

# ON THE MAXIMAL PRIMITIVE IDEAL CORRESPONDING TO THE MODEL NILPOTENT ORBIT

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ABSTRACT. Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$  be a Cartan decomposition of a simple complex Lie algebra corresponding to the Chevalley involution. It is well known that among the set of primitive ideals with the infinitesimal character  $\frac{1}{2}\rho$ , there is a unique maximal primitive ideal  $J$ . Let  $\mathcal{Q} := U(\mathfrak{g})/J$ . Let  $K$  be a connected compact subgroup with Lie algebra  $\mathfrak{k}$  so that the notion of  $(\mathfrak{g}, K)$ -modules is well defined. In this paper we show that  $\mathcal{Q}^K$  is isomorphic to  $U(\mathfrak{k})^K$ . In particular  $\mathcal{Q}^K$  is commutative. A consequence of this result is that if  $W$  is an irreducible  $(\mathfrak{g}, K)$ -module annihilated by  $J$ , then  $W$  is  $K$ -multiplicity free and two such irreducible  $(\mathfrak{g}, K)$ -modules with a common nonzero  $K$ -type are isomorphic.

## 1. STATEMENT OF RESULTS

Let  $\mathfrak{g}$  be a simple complex Lie algebra,  $\mathfrak{a}$  a maximal Cartan subalgebra and  $\Phi$  the corresponding root system. Henceforth we fix a choice of positive roots  $\Phi^+ \subseteq \Phi$ . Let  $\theta$  be the Chevalley involution on  $\mathfrak{g}$ , and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$  the corresponding Cartan decomposition. Let  $K$  be a connected compact Lie group with the complex Lie algebra  $\mathfrak{k}$  such that the adjoint action of  $\mathfrak{k}$  on  $\mathfrak{s}$  can be exponentiated to  $K$ . Then the notion of  $(\mathfrak{g}, K)$ -modules is well defined.

Let  $U(\mathfrak{g})$  be the universal enveloping algebra of  $\mathfrak{g}$  and  $Z(\mathfrak{g})$  its center. Let  $\chi(\lambda) : Z(\mathfrak{g}) \rightarrow \mathbb{C}$  be the Harish-Chandra homomorphism corresponding to  $\lambda \in \mathfrak{a}^*$ . Let  $J_{\chi(\lambda)}$  be the two sided ideal of  $U(\mathfrak{g})$  generated by  $\ker \chi(\lambda)$ . Suppose  $\lambda$  is dominant weight with respect to  $\Phi^+$ . Let  $\chi = \chi(\lambda)$ . We define

$$(1) \quad \Omega(\chi) = \{J \subset U(\mathfrak{g}) : J \text{ is a two sided ideal and } J \supseteq J_{\chi(\lambda)}\}.$$

This is a partially ordered set under the usual set inclusions. A result of Dixmier states that  $\Omega(\chi)$  has an unique maximal element  $J_\lambda$ . We recall that a two-sided ideal of  $U(\mathfrak{g})$  is called primitive if it is the annihilator ideal of an irreducible (left)  $U(\mathfrak{g})$ -module. The ideal  $J_\lambda$  is a primitive ideal and thus it is the *maximal primitive ideal* of the infinitesimal character  $\chi(\lambda)$ . Let  $\rho$  be the half sum of the positive roots. We fix  $\chi = \chi(\frac{1}{2}\rho)$  and we let  $J = J_{\frac{1}{2}\rho}$  be the corresponding maximal primitive ideal. Let  $\mathcal{Q} = U(\mathfrak{g})/J$ .

A vector space  $V$  is a  $U(\mathfrak{g})$ -bimodule if it is both a left and a right  $U(\mathfrak{g})$ -module. If that is the case, then  $\mathfrak{g}$  acts on  $V$  by the adjoint action defined by  $\text{ad}(X)v = Xv - vX$ , where  $v \in V$ . Since  $\mathcal{Q}$  is a  $U(\mathfrak{g})$ -bimodule, we have an adjoint action of  $\mathfrak{g}$  on  $\mathcal{Q}$ . Let  $V_\mu$  denote a finite dimensional representation of  $\mathfrak{g}$  with the highest weight  $\mu$ . Let  $V_\mu^*$  denote its dual representation. Its highest weight is  $-w_0\mu$  where  $w_0$  is the longest element in the Weyl group of  $\mathfrak{g}$ . Let  $\Lambda_r$  denote the root lattice of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ . We can now state Theorem 2.1 in [Mc].

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**Theorem 1.1.** *Let  $\Lambda_r^d$  denote the set of dominant weights in the root lattice such that  $V_\mu \simeq V_\mu^*$ . Then, under the adjoint action of  $\mathfrak{g}$ ,  $\mathcal{Q}$  is multiplicity free and  $\mathcal{Q} = \bigoplus_{\mu \in \Lambda_r^d} V_\mu$ .  $\square$*

The inclusion  $\mathfrak{k} \subset \mathfrak{g}$  induces an algebra homomorphism  $\tilde{t} : U(\mathfrak{k}) \rightarrow \mathcal{Q} = U(\mathfrak{g})/J$ . Let  $U(\mathfrak{k})^K$  and  $\mathcal{Q}^K$  be the subalgebras of  $K$ -invariant elements. Note that  $U(\mathfrak{k})^K$  is simply the center of  $U(\mathfrak{k})$ .

**Theorem 1.2.** *Suppose  $\mathfrak{g}$  is a complex simple Lie algebra. Let  $\mathcal{Q}$  and  $\tilde{t}$  as above. Then  $\tilde{t}$  induces an algebra isomorphism*

$$t : U(\mathfrak{k})^K \rightarrow \mathcal{Q}^K = U(\mathfrak{g})^K/J^K.$$

*In particular  $\mathcal{Q}^K$  is commutative.*

We state an important application of the above theorem.

**Corollary 1.3.** *Let  $W_1$  be an irreducible Harish-Chandra  $(\mathfrak{g}, K)$ -module whose annihilator ideal is equal to  $J$ .*

- (i) *Then  $W_1$  is  $K$ -multiplicity free.*
- (ii) *The set of highest weight  $\mu$  of the  $K$ -types in  $W_1$  is Zariski dense in  $\Lambda_r \otimes \mathbb{C}$ .*
- (iii) *Suppose  $W_2$  is another irreducible Harish-Chandra  $(\mathfrak{g}, K)$ -module whose annihilator ideal is  $J$ . If  $W_1$  and  $W_2$  have a common  $K$ -type, then  $W_1$  and  $W_2$  are isomorphic  $(\mathfrak{g}, K)$ -modules.*

*Proof.* Let  $\tau$  be an irreducible  $K$ -module. For  $i = 1, 2$ , we set  $W_i(\tau) := \text{Hom}_K(\tau, W_i)$ . It is finite dimensional because  $W_i$  is irreducible and hence  $K$ -admissible. The action of  $\mathcal{Q}$  on  $W_i$  induces an action of  $\mathcal{Q}^K$  on  $W_i(\tau)$ . It is a result of Harish-Chandra that  $W_i(\tau)$  is an irreducible  $\mathcal{Q}^K$ -module. Furthermore if both  $W_1(\tau)$  and  $W_2(\tau)$  are nonzero isomorphic  $\mathcal{Q}^K$ -modules, then  $W_1$  and  $W_2$  are isomorphic Harish-Chandra modules. See Proposition 3.5.4 in [Wa].

By Theorem 1.2,  $\mathcal{Q}^K$  is commutative so  $\dim W_1(\tau) = 1$ . This proves (i). Suppose  $\tau$  is a common  $K$ -type as in (iii). The ring  $U(\mathfrak{k})^K \simeq \mathcal{Q}^K$  acts through the infinitesimal character of  $\tau$  on  $W_i(\tau)$ . Hence  $W_1(\tau)$  and  $W_2(\tau)$  are isomorphic  $\mathcal{Q}^K$ -modules so  $W_1$  and  $W_2$  are isomorphic Harish-Chandra modules. This proves (iii).

By the Harish-Chandra homomorphism,  $U(\mathfrak{k})^K \simeq \text{Sym}(\mathfrak{t})^{W_K}$  can be interpreted as the algebra of Weyl group invariant polynomials on  $\Lambda_r \otimes \mathbb{C}$ . Suppose the set of highest weights of the  $K$ -types of  $W_1$  lies in some algebraic subset  $S$ . Then there exists  $f \in \text{Sym}(\mathfrak{t})^{W_K}$  which vanishes on  $S + \rho_K$ . This implies that  $U(\mathfrak{k})^K$  does not act faithfully on the  $K$ -types of  $W_1$ . However this contradicts Theorem 1.2 and proves (ii).  $\square$

In this paper we do not describe modules annihilated by  $J$ . This remains an interesting and challenging problem, solved completely only for  $\mathfrak{sl}_n$  by Lucas [Lu]. For example, if  $\mathfrak{g} = \mathfrak{sl}_2$ , then the irreducible modules annihilated by  $J$  are precisely the components (even and odd) of two Weil representations, four representations in all. Analogs of the even Weil representations in the simply laced case are constructed as follows. Let  $\underline{G}$  be the simply connected algebraic group defined over  $\mathbb{Q}$ , corresponding to  $\mathfrak{g}$ . Let  $G$  be a non-linear 2-fold central extension of  $\underline{G}(\mathbb{R})$ . Now  $K$  is the maximal compact subgroup of  $G$ . Let  $B = MAN$  be a Borel subgroup of  $G$  corresponding to our choice of positive roots. Let  $\delta$  be a pseudo-spherical  $K$ -type as in [A-V]. Then  $\delta$ , restricted to  $M$ , is an irreducible representation of

$M$ . Let  $\Theta$  be the irreducible pseudo-spherical Langlands quotient of the normalized induced principal series representation  $\text{Ind}_B^G(\delta, \frac{1}{2}\rho)$ . The representation  $\Theta$  exhibits many interesting properties and we list some of them.

- The annihilator ideal of  $\Theta$  is  $J$ . One deduces this by proving that  $\Theta$  has maximal  $\tau$ -invariant, using Theorem 4.12 in [V1].
- The representation  $\Theta$  is the local Shimura lift of a trivial representation [A-V].
- The representation  $\Theta$  appears as a local component of an automorphic representation constructed via the residues of Eisenstein series [LS]. In particular,  $\Theta$  is unitarizable.

The representation  $\Theta$  was also studied in [AHV], [H], [Kn], [T] and by Wallach (unpublished). For example, Adams, Huang and Vogan [AHV] give a conjectural description of  $K$ -types for the Lie type  $E_8$ . In particular, they predict that the types are multiplicity free. Corollary 1.3 proves that.

In the last section we generalize Corollary 1.3 to primitive ideals whose associated varieties are contained in the boundary of the model orbit. We would like to thank a referee for providing a crucial information here.

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## 2. INJECTIVITY IN THEOREM 1.2

The main result of this section is to prove that the homomorphism  $t$  in Theorem 1.2 is injective. We first set up some notation. We denote the root space corresponding to a root  $\alpha$  by  $\mathfrak{g}_\alpha$ . For every root  $\alpha$  we pick a nonzero vector  $X_\alpha$  in the root space  $\mathfrak{g}_\alpha$  so that  $X_\alpha$  are a part of a Chevalley basis. Let  $H_\alpha = [X_\alpha, X_{-\alpha}]$ . The Chevalley involution is defined by

$$\theta(X_\alpha) = -X_{-\alpha} \text{ and } \theta(H_\alpha) = -H_\alpha.$$

**The model orbit.** We now introduce the model orbit. Let  $U_v$  be the standard filtration of  $U(\mathfrak{g})$ . Let  $J_v = J \cap U_v$ . Then  $\text{Gr}(J) = \bigoplus_{v=1}^{\infty} (J_v/J_{v-1})$  is an ideal of the symmetric algebra  $\text{Sym}(\mathfrak{g}) = \bigoplus_{v=0}^{\infty} \text{Sym}^v(\mathfrak{g})$ . Using the Killing form, we identify  $\mathfrak{g}^*$  and  $\mathfrak{g}$ . This gives an identification of  $\text{Sym}(\mathfrak{g})$  with the algebra of polynomials on  $\mathfrak{g}$ . Let  $\mathcal{V} \subseteq \mathfrak{g}$  be the variety defined by the ideal  $\text{Gr}(J)$ . By [BB] and Corollary 4.7 in [V2],  $\mathcal{V}$  contains a dense open orbit denoted by  $\mathcal{O}$  which McGovern calls the model orbit [Mc]. A general description of  $\mathcal{O}$  is as follows. Fix  $\Delta$  a set of simple roots. Let  $S$  be a maximal subset of  $\Delta$  consisting of mutually orthogonal roots, such that  $S$  contains at least one short root. Then  $\mathcal{O}$  is the adjoint orbit of

$$(2) \quad X_S = \sum_{\alpha \in S} X_\alpha$$

and it does not depend on the choices made.

Suppose  $\mathfrak{g}$  is a classical Lie algebra. Then nilpotent orbits are classified by partitions. In the following table we give the partition corresponding to the model orbit  $\mathcal{O}$  for each classical Lie algebra.

$\mathfrak{g}$	$A_{2k-1}$	$A_{2k}$	$B_{2k}$	$B_{2k+1}$	$C_n$	$D_{2k}$	$D_{2k+1}$
$\mathcal{O}$	$[2^k]$	$[2^k, 1]$	$[3, 2^{2k-2}, 1^2]$	$[3, 2^{2k}]$	$[2^n]$	$[3, 2^{2k-2}, 1]$	$[3, 2^{2k-2}, 1^3]$

Let  $I = \bigoplus_{v=1}^{\infty} I^v$  be the graded ideal in  $\text{Sym}(\mathfrak{g})$  consisting of all polynomials vanishing on  $\mathcal{O}$ . Note that  $J_v/J_{v-1} \subseteq I^v$  for all  $v$ .

First we state a technical lemma. We recall that  $\theta$  is the Cartan involution on  $\mathfrak{g}$ . Let  $\text{pr} : \mathfrak{g} \rightarrow \mathfrak{k}$  be the projection given by  $\text{pr}(X) = \frac{1}{2}(X + \theta(X))$  for every  $X \in \mathfrak{g}$ .

**Lemma 2.1.** *The image  $\text{pr}(\mathcal{O})$  of the model orbit contains an open subset of  $\mathfrak{k}$ .*

We will postpone the proof of the lemma after Corollary 2.3. We will prove the following proposition using the above lemma.

**Proposition 2.2.** *We have  $U(\mathfrak{k}) \cap J = 0$ . In particular, the algebra homomorphism  $t$  in Theorem 1.2 is an injection.*

*Proof.* We shall first show that  $\text{Sym}(\mathfrak{k}) \cap I = 0$ . Let  $p$  be a non-zero element in  $\text{Sym}(\mathfrak{k})$ . Note that

$$p(X) = p\left(\frac{1}{2}(X + \theta(X))\right)$$

for every  $X \in \mathfrak{g}$ . Since  $p \neq 0$  on a Zariski open subset of  $\mathfrak{k}$ , it follows from Lemma 2.1 that there is an element  $X$  in the model orbit such that  $p(X) \neq 0$ . Thus  $\text{Sym}(\mathfrak{k}) \cap I = 0$ .

Now let  $p$  be a non-zero element in  $U(\mathfrak{k})$ . Suppose that  $p \in U_v$  but  $p \notin U_{v-1}$ . We must show that  $p \notin J_v$ . Let  $\bar{p}$  denote its image in  $U_v/U_{v-1} \simeq \text{Sym}^v(\mathfrak{g})$ . If  $p \in J_v$ , then  $\bar{p} \in J_v/J_{v-1} \subseteq I^v$ , a contradiction since we have already proved that  $\text{Sym}(\mathfrak{k}) \cap I = 0$ .  $\square$

**Corollary 2.3.** *Let  $\mathcal{Q}_v = U_v(\mathfrak{g})/J_v$ . We have*

$$\dim U_v(\mathfrak{k})^K \leq \dim \mathcal{Q}_v^K.$$

*Proof.* Obvious from Proposition 2.2.  $\square$

We now proceed to the proof of Lemma 2.1. Let  $S$  be a maximal subset of orthogonal simple roots as in (2). We set  $\bar{S} = \Delta \setminus S$  so that  $\Delta = S \cup \bar{S}$  is the set of simple roots. We define

$$X^+ = \sum_{\alpha \in S} X_{\alpha} - \sum_{\alpha \in \bar{S}} X_{-\alpha}.$$

Let  $\mathfrak{g}_{\alpha}^{\times} = \mathfrak{g}_{\alpha} \setminus \{0\}$ . Let  $S^+ = S \cup (-\bar{S})$ . Let  $\mathfrak{c} := \prod_{\alpha \in S} \mathfrak{g}_{\alpha}^{\times}$  and let  $\mathfrak{c}^+ := \prod_{\alpha \in S^+} \mathfrak{g}_{\alpha}^{\times}$ . Clearly,  $\mathcal{O}$  contains  $\mathfrak{c}$  and  $X^+$  is contained in  $\mathfrak{c}^+$ .

**Lemma 2.4.** *The model orbit  $\mathcal{O}$  contains  $\mathfrak{c}^+$ .*

*Proof.* The proof is a case by case inspection. We will only prove this when  $\mathfrak{g}$  is of type  $D_n$ , with  $n$  even. In the standard realization of this root system (Plate IX in [B]),

$$S = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$$

and

$$S^+ = S \cup \{-\varepsilon_2 + \varepsilon_3, -\varepsilon_4 + \varepsilon_5, \dots, -\varepsilon_{n-2} + \varepsilon_{n-1}\}.$$

The adjoint actions of  $\exp(\text{ad}_{\mathfrak{g}_{-\varepsilon_2+\varepsilon_4}})$ ,  $\exp(\text{ad}_{\mathfrak{g}_{-\varepsilon_4+\varepsilon_6}})$ ,  $\dots$ ,  $\exp(\text{ad}_{\mathfrak{g}_{-\varepsilon_{n-2}+\varepsilon_n}})$  on  $\mathfrak{c}$  show that  $\mathfrak{c}^+$  is contained in  $\mathcal{O}$ . Other cases are treated in the same manner.  $\square$

**Proposition 2.5.** *Let  $X_0 = \text{pr}(\sum_{\alpha \in \Delta} X_{\alpha})$ . Then  $\mathfrak{k} = [\mathfrak{k}, X_0] + \text{pr}(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha})$ .*

We will first deduce Lemma 2.1 from this proposition. Let  $C^+ = \text{pr}(\mathfrak{c}^+) \subseteq \text{pr}(\mathcal{O})$ . Note that  $X_0 = \text{pr}(X^+)$  and

$$\text{pr}(\bigoplus_{\alpha \in S^+} \mathfrak{g}_\alpha) = \text{pr}(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha).$$

It follows that  $X_0$  is in  $C^+$  and  $\text{pr}(\sum_{\alpha \in \Delta} \mathfrak{g}_\alpha)$  is the tangent space of  $C^+$  at  $X_0$ . Let  $K_c$  be the complexification of  $K$ . Proposition 2.5 and the inverse function theorem imply that  $\text{Ad}_{K_c}(C^+)$  contains an open neighborhood  $U$  of  $X_0$  in  $\mathfrak{k}$ . Since  $\text{pr}(\mathcal{O})$  contains  $\text{Ad}_{K_c}(C^+)$ , we obtain Lemma 2.1.

The rest of this section is devoted to the proof of Proposition 2.5.

If  $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$  is a sum of simple roots, we set  $\text{ht}(\beta) = \sum_{\alpha \in \Delta} m_\alpha \in \mathbb{Z}$ . We also set  $\mathfrak{k}_\beta := \text{pr}(\mathfrak{g}_\beta) \subset \mathfrak{k}$ . For an integer  $m$ , we set  $\mathfrak{g}(m) = \bigoplus_{\text{ht}(\beta)=m} \mathfrak{g}_\beta$ . We set  $\mathfrak{g}(0)$  to be the Cartan subalgebra  $\mathfrak{a}$ . For a positive integer  $m$ , we set  $\mathfrak{k}_m = \bigoplus_{1 \leq \text{ht}(\beta) \leq m} \mathfrak{k}_\beta = \bigoplus_{j=1}^m \text{pr}(\mathfrak{g}(j))$ . For positive integers  $m$  and  $n$ , we have  $[\mathfrak{k}_m, \mathfrak{k}_n] \subseteq \mathfrak{k}_{m+n}$ .

Write  $X_p = \sum_{\alpha \in \Delta} X_\alpha$ . The nilpotent element  $X_p$  generates the principal nilpotent orbit on  $\mathfrak{g}$ . Let  $(X_p, Y_p, H_p)$  denote an  $\mathfrak{sl}_2$  triple where  $H_p = \sum_{\alpha \in \Phi^+} H_\alpha$ . Thus  $[H_p, X_\beta] = 2\text{ht}(\beta)X_\beta$  for all positive roots  $\beta$ . Then  $\mathfrak{g}(m)$  is the  $\text{ad}_{H_p}$ -eigenspace with the eigenvalue  $2m$  and  $\mathfrak{g} = \bigoplus_m \mathfrak{g}(m)$ . It follows from the representation theory of  $\mathfrak{sl}_2$  that for a positive integer  $m$ , we have

$$(3) \quad \mathfrak{g}(m+1) = [\mathfrak{g}(m), X_p].$$

**Lemma 2.6.** *We have  $\mathfrak{k}_{m+1} = [\mathfrak{k}_m, X_0] + \mathfrak{k}_m$ .*

*Proof.* Since  $X_0 \in \mathfrak{k}_1$ , the left hand side contains the right hand side. On the other hand, suppose  $\text{ht}(\beta) = m$ . Then

$$\begin{aligned} [\text{pr} X_\beta, X_0] &= \sum_{\alpha \in \Delta} [X_\beta - X_{-\beta}, X_\alpha - X_{-\alpha}] \\ &= \sum_{\alpha \in \Delta} ([X_\beta, X_\alpha] + [X_{-\beta}, X_{-\alpha}]) - ([X_\beta, X_{-\alpha}] + [X_{-\beta}, X_\alpha]) \\ &\equiv \text{pr} \sum_{\alpha \in \Delta} [X_\beta, X_\alpha] \pmod{\mathfrak{k}_m} \\ &\equiv \text{pr}[X_\beta, X_p] \pmod{\mathfrak{k}_m}. \end{aligned}$$

This implies that  $[\text{pr}(\mathfrak{g}(m)), X_0] + \mathfrak{k}_m$  contains  $\text{pr}[\mathfrak{g}(m), X_p] = \text{pr}(\mathfrak{g}(m+1))$  by (3). Hence  $[\text{pr}(\mathfrak{g}(m)), X_0] + \mathfrak{k}_m$  contains  $\mathfrak{k}_{m+1}$ . This proves our lemma.  $\square$

Using the last lemma, it follows easily by induction on  $m$  that  $\mathfrak{k}_{m+1} = [\mathfrak{k}_m, X_0] + \mathfrak{k}_1$  for all positive integers  $m$ . Since  $\mathfrak{k}_m = \mathfrak{k}$  for sufficient large  $m$ , we get  $\mathfrak{k} = [\mathfrak{k}, X_0] + \mathfrak{k}_1$ . This gives Proposition 2.5 since  $\text{pr}(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) = \mathfrak{k}_1$ .

We note the following interesting corollary to Proposition 2.5 which will not be used in the paper.

**Corollary 2.7.** *Assume that the rank of  $\mathfrak{k}$  is equal to the rank of  $\mathfrak{g}$ . Then  $X_0 = \text{pr}(\sum_{\alpha \in \Delta} X_\alpha)$  is a regular element in  $\mathfrak{k}$ .*

*Proof.* Proposition 2.5 implies that  $\dim([\mathfrak{k}, X_0]) \geq \dim \mathfrak{k} - \text{rank}(\mathfrak{k})$ , and this implies that the dimension of the centralizer of  $X_0$  in  $\mathfrak{k}$  is not more than the rank of  $\mathfrak{k}$ . However the dimension of a centralizer of any element is at least the rank, and it is the rank if and only if the element is regular.  $\square$

### 3. SURJECTIVITY IN THEOREM 1.2

We have shown in the last section that the homomorphism  $t$  in Theorem 1.2 is an injection. In order to complete the proof of Theorem 1.2 it suffices to show that the inequality in Corollary 2.3 is an equality. To that end, we shall first compute the dimension of  $U_v(\mathfrak{k})^K$  or, equivalently, the dimension of  $\text{Sym}_v(\mathfrak{k})^K$  where  $\text{Sym}_v(\mathfrak{k}) = \bigoplus_{i=0}^v \text{Sym}^i(\mathfrak{k})$ . Let  $\mathfrak{t}$  be a Cartan subalgebra of  $\mathfrak{k}$ . Using the Killing form we identify  $\text{Sym}(\mathfrak{t})$  with the algebra of polynomials on  $\mathfrak{t}$ . Let  $W_K$  be the Weyl group for the pair  $(\mathfrak{k}, \mathfrak{t})$ . By a result of Chevalley, restricting polynomial functions from  $\mathfrak{k}$  to  $\mathfrak{t}$  gives an isomorphism

$$\text{Sym}(\mathfrak{k})^K \cong \text{Sym}(\mathfrak{t})^{W_K}.$$

Let  $m = \dim \mathfrak{t}$ . It is well known that  $\text{Sym}(\mathfrak{t})^{W_K}$  is generated by  $m$  algebraically independent homogeneous polynomials on  $\mathfrak{t}$  of degrees  $d_1, \dots, d_m$  respectively. Here  $d_i$  are the degrees of the Weyl group  $W_K$  (See Section 3.7 in [Hu]). We will list them in Table 1 below.

TABLE 1

$\Phi$	$\mathfrak{k}$	$m = \dim \mathfrak{t}$	$d_1, \dots, d_m$
$A_{2k-1}$	$\mathfrak{so}(2k)$	$k$	$2, 4, 6, \dots, 2k-2, k$
$A_{2k}$	$\mathfrak{so}(2k+1)$	$k$	$2, 4, 6, \dots, 2k$
$B_{2k}$	$\mathfrak{so}(2k) \oplus \mathfrak{so}(2k+1)$	$2k$	$2, 2, 4, 4, \dots, 2k-2, 2k-2, 2k, k$
$B_{2k+1}$	$\mathfrak{so}(2k+1) \oplus \mathfrak{so}(2k+2)$	$2k+1$	$2, 2, 4, 4, \dots, 2k, 2k, k+1$
$C_m$	$\mathfrak{gl}(m)$	$m$	$1, 2, 3, \dots, m$
$D_{2k}$	$\mathfrak{so}(2k) \oplus \mathfrak{so}(2k)$	$2k$	$2, 2, 4, 4, \dots, 2k-2, 2k-2, k, k$
$D_{2k+1}$	$\mathfrak{so}(2k+1) \oplus \mathfrak{so}(2k+1)$	$2k$	$2, 2, 4, 4, \dots, 2k, 2k$
$E_6$	$\mathfrak{sp}(8)$	4	2, 4, 6, 8
$E_7$	$\mathfrak{sl}(8)$	7	2, 3, 4, 5, 6, 7, 8
$E_8$	$\mathfrak{so}(16)$	8	2, 4, 6, 8, 8, 10, 12, 14
$F_4$	$\mathfrak{sl}(2) \oplus \mathfrak{sp}(6)$	4	2, 2, 4, 6
$G_2$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	2	2, 2

Let  $s_v$  and  $t_v$  denote the cardinalities of the sets

$$(4) \quad \begin{aligned} S_v &= \{(n_1, n_2, \dots, n_m) \in \mathbb{Z}^m : n_i \geq 0 \text{ and } v \leq \sum_{i=1}^m d_i n_i\} \text{ and} \\ T_v &= \{(n_1, n_2, \dots, n_m) \in \mathbb{Z}^m : n_i \geq 0 \text{ and } v = \sum_{i=1}^m d_i n_i\} \end{aligned}$$

respectively. Then  $\dim \text{Sym}^v(\mathfrak{k})^K = t_v = s_v - s_{v-1}$  and  $\dim \text{Sym}_v(\mathfrak{k})^K = s_v$ . This gives  $\dim U_v(\mathfrak{k})^K = s_v$ .

We need to show that  $\dim Q_v^K = s_v$ . Let  $V_\mu$  be as in Theorem 1.1 and let  $B = MAN$  be the Borel subgroup. By the Cartan-Helgason theorem,  $\dim V_\mu^K \leq 1$ , and it is one if and only if  $M$  acts trivially on the highest weight vector of  $V_\mu$ . Let  $n = \dim \mathfrak{a}$  denote the rank of  $\mathfrak{g}$ . We will follow the enumeration and notation of Plate IX in [B]. Let  $\varpi_i$  denote the  $i$ -th fundamental weight. With the help of the table on page 587 in [GW], we see that  $V_\mu$  contains

a  $K$ -fixed vector if and only if  $\mu = n_1w_1 + n_2w_2 + \dots + n_mw_m$  where  $n_i$  is a nonnegative integer and  $w_1, \dots, w_m$  are linearly independent weights listed in Table 2 below. Note that the number of weights  $w_i$ 's is  $m = \dim \mathfrak{t}$ . The last column of Table 2 lists  $\mu$  more explicitly using the standard notation for the classical Lie algebras. The  $a_i$ 's in the table are integers.

TABLE 2

$\Phi$	$m$	$w_1, w_2, \dots, w_m$	$\mu = n_1w_1 + \dots + n_mw_m$
$A_{2k-1}$	$k$	$2(\varpi_1 + \varpi_{2k}), 2(\varpi_2 + \varpi_{2k-1}), \dots,$ $2(\varpi_k + \varpi_{k+1}), 2\varpi_k$	$\mu = \sum_{i=1}^{n+1} a_i \varepsilon_i,$ $a_i + a_{n+2-i} = 0,$ $a_i \equiv a_j \pmod{2}.$
$A_{2k}$	$k$	$2(\varpi_1 + \varpi_{2k-1}), 2(\varpi_2 + \varpi_{2k-2}), \dots,$ $2(\varpi_k + \varpi_{k+1})$	Same as above and $a_k = 0.$
$B_m$	$m = n$	$2\varpi_1, 2\varpi_2, \dots, 2\varpi_m$	$\mu = \sum_{i=1}^m a_i \varepsilon_i, a_i \geq 0,$ $a_i \equiv a_j \pmod{2}.$
$C_m$	$m = n$	$2\varpi_1, 2\varpi_2, \dots, 2\varpi_m$	$\mu = \sum_{i=1}^m 2a_i \varepsilon_i, a_i \geq 0$
$D_{2k}$	$2k$	$2\varpi_1, 2\varpi_2, \dots, 2\varpi_{2k}$	$\mu = \sum_{i=1}^{2k} a_i \varepsilon_i,$ $a_i \geq 0$ for $i \neq 2k,$ $a_i \equiv a_j \pmod{2}.$
$D_{2k+1}$	$2k$	$2\varpi_1, 2\varpi_2, \dots, 2\varpi_{2k-1}, 2\varpi_{2k} + 2\varpi_{2k+1}$	$\mu = \sum_{i=1}^{2k} 2a_i \varepsilon_i, a_i \geq 0.$
$E_6$	4	$2\varpi_2, 2(\varpi_1 + \varpi_6), 2\varpi_4, 2(\varpi_3 + \varpi_5)$	
$E_7$	7	$2\varpi_1, 2\varpi_7, 2\varpi_6, 2\varpi_2, 2\varpi_3, 2\varpi_5, 2\varpi_4$	
$E_8$	8	$2\varpi_8, 2\varpi_1, 2\varpi_7, 2\varpi_2, 2\varpi_6, 2\varpi_3,$ $2\varpi_5, 2\varpi_4$	
$F_4$	4	$2\varpi_1, 2\varpi_4, 2\varpi_3, 2\varpi_2$	
$G_2$	2	$2\varpi_1, 2\varpi_2$	

**Lemma 3.1.** *Suppose  $\mathfrak{g}$  is a simple Lie algebra. Let  $\mu = n_1w_1 + \dots + n_mw_m$  as in Table 2. If  $V_\mu$  is a  $\mathfrak{g}$ -submodule in  $\text{Sym}^v(\mathfrak{g})$ , then  $v \geq n_1d_1 + n_2d_2 + \dots + n_md_m$ .*

We begin the proof of Lemma 3.1. First we review Section 4 of [Jo]. Let  $\mathfrak{g}$  be a simple Lie algebra. Let  $\tilde{\alpha}$  be the highest root in the positive root system  $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{a})$ . Let  $\Phi_{\perp \tilde{\alpha}}^+$  denote the subset of all positive roots that is orthogonal to  $\tilde{\alpha}$ . We may write  $\Phi_{\perp \tilde{\alpha}}^+ = \bigcup_i \Phi_i^+$  as a disjoint union of simple root systems. We pick the highest root  $\tilde{\alpha}_i$  from each  $\Phi_i^+$ . This process is repeated over and over again and it has to stop eventually. Then all the highest roots would form a maximal set  $\mathcal{K}$  of strong orthogonal positive roots in  $\Phi^+$ . The set  $\mathcal{K}$  is given in Tables I and II in [Jo] for every simple root system. It is a well known fact that the cardinality of  $\mathcal{K}$  is equal to the rank of  $\mathfrak{k}$ .

Let  $\mu$  be a dominant weight in the root lattice of  $\Phi^+$ . Let  $(|)$  denote the inner product on the root lattice. We define

$$|\mu|_{\mathcal{K}} = \sum_{\beta \in \mathcal{K}} \frac{(\mu|\beta)}{(\beta|\beta)}.$$

We note that  $|\mu + \mu'|_{\mathcal{K}} = |\mu|_{\mathcal{K}} + |\mu'|_{\mathcal{K}}$ . We can now state Lemma 4.13(iii) in [Jo].

**Lemma 3.2.** *Suppose  $V_\mu$  is a  $\mathfrak{g}$ -submodule in  $\text{Sym}^v(\mathfrak{g})$ . Then  $v \geq |\mu|_{\mathcal{K}}$ . □*

We will compute  $|\mu|_{\mathcal{K}}$  explicitly. We write the weight  $\mu = \sum_{i=1}^r a_i \varepsilon_i$  for  $a_i \in \mathbb{R}$  as in [B]. We also refer to  $\mu$  in the last column in Table 2 for classical  $\mathfrak{g}$ . We list  $r$  and  $|\mu|_{\mathcal{K}}$  in Table 3 below.

Table 3

$\Phi$	$r$	$ \mu _{\mathcal{K}}$
$A_{2k-1}$	$2k$	$\frac{1}{2}(a_1 + a_2 + \dots + a_k - a_{k+1} - a_{k+2} - \dots - a_{2k})$
$A_{2k}$	$2k + 1$	$\frac{1}{2}(a_1 + a_2 + \dots + a_k - a_{k+2} - a_{k+3} - \dots - a_{2k+1})$
$B_{2k}$	$2k$	$a_1 + a_3 + \dots + a_{2k-1}$
$B_{2k+1}$	$2k + 1$	$a_1 + a_3 + \dots + a_{2k+1}$
$C_m$	$m$	$\frac{1}{2}(a_1 + a_2 + \dots + a_m)$
$D_{2k}$	$2k$	$a_1 + a_3 + \dots + a_{2k-1}$
$D_{2k+1}$	$2k + 1$	$a_1 + a_3 + \dots + a_{2k-1}$
$E_6$	8	$\frac{1}{2}(a_8 - a_7 - a_6 + a_5 + a_4 + a_3 - a_2 - a_1)$
$E_7$	8	$\frac{1}{2}(a_8 - a_7) + a_6 + a_4 + a_2$
$E_8$	8	$a_8 + a_6 + a_4 + a_2$
$F_4$	4	$a_1 + a_3$
$G_2$	3	$\frac{1}{3}(a_1 - 2a_2 + a_3)$

Using the above table, it is easy to check that  $|w_i|_{\mathcal{K}} = d_i$  where  $d_i$  is the degree in Table 1 and  $w_i$  is the weight in Table 2. Suppose  $\mu = \sum_i^m n_i w_i$  as in Lemma 3.1. Since  $|w|_{\mathcal{K}}$  is linear in  $w$ , we have  $|\mu|_{\mathcal{K}} = \sum_{i=1}^m n_i |w_i|_{\mathcal{K}} = \sum_{i=1}^m n_i d_i$ . Now Lemma 3.1 follows immediately from Lemma 3.2.

We recall  $s_v = \dim U_v(\mathfrak{k})^K$  and  $t_v = s_v - s_{v-1}$  in (4).

**Lemma 3.3.** *Let  $\mu = n_1 w_1 + \dots + n_m w_m$  be a highest weight of a  $K$ -spherical  $\mathfrak{g}$ -module, as in Table 2 above. Then  $V_\mu$  is a  $\mathfrak{g}$ -submodule in  $\mathcal{Q}_v/\mathcal{Q}_{v-1}$  where  $v = n_1 d_1 + \dots + n_m d_m$ , and  $\dim \mathcal{Q}_v^K = s_v$  for all  $v$ .*

*Proof.* Let  $V_\mu$  be a spherical  $\mathfrak{g}$ -submodule  $V_\mu$  in  $\mathcal{Q}_v/\mathcal{Q}_{v-1}$ . Write  $\mu = n_1 w_1 + \dots + n_m w_m$  as in Table 2. Since  $\mathcal{Q}_v/\mathcal{Q}_{v-1}$  is a quotient of  $\text{Sym}^v(\mathfrak{g})$ , Lemma 3.1 implies that  $v \geq n_1 d_1 + n_2 d_2 + \dots + n_m d_m$ . Since  $s_v \leq \dim \mathcal{Q}_v^K$  for all  $v$ ,  $V_\mu$  must occur in  $\mathcal{Q}_v/\mathcal{Q}_{v-1}$  where  $v = n_1 d_1 + \dots + n_m d_m$ . It follows that  $\dim(\mathcal{Q}_v/\mathcal{Q}_{v-1})^K = t_v$  and  $\dim \mathcal{Q}_v^K = s_v$ .  $\square$

This completes the proof of Theorem 1.2.

#### 4. THE BOUNDARIES OF THE MODEL ORBITS

In this section we will extend Theorem 1.2 to primitive ideals associated to varieties that lie in the boundary  $\partial \mathcal{O}_{\text{mod}}$  of the model orbit  $\mathcal{O}_{\text{mod}}$ . Let  $\mathcal{O}$  be an such an orbit in the boundary. We will assume that  $\mathcal{O}$  is not the zero orbit. Let  $R(\overline{\mathcal{O}})$  denote the algebra of regular functions on  $\overline{\mathcal{O}}$ , the Zariski closure of  $\mathcal{O}$ . It is well known that  $R(\mathcal{O})$  is the normalization of  $R(\overline{\mathcal{O}})$ ; for orbits contained in  $\mathcal{O}_{\text{mod}}$  it is also known that  $\overline{\mathcal{O}}$  is normal apart from the model orbit itself in type  $G_2$ . Even in this case it is still true that  $R(\overline{\mathcal{O}_{\text{mod}}})$  surjects onto  $R(\mathcal{O})$  for any orbit  $\mathcal{O}$  lying in the boundary of  $\mathcal{O}_{\text{mod}}$ .

Suppose we are given a semisimple  $\mathfrak{g}$ -module  $W = \bigoplus_i W_i$  where each  $W_i$  is an irreducible representation of  $\mathfrak{g}$ . Let  $W_{\text{sph}}$  denote the subrepresentation of  $W$  which is a direct sum of



all the  $K$ -spherical irreducible summands  $W_i$ . Let  $J$  denote a primitive ideal such that  $\text{Gr}J$  cuts out  $\overline{\mathcal{O}}$  in  $\mathfrak{g}^* \simeq \mathfrak{g}$ . In particular,  $R(\overline{\mathcal{O}}) = \text{Sym}(\mathfrak{g})/\text{Rad}(\text{Gr}(J))$  where  $\text{Rad}(\text{Gr}(J))$  denotes the radical ideal of  $\text{Gr}(J)$ .

**Proposition 4.1.** *Let  $\mathcal{O}$  be a nilpotent orbit in the boundary of the model orbit. Let  $J$  be a primitive such that  $\text{Gr}J$  cuts out  $\overline{\mathcal{O}}$ . Let  $\mathcal{Q} = U(\mathfrak{g})/J$ . Suppose  $(\mathcal{Q})_{\text{sph}} \simeq (R(\mathcal{O}))_{\text{sph}}$  as  $\mathfrak{g}$ -modules. Then the canonical map  $U(\mathfrak{k})^K \rightarrow \mathcal{Q}^K$  is a surjection and Corollary 1.3 applies to  $J$  too.*

*Proof.* Let  $R^v(\mathcal{O})$  denote the space of regular functions of degree  $v$ . Let  $J_{\text{mod}} = J_{\frac{1}{2}\rho}$  be the model ideal. Since  $R(\overline{\mathcal{O}}_{\text{mod}})$  is a quotient of  $\text{Sym}(\mathfrak{g})/\text{Gr}(J_{\text{mod}})$ , our main result shows that the natural map  $(\text{Sym}^v(\mathfrak{k}))^K \rightarrow R^v(\overline{\mathcal{O}}_{\text{mod}})^K$  is surjective. Since  $R(\mathcal{O})$  is a quotient of  $R(\overline{\mathcal{O}}_{\text{mod}})$ , we have surjections

$$U_v(\mathfrak{k})^K \rightarrow (\text{Sym}^v(\mathfrak{k}))^K \rightarrow R^v(\mathcal{O})^K.$$

The composite surjection factors through

$$U_v(\mathfrak{k})^K/U_{v-1}(\mathfrak{k})^K \xrightarrow{p} \mathcal{Q}_v^K/\mathcal{Q}_{v-1}^K \xrightarrow{q} R^v(\mathcal{O})^K.$$

By the assumption the second map  $q$  is bijective. We conclude that the first map  $p$  is a surjection. By induction on  $v$ , we conclude that  $U_v(\mathfrak{k})^K \rightarrow \mathcal{Q}_v^K$  is surjective. This proves the proposition.  $\square$

There are many known primitive ideals satisfying the above proposition, for example, the Joseph ideal and the ideals for classical Lie algebras constructed by Brylinski [Br].

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