$.

Proof. It is easy to prove the first assertion. For the second assertion, let $\{X_n\}$ and $\{Y_n\}$ be the respective sequences of random variables obtained from $\{Z_{jk}\}$ and $\{Y_{jk}\}$ by ordering the index set $\{(j,k)\colon j,k=1,2,\ldots\}$ according to the following rule: $(j,k)<(l,m)$ if $j+k<l+m$ or if $j+k=l+m$ but $j<l$. Then it is not difficult to see that $n^{-1}\,\mathrm{var}\left(\sum_{j=1}^n X_j\right) \to 25$ and that $n^{-\frac{1}{2}} \sum_{j=1}^n X_j$ can be expressed as the n^{th} row sum of the triangular array $X_{n1}, \ldots, X_{nr_n}$, $n \geqq 1$, where $r_n \sim 5n$ as $n \to \infty$ and for each $n \geqq 1$, $(n^{\frac{1}{2}} X_{n1}, \ldots, n^{\frac{1}{2}} X_{nr_n})$ is a permutation of

$(Y_1, \ldots, Y_1, \ldots, Y_{k_n}, \ldots, Y_{k_n})$ for some k_n. It is easily seen that each Y_j in $(Y_1, \ldots, Y_1, \ldots, Y_{k_n}, \ldots, Y_{k_n})$ occurs m_{nj} times with $1 \leqq m_{nj} \leqq 5$, but the number of j for which $m_{nj}=1$ is of the order $n^{\frac{1}{2}}$ as $n \to \infty$. Since this is the case and those Y_j's for which $m_{nj}=1$ are independent of the other Y_j's, our problem reduces to essentially a special case of Example 3.1. This proves the second assertion.

We wish to remark that it is not possible to group the random variables in both examples as in [8] (Theorems 5.1 and 5.2) or in [21], since the permutation in Example 3.1 and the enumeration in Example 3.2 are not specified. It is not difficult to see that even if they are specified, such grouping need not be possible. On the other hand, the asymptotic normality in both examples follows from Corollary 4.3 even if the Y_n's in Example 3.1 and the Y_{jk}'s in Example 3.2 are not normally distributed, provided that in each case they are independent and identically distributed with zero mean and finite variance. In Example 3.2, each Z_{jk} may even be any Borel measurable function of $\{Y_{lm}: |l-j|+|m-k| \leqq 1\}$ provided that the function is the same for all (j,k) (except when $j=1$ or $k=1$) and that Z_{jk} has zero mean and finite variance and $n^{-1} \operatorname{var}\left(\sum_{j=1}^{n} X_j\right)$ converges to a positive number.

4. A Dependent Central Limit Problem

We first prove the main theorem which concerns the central limit problem for random variables satisfying a dependence assumption more general than finite dependence. We shall later prove another theorem which concerns only asymptotic normality but under even more general dependence assumption.

Theorem 4.1. *Let $X_{n1}, \ldots, X_{nr_n}$, $n \geqq 1$, be a triangular array of random variables with means $\mu_{n1}, \ldots, \mu_{nr_n}$ and variances $\sigma_{n1}^2, \ldots, \sigma_{nr_n}^2$ respectively. Assume that for every $n \geqq 1$ and $1 \leqq j \leqq r_n$ there exist subsets A_{nj} and B_{nj} of $\{1, \ldots, r_n\}$ such that $\{X_{nj}\}$ is independent of $\{X_{nk}: k \in A_{nj}^c\}$ and $\{X_{nk}: k \in A_{nj}\}$ is independent of $\{X_{nk}: k \in B_{nj}^c\}$, and such that both $\sum_{k \in A_{nj}} X_{nk}$ and $\sum_{k \in B_{nj}} X_{nk}$ converge in probability to zero uniformly in j for $1 \leqq j \leqq r_n$ as $n \to \infty$. Let*

$$Y_{nj} = \sum_{k \in A_{nj}} X_{nk}, \qquad \tau_{nj}^2 = \operatorname{var}(Y_{nj}), \qquad \beta_n = \sum_{j=1}^{r_n} \mu_{nj}$$

and define the finite signed measure v_n by

$$v_n(A) = \sum_{j=1}^{r_n} \operatorname{cov}(X_{nj}, Y_{nj} I(Y_{nj} \in A))$$

for every $A \in \mathscr{B}(\mathbb{R})$. Suppose

$$\sup_n \sum_{j=1}^{r_n} \sigma_{nj} \tau_{nj} < \infty. \tag{4.1}$$

Then $\mathscr{L}\left(\sum_{j=1}^{r_n} X_{nj}\right)$ converges weakly to the infinitely divisible distribution with characteristic function ϕ given by

$$\phi(t) = \exp\{it\beta + \int (e^{itx} - 1 - itx) x^{-2} \, dv(x)\} \tag{4.2}$$

where ν is a finite measure, if and only if

$$\beta_n \to \beta \tag{4.3}$$

and

$$\nu_n \text{ converges weakly to } \nu. \tag{4.4}$$

Proof. Let $W_n = \sum_{j=1}^{r_n} X_{nj}$. By (4.1), $\sup_n \operatorname{var}(W_n) < \infty$ and so $\{W_n - \beta_n\}$ is uniformly integrable. If $\beta_n \to \beta$, then $\{W_n\}$ is also uniformy integrable. One the other hand, if $\mathscr{L}(W_n)$ converges weakly to a probability measure, then $\{\mathscr{L}(W_n)\}$ is tight. This together with the uniform integrability of $\{W_n - \beta_n\}$ imply that $\{\beta_n\}$ is bounded. It follows that $\{W_n\}$ is uniformly integrable. Hence the uniform integrability condition in Lemma 1.2 is satisfied under the hypotheses of the theorem.

Now let

$$f(w) = e^{itw}, \qquad M_{nj} = EY_{nj},$$

$$W_n = \sum_{j=1}^{r_n} X_{nj}, \qquad U_{nj} = \sum_{k \in A_{nj}^c} X_{nk}$$

$$V_{nj} = \sum_{k \in B_{nj}^c} X_{nk}, \qquad Z_{nj} = \sum_{k \in B_{nj}} X_{nk},$$

$$T_{nj} = \sum_{k \in B_{nj} - A_{nj}} X_{nk}, \qquad \xi_{nj} = e^{itY_{nj}} - 1,$$

$$\eta_{nj} = e^{itZ_{nj}} - 1, \qquad \zeta_{nj} = e^{itT_{nj}} - 1,$$

where t and w are real numbers. Then we have for sufficiently large n,

$$\begin{aligned}
EW_n f(W_n) &= \sum_{j=1}^{r_n} E\{X_{nj}[f(W_n) - f(U_{nj})]\} + \sum_{j=1}^{r_n} \mu_{nj} Ef(U_{nj}) \\
&= \left\{\beta_n + \sum_{j=1}^{r_n} E[(X_{nj} - \mu_{nj})\,\xi_{nj}]\right\} Ef(W_n) \\
&\quad + \sum_{j=1}^{r_n} E\{X_{nj}\xi_{nj}[f(U_{nj}) - f(V_{nj})]\} \\
&\quad - \sum_{j=1}^{r_n} \mu_{nj} E\{\xi_{nj}[f(U_{nj}) - f(V_{nj})]\} \\
&\quad - \sum_{j=1}^{r_n} [EX_{nj}\xi_{nj}][Ef(W_n) - Ef(V_{nj})] \\
&\quad + \sum_{j=1}^{r_n} \mu_{nj}[E\xi_{nj}][Ef(W_n) - Ef(V_{nj})] \\
&= \{\beta_n + \textstyle\int (e^{itx} - 1)\, x^{-1}\, d\nu_n(x)\}\, Ef(W_n) \\
&\quad + \sum_{j=1}^{r_n} \operatorname{cov}(X_{nj}, \xi_{nj}\zeta_{nj} f(V_{nj})) \\
&\quad - \sum_{j=1}^{r_n} \operatorname{cov}(X_{nj}, \xi_{nj})\, E\eta_{nj} f(V_{nj}).
\end{aligned}$$

Let R_{n1} and R_{n2} denote the last two sums respectively. By (4.1), $\sup_n \|v_n\| < \infty$. Therefore by Lemma 1.2, it suffices to prove that $\lim_{n\to\infty} R_{nj}=0$ for $j=1,2$. Now using the independence between X_{nj} and $\zeta_{nj} f(V_{nj})$, we have for $1 \leqq j \leqq r_n$,

$$\begin{aligned}&\operatorname{cov}(X_{nj}, \xi_{nj} \zeta_{nj} f(V_{nj}))\\&\quad = E\{(X_{nj}-\mu_{nj})(e^{it(Y_{nj}-M_{nj})}-1)\, e^{itM_{nj}} \zeta_{nj} f(V_{nj})\}.\end{aligned}$$

So by the Cauchy-Schwarz inequality and the independence between X_{nj} and $\zeta_{nj} f(V_{nj})$ again, we have for $1 \leqq j \leqq r_n$,

$$\begin{aligned}&|\operatorname{cov}(X_{nj}, \xi_{nj} \zeta_{nj} f(V_{nj}))|\\&\quad \leqq |t|\, \tau_{nj} \{E[(X_{nj}-\mu_{nj})^2 |\zeta_{nj}|^2]\}^{\frac{1}{2}}\\&\quad = |t|\, \sigma_{nj} \tau_{nj} [E|\zeta_{nj}|^2]^{\frac{1}{2}}\\&\quad \leqq |t|\, \sigma_{nj} \tau_{nj} [\max_{1 \leqq k \leqq r_n} E|\zeta_{nk}|^2]^{\frac{1}{2}}.\end{aligned}$$

This together with (4.1) and the dependence assumption imply that $\lim_{n\to\infty} R_{n1}=0$. A similar but much simpler argument proves that $\lim_{n\to\infty} R_{n2}=0$. This proves the theorem.

We deduce a few corollaries. The first is easy and so we state it without proof.

Corollary 4.1. *Let $X_{n1}, \dots, X_{nr_n}$, $n \geqq 1$, be a finitely dependent, uniformly infinitesimal array of random variables with means $\mu_{n1}, \dots, \mu_{nr_n}$ and variances $\sigma_{n1}^2, \dots, \sigma_{nr_n}^2$ respectively. For each $n \geqq 1$ and $1 \leqq j \leqq r_n$, let $\{X_{nk}\colon k \in A_{nj}\}$ be a dependent set of $\{X_{nj}\}$. Define Y_{nj}, β_n and v_n as in Theorem* 4.1. *Suppose* (4.1) *holds. Then $\mathscr{L}\left(\sum_{j=1}^{r_n} X_{nj}\right)$ converges weakly to the infinitely divisible distribution with characteristic function ϕ given by* (4.2) *if and only if* (4.3) *and* (4.4) *hold.*

The result in the next corollary is due to Diananda [6].

Corollary 4.2. *Let $X_{n1}, \dots, X_{nr_n}$, $n \geqq 1$, be a uniformly infinitesimal array of random variables with zero means and finite variances such that each row is m-dependent. Let $W_n = \sum_{j=1}^{r_n} X_{nj}$. Supposes for each $n \geqq 1$, $\operatorname{var}(W_n)=1$, $\sup_n \sum_{j=1}^{r_n} \operatorname{var}(X_{nj}) < \infty$, and for each $\varepsilon > 0$,*

$$\lim_{n\to\infty} \sum_{j=1}^{r_n} EX_{nj}^2 I(|X_{nj}| > \varepsilon) = 0.$$

Then $\mathscr{L}(W_n)$ converges weakly to $N(0,1)$.

Proof. Take $A_{nj} = \{k\colon\ 1 \leqq k \leqq r_n,\ |k-j| \leqq m\}$. It is easy to see that $\sup_n \sum_{j=1}^{k_n} \operatorname{var}(X_{nj}) < \infty$ implies (4.1). Hence it suffices to show that the Lindeberg condition implies the weak convergence of v_n to the Dirac measure at zero.

Indeed,

$$\nu_n(\mathbb{R}) = \sum_{j=1}^{r_n} \operatorname{cov}(X_{nj}, Y_{nj}) = \operatorname{var}(W_n) = 1,$$

and for every ε, $\delta > 0$,

$$\begin{aligned} &|\nu_n|(\{x: |x| > \delta\}) \\ &\leqq \sum_{j=1}^{r_n} E|X_{nj} Y_{nj}| I(|Y_{nj}| > \delta) \\ &\leqq \varepsilon \sum_{j=1}^{r_n} E|Y_{nj}| I(|Y_{nj}| > \delta) + \sum_{j=1}^{r_n} E|X_{nj} Y_{nj}| I(|X_{nj}| > \varepsilon) \\ &\leqq \varepsilon \delta^{-1} \sum_{j=1}^{r_n} E Y_{nj}^2 + L_n^{\frac{1}{2}}(\varepsilon) \left(\sum_{j=1}^{r_n} E Y_{nj}^2 \right)^{\frac{1}{2}} \\ &\leqq \varepsilon \delta^{-1} (2m+1)^2 M_n + (2m+1) L_n^{\frac{1}{2}}(\varepsilon) M_n^{\frac{1}{2}} \end{aligned}$$

where

$$M_n = \sum_{j=1}^{r_n} \operatorname{var}(X_{nj}) \quad \text{and} \quad L_n(\varepsilon) = \sum_{j=1}^{r_n} E X_{nj}^2 I(|X_{nj}| > \varepsilon).$$

This proves the corollary.

Corollary 4.3. *Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables with zero mean and finite variance. For every $j \geqq 1$, let A_j and B_j be finite subsets of the positive integers such that $\{X_j\}$ is independent of $\{X_k: k \in A_j^c\}$ and $\{X_k: k \in A_j\}$ is independent of $\{X_k: k \in B_j^c\}$ and such that $\sup_{1 \leqq j < \infty} |B_j| < \infty$. Then $\mathscr{L}\left(n^{-\frac{1}{2}} \sum_{j=1}^{n} X_j\right)$ converges weakly to $N(0, \sigma^2)$ if and only if $n^{-1} \operatorname{var}\left(\sum_{j=1}^{n} X_j\right) \to \sigma^2$.*

Proof. Consider the array $X_{n1}, \ldots, X_{nn}$, $n \geqq 1$, where $X_{nj} = n^{-\frac{1}{2}} X_j$ for $1 \leqq j \leqq n$. It is clear that the dependence assumption in Theorem 4.1 is satisfied with $A_{nj} = A_j \cap \{1, \ldots, n\}$ and $B_{nj} = B_j \cap \{1, \ldots, n\}$ and that (4.1) is also satisfied. It remains to show that for every $\delta > 0$, $|\nu_n|(\{x: |x| > \delta\}) \to 0$. As the argument for this is almost identical to the corresponding one in the proof of Corollary 4.2, we omit it here. Hence the corollary.

In Corollary 4.1, if for each $n \geqq 1$, $X_{n1}, \ldots, X_{nr_n}$ are m-dependent, then we may take $A_{nj} = \{k: 1 \leqq k \leqq r_n, |k - j| \leqq m\}$ (as we have done so in the proof of Corollary 4.2). For m-dependent arrays, (4.1) is definitely weaker than the condition $\sup_n \sum_{j=1}^{r_n} \operatorname{var}(X_{nj}) < \infty$ assumed in Corollary 4.2. There are examples for which (4.1) holds but $\sup_n \sum_{j=1}^{r_n} \operatorname{var}(X_{nj}) = \infty$ (see the example in [9]). Of course Corollary 4.1 reduces to the classical result for sums of independent (that is 0-dependent) random variables with finite variances.

Note that every finitely dependent sequence satisfies the dependence assumption is Corollary 4.3. In this respect we wish to mention, but only briefly, that

Corollary 4.3 covers Examples 3.1 and 3.2 as well as those mixing sequences constructed in Theorem 1 of [16].

In Theorem 4.1, it can be shown that if $\mathscr{L}(W_n)$ (where $W_n = \sum_{j=1}^{r_n} X_{nj}$) converges weakly to a probability measure then v_n converges weakly. However it is not clear, nor have we been able to determine, whether the weak limit of v_n must necessarily be a (positive) measure. If so, then it can be shown that the class of all possible weak limits of $\mathscr{L}(W_n)$ which are probability measures is the class of all infinitely divisible distributions with finite variance. Otherwise, in view of Example 1.1, the class of all possible weak limits of $\mathscr{L}(W_n)$ which are probability measures is not contained in the class of infinitely divisible distributions. In this case it would seem interesting to identify the class.

We now prove a theorem which concerns only asymtotic normality.

Theorem 4.2. *Let $X_{n1}, \ldots, X_{nr_n}$, $n \geqq 1$, be a triangular array of random variables with zero means and variances $\sigma_{n1}^2, \ldots, \sigma_{nr_n}^2$ respectively. For every $n \geqq 1$ and $1 \leqq j \leqq r_n$ let A_{nj} be a subset of $\{1, \ldots, r_n\}$, and for every $k \in A_{nj}$ let B_{njk} be a subset of $\{1, \ldots, r_n\}$, such that $\{X_{nj}\}$ is independant of $\{X_{nl}: l \in A_{nj}^c\}$ and $\{X_{nj}, X_{nk}\}$ is independent of $\{X_{nl}: l \in B_{njk}^c\}$, and such that both $\sum_{l \in A_{nj}} X_{nl}$ and $\sum_{l \in B_{njk}} X_{nl}$ converge in probability to zero uniformly in j and k for $1 \leqq j \leqq r_n$ and $k \in A_{nj}$ as $n \to \infty$. Let*

$$Y_{nj} = \sum_{k \in A_{nj}} X_{nk}, \qquad W_n = \sum_{j=1}^{r_n} X_{nj}.$$

Suppose

$$\sup_n \sum_{j=1}^{r_n} \sum_{k \in A_{nj}} \sigma_{nj}\sigma_{nk} < \infty \tag{4.5}$$

and for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \sum_{j=1}^{r_n} E|X_{nj} Y_{nj}| I(|Y_{nj}| > \varepsilon) = 0. \tag{4.6}$$

Then $\mathscr{L}(W_n)$ converges weakly to $N(0, \sigma^2)$ if and only if $\operatorname{var}(W_n) \to \sigma^2$.

Proof. Let

$$Z_{njk} = \sum_{l \in B_{njk}} X_{nl}, \qquad U_{nj} = \sum_{k \in A_{nj}^c} X_{nk},$$

$$V_{njk} = \sum_{l \in B_{njk}^c} X_{nl}, \qquad M_n = \sum_{j=1}^{r_n} \sum_{k \in A_{nj}} \sigma_{nj}\sigma_{nk},$$

$$L_n(\varepsilon) = \sum_{j=1}^{r_n} E|X_{nj} Y_{nj}| I(|Y_{nj}| > \varepsilon).$$

Then for every real number t, we have

$$
\begin{aligned}
E W_n e^{itW_n} = & \, i t[\operatorname{var}(W_n)] E e^{itW_n} \\
& + i t \sum_{j=1}^{r_n} \sum_{k \in A_{nj}} E\{X_{nj} X_{nk} (E e^{itV_{njk}} - E e^{itW_n})\} \\
& + i t \sum_{j=1}^{r_n} \sum_{k \in A_{nj}} E\{X_{nj} X_{nk} (e^{itU_{nj}} - e^{itV_{njk}})\} \\
& + \sum_{j=1}^{r_n} E\{X_{nj} (e^{itY_{nj}} - 1 - i t Y_{nj}) e^{itU_{nj}}\} \\
= & \{\textstyle\int (e^{itx} - 1) x^{-1} d v_n(x)\} E e^{itW_n} \\
& + R_{n1} + R_{n2} + R_{n3}
\end{aligned}
$$

where R_{n1}, R_{n2} and R_{n3} denote the last three sums respectively and v_n is a measure of mass $\operatorname{var}(W_n)$ concentrated at zero. Now (4.5) implies $\sup_n \operatorname{var}(W_n) < \infty$ which in turn implies that $\{W_n\}$ is uniformly integrable and that $\sup_n \|v_n\| < \infty$. Therefore by Lemma 1.2, it suffices to show that $\lim_{n\to\infty} R_{nj} = 0$ for $j = 1$, 2 and 3. Using arguments very similar to those in the proof of Theorem 4.1, it can easily be shown that $\lim_{n\to\infty} R_{nj} = 0$ for $j = 1$, 2. For R_{n3}, we have for every $\varepsilon > 0$,

$$
\begin{aligned}
|R_{n3}| & \leqq \tfrac{1}{2} \varepsilon t^2 \sum_{j=1}^{r_n} E|X_{nj} Y_{nj}| + 2|t| L_n(\varepsilon) \\
& \leqq \tfrac{1}{2} \varepsilon t^2 M_n + 2|t| L_n(\varepsilon).
\end{aligned}
$$

This together with (4.5) and (4.6) imply that $\lim_{n\to\infty} R_{n3} = 0$. This proves the theorem.

Note that the triangular array in Theorem 4.2 includes that in Theorem 4.1 as well as arrays formed by the summands of the standardized U-statistics. On the other hand the triangular array in Theorem 4.1 does not include the latter. The U-statistics form an important class of statistics in the theory of nonparametric inference (see Puri and Sen [22], pp. 51–66). Their asymptotic normality is well known and is usually proved by replacing the statistic concerned, say U_n, by a sum of independent random variables which is asymptotically equivalent to U_n and to which the central limit theorem can be applied. This approach is due to Hoeffding [12]. Although we shall not discuss it here, we wish to remark that the asymptotic normality of the U-statistics can also be deduced from Theorem 4.2.

5. A Combinatorial Central Limit Problem

For every $n \geqq 2$, let $X_{njk}, j, k = 1, \ldots, n$, be a square array of independent random variables with finite variances, and let $\pi_n = (\pi_n(1), \ldots, \pi_n(n))$ be a random permutation of $(1, \ldots, n)$ independent of the X_{njk}'s. (By a random permutation of $(1, \ldots, n)$ we mean an n-dimensional random vector which takes on each permutation of $(1,\dots,n)$ with probability $1/n!$.) We consider the central limit problem for the sums $W_n=\sum_{j=1}^{n} X_{nj\pi_n(j)}$, $n\geqq 2$.

A special special case of W_n is the statistic $\xi_n=\sum_{j=1}^{n} c_{nj\pi_n(j)}$ where c_{njk}, j, $k=1,\dots,n$, is a square array of real numbers. A further special case is the statistic $\eta_n=\sum_{j=1}^{n} a_{nj}b_{n\pi_n(j)}$ where a_{nj} and b_{nj}, $j=1,\dots,n$, are two sequences of real numbers. Both statistics ξ_n and η_n occur in permutation tests in nonparametric inference (see, for example, Puri and Sen [22], pp. 66–85). The asymptotic normality of η_n was considered by Wald and Wolfowitz [29], Noether [19], Hajek [10] and Robinson [23], and that of ξ_n by Hoeffding [13] and Motoo [18]. Chen [5] considered the Poisson approximation for W_n in the case where the X_{njk}'s are 0–1 variables. More recently von Bahr [1] considered the normal approximation for W_n in the general case but assumed the existence of third moments of the X_{njk}'s. Taking a different approach, Ho and Chen [11] also considered the normal approximation for W_n in the general case assuming only the existence of second moments of the X_{njk}'s. In this section we prove the following theorem.

Theorem 5.1. *For $n\geqq 2$, let $X_{njk}, j,k=1,\dots,n$, be a square array of independent random variables with $EX_{njk}=c_{njk}$ and with finite variances, let $\pi_n=(\pi_n(1),\dots,\pi_n(n))$ be a random permutation independent of the X_{njk}'s and let $W_n=\sum_{j=1}^{n} X_{nj\pi_n(j)}$. Suppose*

$$\sum_{j=1}^{n} c_{njk}=0 \text{ for } 1\leqq k\leqq n \quad\text{and}\quad \sum_{k=1}^{n} c_{njk}=0 \text{ for } 1\leqq j\leqq n, \tag{5.1}$$

$$\sup_n \operatorname{var}(W_n)<\infty, \tag{5.2}$$

and for every $\varepsilon>0$,

$$\lim_{n\to\infty}\alpha_{n\varepsilon}=0 \tag{5.3}$$

where

$$\alpha_{n\varepsilon}=\max_{1\leqq j,k\leqq n}\left\{n^{-1}\sum_{j=1}^{n}P(|X_{njk}|>\varepsilon),\ n^{-1}\sum_{k=1}^{n}P(|X_{njk}|>\varepsilon)\right\}.$$

Then the class of all possible weak limits of $\mathscr{L}(W_n)$ is the class of all infinitely divisible distributions with zero means and finite variances. Furthermore $\mathscr{L}(W_n)$ converges weakly to the infinitely divisible distribution with charactersitic function ϕ given by

$$\phi(t)=\exp\{\textstyle\int(e^{itx}-1-itx)x^{-2}\,d\nu(x)\} \tag{5.4}$$

where ν is a finite measure, if and only if ν_n converges weakly to ν where ν_n is the finite measure defined by

$$\nu_n(A)=n^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n} EX_{njk}^2 I(X_{njk}\in A)$$

for every $A\in\mathscr{B}(\mathbb{R})$.

We first state a lemma (Lemma 4.3 of [11] without proof.

Lemma 5.1. *With the same notation as in Theorem* 5.1,

$$\operatorname{var}(W_n)=(n-1)^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{njk}^2+n^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n}\operatorname{var}(X_{njk}).$$

Proof of Theorem 5.1. From now on we drop the subscript n for simplicity but will pick it up whenever we need it. Let the random indices I, J, K, L, M and the random permutations π, ρ, τ be as defined in [11] ((4.1)–(4.7)). This constructions was first used in [5] ((2.3)–(2.9)) but in slightly different notation. We reproduce the defining properties of these random indices and permutations here for ease of reference.

For each $n\geqq 2$, the random indices I, J, K, L, M are each uniformly distributed on $\{1, \ldots, n\}$, and $\pi=(\pi(1), \ldots, \pi(n))$, $\rho=(\rho(1), \ldots, \rho(n))$ and $\tau=(\tau(1), \ldots, \tau(n))$ are random permutations of $(1, \ldots, n)$ such that

$$\{I,J,K,L,M,\pi,\rho,\tau\}\quad\text{is independent of }\{X_{jk}:j,k=1,\ldots,n\};\tag{5.5}$$

$$(I,K)\text{ and }(L,M)\text{ are uniformly distributed on }\{(i,k):i\neq k;\ i,k=1,\ldots,n\};\tag{5.6}$$

$$J,\ (I,K),\ (L,M)\text{ and }\tau\text{ are mutually independent;}\tag{5.7}$$

$$J,\ (I,K)\text{ and }\rho\text{ are mutually independent;}\tag{5.8}$$

$$I\text{ and }\pi\text{ are mutually independent;}\tag{5.9}$$

$$\rho(\alpha)=\begin{cases}\tau(\alpha), & \alpha\neq I,K,\quad \tau^{-1}(L),\quad \tau^{-1}(M)\\ L, & \alpha=I\\ M, & \alpha=K\\ \tau(I), & \alpha=\tau^{-1}(L)\\ \tau(K), & \alpha=\tau^{-1}(M)\end{cases}\tag{5.10}$$

and

$$\pi(\alpha)=\begin{cases}\rho(\alpha), & \alpha\neq I,\quad \rho^{-1}(J)\\ J, & \alpha=I\\ \rho(I), & \alpha=\rho^{-1}(J)\end{cases}\tag{5.11}$$

where $\rho(\rho^{-1}(\alpha))=\rho^{-1}(\rho(\alpha))=\alpha$ and $\tau(\tau^{-1}(\alpha))=\tau^{-1}(\tau(\alpha))=\alpha$ for $\alpha=1,\ldots,n$.

Also let

$$V^{**}=\sum_{j\neq I,K} X_{j\rho(j)}, \qquad V=\sum_{j=1}^{n} X_{j\tau(j)},$$

$$Z=n^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n} X_{jk},$$

$$c_{j+}=\sum_{k=1}^{n} c_{jk}, \qquad c_{+k}=\sum_{j=1}^{n} c_{jk}.$$

We use the following identity taken from the intermediate steps of (4.10) in [11],

$$\begin{aligned} &E\{(W-Z)f(W)\}\\ &=\tfrac{1}{2}(n-1)\,E\{(X_{IL}+X_{KM})[f(V^{**}+X_{IL}+X_{KM})-f(V^{**}+X_{IM}+X_{KL})]\} \end{aligned} \tag{5.12}$$

where f is any bounded Borel measurable real-valued function defined on $\mathbb{R}$. Now, using (5.10), we have

$$V^{**}=X_{\tau^{-1}(L),\tau(I)}+X_{\tau^{-1}(M),\tau(K)}+\sum_{j\neq I,K,\tau^{-1}(L),\tau^{-1}(M)} X_{j\tau(j)}.$$

Hence V^{**} depends on I, K, L, M, τ and the X_{jk}'s but does not contain any X_{jk} from the I^{th} or K^{th} row. By (5.5) and (5.7), (I,K), $(L,\ M)$, τ and the X_{jk}'s are mutually independent. It follows that V^{**} and $(X_{IM},\ X_{KL},\ X_{IL},\ X_{KM})$ are conditionally independent given $(I,\ K,L,\ M)$. Let $\hat{E}$ denote $E^{I,K,L,M}$. Then (5.12) yields

$$\begin{aligned} &E\{(W-Z)e^{itW}\}\\ &=\tfrac{1}{2}(n-1)\,E\{[\hat{E}(X_{IL}+X_{KM})(e^{it(X_{IL}+X_{KM})}-e^{it(X_{IM}+X_{KL})})][\hat{E}e^{itV^{**}}]\} \end{aligned} \tag{5.13}$$

for every real number t. Now define the finite measure $\tilde{v}_n$ by

$$\tilde{\gamma}_n(\mathrm{A})=\tfrac{1}{2}(n-1)\,E(X_{IL}+X_{KM})^2\, I(X_{IL}+X_{KM}\in A)$$

for every $A\in\mathscr{B}(\mathbb{R})$, and let

$$\begin{aligned} R_{n1}&=EZ\,e^{itW};\\ R_{n2}&=-\tfrac{1}{2}(n-1)\,E\{[\hat{E}(X_{IL}+X_{KM})(e^{it(X_{IM}+X_{KL})}-1)][\hat{E}e^{itV^{**}}]\};\\ R_{n3}&=\tfrac{1}{2}(n-1)\,E\{\hat{E}(X_{IL}+X_{KM})(e^{it(X_{IL}+X_{KM})}-1)][\hat{E}(e^{itV^{**}}-e^{itV})]\};\\ R_{n4}&=\{\textstyle\int(e^{itx}-1)\,x^{-1}\,d(\tilde{v}_n-v_n)(x)\}\,E\,e^{itW}. \end{aligned}$$

Then (5.13) yields

$$\begin{aligned} &EW_n e^{itW_n}-\{\textstyle\int(e^{itx}-1)\,x^{-1}\,dv_n(x)\}\,E\,e^{itW_n}\\ &\qquad=R_{n1}+R_{n2}+R_{n3}+R_{n4} \end{aligned} \tag{5.14}$$

where, by (5.5), (5.7) and the fact that $\mathscr{L}(V)=\mathscr{L}(W)$, $\hat{E}e^{itV}=E\,e^{itV}=E\,e^{itW}$.

There are errors in these lemmas. But they can be corrected without much effort.

Now for each $n \geqq 2$, W_n includes in terms of distribution sums of independent random variables as special cases (by letting $X_{nj1}, \ldots, X_{njn}$ have the same distribution for each j). By (5.1) and (5.2), it follows that the class of all possible weak limits of $\mathscr{L}(W_n)$ contains the class of all infinitely divisible distributions with zero means and finite variances. Next (5.1) and (5.2) imply that $\{W_n\}$ is uniformly integrable, and (5.1), (5.2) and Lemma 5.1 imply that $\sup_n \|\nu_n\| < \infty$. Furthermore, for every $n \geqq 2$, ν_n is a (positive) measure. Therefore by Lemma 1.2, the theorem is proved if we show that $\lim_{n\to\infty} R_{nj} = 0$ for $j = 1, 2, 3, 4$. We break up the rest of the proof into lemmas.

Lemma 5.2

$$\lim_{n\to\infty} R_{n1} = 0.$$

Proof. By [11] (Lemma 4.1) and (5.1), $EZe^{itW} = n^{-1}E(W - E^{\pi}W)e^{itW}$ so that $|R_{n1}| \leqq 2n^{-1}(EW^2)^{\frac{1}{2}}$ which by (5.1) and (5.2) tends to zero as $n \to \infty$.

The next lemma follows from (5.3).

Lemma 5.3. *X_{nIM}, X_{nKL} and $U_n = V_n - V_n^{**}$ converge to zero in probability as $n \to \infty$.*

The next three lemmas follow from (5.1), Lemma 5.1 and direct computations.

Lemma 5.4.

$$(n-1)\,E(c_{IL} + c_{KM})^2 \leqq 4\,\mathrm{var}(W).$$

Lemma 5.5.

$$\lim_{n\to\infty} (n-1)\,E(X_{IL} + X_{KM})(X_{IM} + X_{KL}) = 0.$$

Lemma 5.6.

$$(n-1)^2\,E(c_{IL} + c_{KM})^2 (c_{IM} + c_{KL})^2 \leqq 16[\mathrm{var}(W)]^2.$$

Lemma 5.7.

$$\lim_{n\to\infty} (n-1)\,E(X_{IL} + X_{KM})[e^{it(X_{IM} + X_{KL})} - 1] = 0.$$

Proof. The left hand side equals

$$\lim_{n\to\infty} i(n-1) \int_0^t E(X_{IL} + X_{KM})(X_{IM} + X_{KL})\, e^{i\xi(X_{IM} + X_{KL})}\, d\xi$$

which by Lemma 5.5

$$= \lim_{n\to\infty} i(n-1) \int_0^t E(X_{IL} + X_{KM})(X_{IM} + X_{KL})[e^{i\xi(X_{IM} + X_{KL})} - 1]\, d\xi.$$

Now

$$\begin{aligned}(n-1)|E(X_{IL}+X_{KM})(X_{IM}+X_{KL})[e^{i\xi(X_{IM}+X_{KL})}-1]|\\ \leqq(n-1)\,E\{[\hat{E}(X_{IL}+X_{KM})^2(X_{IM}+X_{KL})^2]^{\frac{1}{2}}[\hat{E}|e^{i\xi(X_{IM}+X_{KL})}-1|^2]^{\frac{1}{2}}\}\\ =(n-1)\,E\{[\hat{E}(X_{IL}+X_{KM})^2\,\hat{E}(X_{IM}+X_{KL})^2]^{\frac{1}{2}}[\hat{E}|e^{i\xi(X_{IM}+X_{KL})}-1|^2]^{\frac{1}{2}}\}\\ \leqq(n-1)[E(c_{IL}+c_{KM})^2(c_{IM}+c_{KL})^2]^{\frac{1}{2}}[E|e^{i\xi(X_{IM}+X_{KL})}-1|^2]^{\frac{1}{2}}\end{aligned}$$

The lemma then follows from (5.2) and Lemmas 5.3 and 5.6.

Lemma 5.8.

$$\lim_{n\to\infty} R_{n2}=0.$$

Proof. By Lemma 5.7,

$$\begin{aligned}R_{n2}=-\tfrac{1}{2}(n-1)\,E\{[\hat{E}(X_{IL}+X_{KM})(e^{it(X_{IM}+X_{KL})}-1)]\\ \times[\hat{E}(e^{itV^{**}}-e^{itV})]\}.\end{aligned}$$

Now the absolute value of the right hand side

$$\leqq\tfrac{1}{2}(n-1)\,|t|\,[E(X_{IL}+X_{KM})^2(X_{IM}+X_{KL})^2\,E|e^{itU}-1|^2]^{\frac{1}{2}}$$

which by (5.2) and Lemmas 5.3 and 5.6 tends to zero as $n\to\infty$. Hence the lemma.

Lemma 5.9.

$$\lim_{n\to\infty} R_{n3}=0.$$

Proof.

$$\begin{aligned}2|R_{n3}|\leqq(n-1)\,E\{\hat{E}|e^{itU}-1|\,\hat{E}(X_{IL}+X_{KM})^2\}\\ =(n-1)\,E\{(c_{IL}+c_{KM})^2\,\hat{E}|e^{itU}-1|\}.\end{aligned}$$

Now for every $\varepsilon>0$.

$$\hat{E}|e^{itU}-1|\leqq|t|\varepsilon+C\alpha_{n\varepsilon}$$

where C is some constant. Combining (5.3) and Lemma 5.4, we prove the lemma.

Finally we note that to prove $\lim_{n\to\infty} R_{n4}=0$, it suffices to show that

$$\lim_{n\to\infty}(n-1)\,EX_{IL}e^{itX_{IL}}(e^{itX_{KM}}-1)=0.$$

Now

$$\begin{aligned}(n-1)\,EX_{IL}e^{itX_{IL}}(e^{itX_{KM}}-1)\\ =(n-1)\,E\{E^{I,K}X_{IL}e^{itX_{IL}}\,E^{I,K}(e^{itX_{KM}}-1)\}.\end{aligned}$$

For every $\varepsilon>0$,

$$E^{I,K}|e^{itX_{KM}}-1|\leqq|t|\varepsilon+2\max_{1\leqq j\leqq n} n^{-1}\sum_{k=1}^{n}P(|X_{jk}|>\varepsilon).$$

We also have

$$\begin{aligned}(n-1)E\,|E^{I,K}X_{IL}e^{itX_{IL}}|\\ &=(n-1)E\,|E^{I}X_{IL}(e^{itX_{IL}}-1)|\\ &\leqq(n-1)|t|\,EX_{IL}^{2}.\end{aligned}$$

By (5.3) and Lemma 5.1, we prove $\lim_{n\to\infty} R_{n4}=0$. Hence the theorem.

In the case where the limiting distribution is normal, individual centering can be reduced to overall centering. Let $\bar{c}_{nj+}=c_{nj+}/n$, $\bar{c}_{n+k}=c_{n+k}/n$ and $\bar{c}_{n++}=c_{n++}/n^2=\sum_{j=1}^{n}\sum_{k=1}^{n}c_{njk}/n^2$.

Corollary 5.1. *Let X_{njk}, π_n and W_n be as defined in Theorem 5.1. Suppose for every $n\geqq 2$, $EW_n=0$ and $\operatorname{var}(W_n)=1$. If for every $\varepsilon>0$,*

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n}EX_{njk}^{2}I(|X_{njk}|>\varepsilon)=0, \tag{5.15}$$

then $\mathscr{L}(W_n)$ converges weakly to the standard normal distribution. The condition (5.15) is also necessary provided we further assume

$$\lim_{n\to\infty}\max_{1\leqq j,k\leqq n}(|\bar{c}_{nj+}|,|\bar{c}_{n+k}|)=0 \tag{5.16}$$

and for every $\varepsilon>0$,

$$\lim_{n\to\infty}\alpha_{n\varepsilon}=0 \tag{5.17}$$

where

$$\alpha_{n\varepsilon}=\max_{1\leqq j,k\leqq n}\left\{n^{-1}\sum_{j=1}^{n}P(|Y_{njk}|>\varepsilon),\ n^{-1}\sum_{k=1}^{n}P(|Y_{njk}|>\varepsilon)\right\}$$

and

$$Y_{njk}=X_{njk}-\bar{c}_{nj+}-\bar{c}_{n+k}.$$

We first state without proof a lemma, using the same notation.

Lemma 5.10. *The condition (5.15) implies*

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n}\sum_{k=1}^{n}EY_{njk}^{2}I(|Y_{njk}|>\varepsilon)=0. \tag{5.18}$$

Conversely, (5.18) implies (5.15) provided (5.16) holds.

Proof of Corollary 5.1. First we note that (5.15) implies (5.16) and (5.17). Let $EY_{njk}=d_{njk}$. The condition $EW_n=0$ for $n\geqq 2$ implies that $c_{n++}=0$ for $n\geqq 2$ which in turn implies that

$$d_{nj+}=d_{n+k}=0 \quad \text{for } 1\leqq j,k\leqq n \text{ and } n\geqq 2 \tag{5.19}$$

and that

$$\sum_{j=1}^{n} Y_{nj\pi_n(j)} = \sum_{j=1}^{n} X_{nj\pi_n(j)}. \tag{5.20}$$

Now (5.15) implies (5.16) and (5.17). It follows from (5.17), (5.19), (5.20), Lemma 5.10 and Theorem 4.1 that (5.15) is sufficient for $\mathscr{L}(W_n)$ to converge weakly to the standard normal distribution. The converse is also clear from (5.17), (5.19), (5.20), Lemma 5.10 and Theorem 5.1. This proves the corollary.

We wish to remark that the sufficiency of (5.15) also follows from [11].

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