ON LOCAL LIFTS FROM $G_2(\mathbb{R})$ TO $Sp_6(\mathbb{R})$ AND $F_4(\mathbb{R})$

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Abstract. Let $G_2(\mathbb{R}) \times Sp_6(\mathbb{R})$ and $G_2(\mathbb{R}) \times F_4(\mathbb{R})$ be split dual pairs in split $E_7(\mathbb{R})$ and $E_8(\mathbb{R})$, respectively. It is known that the exceptional correspondences for these dual pairs are functorial on the level of infinitesimal characters. In this paper we show that these dual pair correspondences are functorial for the minimal K-types of principal series representations.

1. Split real groups of type E_n

The Cartan decomposition for split real groups of type E_n can be described by Jordan algebras of rank 4, as it has been shown by Kostant and Brylinski in [3]. To this end, let $J = J_n(Q)$ be a Jordan algebra of $n \times n$-hermitian symmetric matrices over a composition algebra Q. To each Jordan algebra J one can attach a simple Lie algebra $\mathfrak{k} = \mathfrak{k}(J)$ with a short $\mathbb{Z}$-filtration

$$\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

such that $\mathfrak{k}_1 \cong J$. The algebra $\mathfrak{k}$ has n strongly orthogonal roots $\alpha_1, \ldots, \alpha_n$ corresponding to the diagonal entries of J. Let

$$\psi = \frac{1}{2}(\alpha_1 + \ldots + \alpha_n)$$

Of special interest to us is the case $n = 4$, in which case $\langle \psi, \psi \rangle = 2$. Let $\mathfrak{p}$ be the irreducible $\mathfrak{k}$-module of highest weight ψ. Then the exceptional lie algebras of type E_n have Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{p} \cong V_\psi$, as a $\mathfrak{k}$-module, and $\mathfrak{k} = \mathfrak{k}(J_4(Q))$ where Q is a composition algebra over $\mathbb{C}$ of dimension 1, 2 and 4 for E_6, E_7 and E_8, respectively. The minimal representation (the corresponding $(\mathfrak{g}, K)$-module) has K-types

$$V = \oplus_{i=0}^{\infty} V_{i\psi}$$

This $(\mathfrak{g}, K)$-module corresponds to a representation of the simply connected Chevalley group of type E_n. This representation is faithful except for E_7 when the center μ_2 acts trivially.

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The dual pairs. Simply connected Chevalley group $\mathrm{E}_n(\mathbb{R})$ contains a split dual pair $H \times \mathrm{G}_2$ where H is $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{Sp}_6(\mathbb{R})$ and $\mathrm{F}_4(\mathbb{R})$ respectively. Let K_1 and $K_2 = \mathrm{SU}_{2,l} \times_{\mu_2} \mathrm{SU}_{2,s}$ denote the maximal compact subgroup of H and $\mathrm{G}_{2,2}$ respectively. The two factors of K_2 correspond to a pair of perpendicular roots, one long and one short, as the subscripts indicate. The possible K and K_1 are tabulated below.

Split group	K	$\mathfrak{p} = V(\psi)$	H	K_1
E_6	Sp_8/μ_2	$V(\varpi_4) = \wedge^4\mathbb{C}^8 - \wedge^2\mathbb{C}^8$	$\mathrm{SL}_3(\mathbb{R})$	SO_3
E_7	SU_8/μ_2	$V(\varpi_4) = \wedge^4\mathbb{C}^8$	$\mathrm{Sp}_6(\mathbb{R})$	U_3
E_8	Spin_{16}/μ_2	$V(\varpi_8)$	F_4	$\mathrm{SU}_2 \times_{\mu_2} \mathrm{Sp}_6$

For root systems and weights, we follow the enumeration of Bourbaki [1].

We shall now describe how $K_1 \times K_2$ embeds into K. Let $\mathfrak{k}_2 = \mathfrak{sl}(2)_s + \mathfrak{sl}(2)_l$ be the complexified Lie algebra of K_2. Recall that $\mathfrak{k} = \mathfrak{k}(J_4(Q))$ has four strongly orthogonal roots. The long root $\mathfrak{sl}(2)_l$ embeds as $\mathfrak{sl}(2)$ corresponding to the root α_1 and the short root $\mathfrak{sl}(2)_s$ embeds diagonally into three $\mathfrak{sl}(2)$ corresponding to the remaining three roots. The centralizer of $\mathfrak{k}_2$ in $\mathfrak{k}$ is $\mathfrak{k}_1$.

The Langlands quotients. Let π_1 and π_2 be irreducible Harish-Chandra modules of $\mathfrak{h}$ and $\mathfrak{g}_2$ respectively. Let

$$V_{\min} \to \pi_1 \boxtimes \pi_2 \tag{1}$$

be a nonzero morphism of $(\mathfrak{h} \times \mathfrak{g}_2)$-modules. As it has been established in [4] (see also [7]), the infinitesimal character of π_1 determines the infinitesimal character of π_2, and conversely. If $x\varpi_1 + y\varpi_2$ is the infinitesimal character of π_2, then π_1 has infinitesimal character $x\varpi_1 + (x+3y)\varpi_2$, $(x+2y, x+y, y)$, $x\varpi_4 + y\varpi_3 + \rho(\mathfrak{sl}_3)$ for $\mathfrak{h} = \mathfrak{sl}_3, \mathfrak{sp}_6, \mathfrak{f}_4$ respectively. We would like to refine this information in the case when π_1 and π_2 are Langlands quotients of principal series representations. More precisely, let $B_1 = M_1A_1N_1$ and $B_2 = M_2A_2N_2$ denote Borel subgroups of H and G_2, respectively. Note that $M_1 = \mu_2^{r(H)}$, where $r(H)$ is the rank of H. Likewise, $M_2 = \mu_2^2$. Let σ_i be a representation of M_i and let λ_i be a dominant weight of the Lie algebra $\mathfrak{a}_i$. We set $I_1(\sigma_1, \lambda_1)$ to be the Harish-Chandra module of the normalized induced representation

$$\mathrm{Ind}_{M_1A_1N_1}^{H}(\sigma_1 \otimes a^{\lambda_1}).$$

Similarly we define $I_2(\sigma_2, \lambda_2)$ which is a Harish-Chandra module of G_2. Next, we specify a minimal K_i type, denoted by $\tau(\sigma_i)$, contained in the principal series $I_i(\sigma_i, \lambda_i)$. This K_i-type depends only on the Weyl group conjugation class of σ_i, and the restriction of $\tau(\sigma_i)$ to M_i is a direct sum, with multiplicity one, of all characters of M_i Weyl group conjugated to σ_i. In particular, $\tau(\sigma_i')$ is contained in $I_i(\sigma_i'', \lambda_i)$ if and only if the characters σ_i' and σ_i'' are conjugated by the Weyl group. This minimal K_i-type is also known as the fine K_i-type and small K_i-type in [10]. The minimal K_i-type $\tau(\sigma_i)$ is contained in the unique irreducible quotient $J_i(\sigma_i, \lambda_i)$ of $I_i(\sigma_i, \lambda_i)$. If σ_i is the trivial character of M_i, then $\tau(\sigma_i)$ is the trivial K_i-type and $J_i(\sigma_i, \lambda_i)$ is a spherical representation. Other cases are tabulated below. These were also computed in Table 5.8 in [9].

Table 1

G_2	$SL_3(\mathbb{R})$	$Sp_6(\mathbb{R})$	F_4
$\mathbb{C} \boxtimes \mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \boxtimes \mathbb{C}$
$\mathbb{C} \boxtimes S^2(\mathbb{C}^2)$	$\mathbb{C}^3$	$\wedge^2\mathbb{C}^3$	$\mathbb{C}^2 \boxtimes \mathbb{C}^6$,
			$S^2(\mathbb{C}^2) \boxtimes \mathbb{C}$

Remark: Note that the list does not include two minimal U_3-types: $\mathbb{C}^3$ and $\wedge^3\mathbb{C}^3$. This is because μ_2 the center of E_7, which is also the center of $Sp_6(\mathbb{R})$, acts trivially on the minimal representation.

Theorem 1. *Suppose $V_{\min} \to J_1(\sigma_1, \lambda_1) \boxtimes J_2(\sigma_2, \lambda_2)$ is a nonzero morphism of $(\mathfrak{h} \times \mathfrak{g}_2)$-modules. Let τ_i be the minimal K_i-type of $J_i(\sigma_i, \lambda_i)$, $i = 1, 2$. Then τ_1 and τ_2 are on the same row of the above table. Moreover, if $H = F_4$, then representations with the minimal K_1-type $S^2(\mathbb{C}^2) \boxtimes \mathbb{C}$ do not appear as quotients of $V_{\min}$.*

We remark that an irreducible spherical representation is uniquely determined by its infinitesimal character. In §3, we will discuss the situations when the infinitesimal characters are generic . In that case we have more precise results (Theorem 6 and Theorem 9).

Since $J_1(\sigma_1, \lambda_1)$ and $J_2(\sigma_2, \lambda_2)$ are generated by their minimal K_i-types and $V_{\min}|_K = \sum_n V_K(n\lambda_0)$, the above theorem follows immediately from the following technical lemma.

Lemma 2. *Let τ_1 and τ_2 be nontrivial minimal K_1-type and K_2-type respectively as in the third row of the above table. Then*

(i) $\mathrm{Hom}_{K_1 \times K_2}(1_{K_1} \boxtimes \tau_2, V_{n\psi}) = 0$ *for all* $n \in \mathbb{N}$ *and*

(ii) $\mathrm{Hom}_{K_1 \times K_2}(\tau_1 \boxtimes 1_{K_2}, V_{n\psi}) = 0$ *for all* $n \in \mathbb{N}$.

Furthermore, in the case $H = F_4$, $\mathrm{Hom}_{K_1}(S^r(\mathbb{C}^2) \boxtimes S^s(\mathbb{C}^6), V_{n\psi}) = 0$ *unless $r \leq s$ and $r \equiv s$* mod 2.

This was proven in [2] in the case E_6, and in this paper we will only deal with E_7 and E_8. Note that the group $K_1 \times K_2$ is *much smaller* than K. In particular, the lemma does not follow from any of the known, classical, branching rules. We also do not develop any new branching rules. In order to illustrate the main idea consider the first case of the above Lemma. As a first step, we calculate $K_1 \times SU_{2,l}$ invariants in $V_{n\psi}$. This is done on a case by case basis and is the most difficult part of this paper. Part (i) of the lemma states that the representation $S^2(\mathbb{C}^2)$ does not appear in the space of $K_1 \times SU_{2,l}$ invariants. Since the highest weight of $S^2(\mathbb{C}^2)$ is 2 it suffices to show that the dimension of the weight 4 space is the same as the dimension of the weight 2 space. In fact, using a nice trick, one can show that the dimension of the weight 2 space is the same as the dimension of the weight -4 space. A similar trick works for the case (ii). In addition, in the case of E_8, we show that the K_1-types $S^r(\mathbb{C}^2) \boxtimes S^s(\mathbb{C}^6)$ does not appear in the minimal representation unless $s \geq r$ and $r \equiv s$ (mod 2). This gives severe restrictions on possible F_4-quotients of the minimal representation.

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2. Generic principal series representations

In this section, we will show that if the infinitesimal character λ_1 of π_1 in (1) is generic (which we will define below), then π_i is isomorphic to $I_i(\sigma_1, \lambda_1)$ (for $i = 1,\ 2$) which is irreducible.

First we recall a theorem of Speh and Vogan [8]. There is also a converse statement but we do not need it here.

Theorem 3. *Let $P = MAN$ be a cuspidal parabolic subgroup of a real reductive group G. Let T_M be a compact Cartan subgroup of M. Let H_λ be a discrete series representation of M with infinitesimal character λ. We consider the normalized induced principal series representation*

$$\mathrm{Ind}_P^G(H_\lambda \boxtimes a^\nu \boxtimes 1)$$

of G with infinitesimal character $\gamma = (\lambda, \nu)$. Suppose γ is nonsingular. If the principal series representation above is reducible, then there exists a non-compact root α of $T_M A$ such that $n = 2(\alpha, \gamma)/(\alpha, \alpha)$ is a positive integer.

The above theorem leads us to the following definition: A weight γ is called *algebraically integral* with respect to a root α if $2(\alpha, \gamma)/(\alpha, \alpha)$ is an integer. In this paper, we say that the weight γ is *generic* if it is not algebraically integral to any root α in the root system.

Lemmas 4 and 5 below follow from Theorem 3.

Lemma 4. *(generic representations of* $\mathrm{G}_2(\mathbb{R})$*)*

- *The following two statements are equivalent:*
  (i) *The weight $\lambda_2 := x\varpi_1 + y\varpi_2$ is generic with respect to the root system of* G_2.
  (ii) *None of the following six numbers are integers: $x, y, x+y, x+2y, x+3y, 2x+3y$.*
- *If π_2 is an irreducible Harish-Chandra module of* G_2 *with infinitesimal character λ_2 satisfying either (i) or (ii) above, then π_2 is the irreducible principal series representation $I_2(\sigma_2, \lambda_2)$ for some character σ_2 of M_2.*

Proof. Items (i) and (ii) are equivalent by considering $2(\alpha, \lambda_2)/(\alpha, \alpha)$ for all roots α of G_2. Given π_2 in the lemma, then it is the Langlands quotient of $I_2(\sigma_2, \lambda_2)$ for some character σ_2 of M_2. By Theorem 3, $I_2(\sigma_2, \lambda_2)$ is irreducible. This proves the lemma. □

Lemma 5. *(generic representations of* $\mathrm{Sp}_6(\mathbb{R})$*)*

- *The following two statements are equivalent:*
  (i) *The weight $\lambda_1 := (x + 2y, x + y, y)$ is generic with respect to the root system of* $\mathrm{Sp}_6(\mathbb{R})$.
  (ii) *None of the following six numbers are integers: $x, y, x+y, x+2y, x+3y, 2x+3y$.*
- *If π_1 is an irreducible Harish-Chandra module of* $\mathrm{Sp}_6(\mathbb{R})$ *with infinitesimal character λ_1 satisfying either (i) or (ii) above, then π_1 is the irreducible principal series representation $I_1(\sigma_1, \lambda_1)$ for some character σ_1 of M_1.* □

The proof of the lemma is similar to the previous one. We will call π_1 and π_2 in the above two lemmas generic principal series representation.

In the notation of the above two lemmas, the correspondence of infinitesimal characters for the dual pair $G_2 \times Sp_6(\mathbb{R})$ is given by $\lambda_1 \leftrightarrow \lambda_2$. We can incorporate these into Theorem 1 and we have the following result.

Theorem 6. *Suppose $\pi_1 \boxtimes \pi_2$ is a quotient of the minimal representation of E_7. Then π_1 is a generic spherical principal series representation of $Sp_6(\mathbb{R})$ with infinitesimal character λ_1 if and only if π_2 is a generic spherical principal series representation of G_2 with infinitesimal character λ_2.* □

Representations of split F_4. Suppose π_1 is a representation of the split F_4 whose infinitesimal character is $\lambda_1 := x\varpi_1 + y\varpi_2 + \rho(\mathfrak{sl}_3)$ where x and y satisfies Lemma 4(ii). We would like to know all possible Langlands parameters it can have. We assume that π_1 is the quotient of the principal series representation

$$\mathrm{Ind}_{MAN}^{F_4}(H_\lambda \boxtimes a^\nu \boxtimes 1) \tag{2}$$

where MAN is a cuspidal representation of F_4 and H_λ is a discrete series representation of M with Harish-Chandra parameter λ.

Lemma 7. *Suppose π_1 satisfies the above assumptions, then the parabolic subgroup $P = MAN$ in (2) is either*

(i) *the Borel subgroup, or*
(ii) *it is the parabolic subgroup corresponding to the long simple root α_1 or α_2, and the connected component of M is $M^0 = SL_2(\mathbb{R})$. The discrete series H_λ of M has Harish-Chandra parameter $\lambda = 1$ or 2.*

Proof. We assume that P is not the Borel subgroup. Let α be a simple root in M. Then $\lambda + \rho(M)$ is algebraically integral with respect to α. Let $\lambda_1 := x\varpi_1 + y\varpi_2 + \rho(\mathfrak{sl}_3)$. From the consideration of infinitesimal characters, $w(\lambda_1) = (\lambda, \nu)$ for some w in the Weyl group. Since $(0, \nu)$ is perpendicular to α and $\rho(M)$ is algebraically integral with respect to α, we conclude that

$$2(\lambda_1, \alpha')/(\alpha', \alpha') \in \mathbb{Z} \tag{3}$$

where $\alpha' = w^{-1}\alpha$. Suppose α' is short root, then a check by hand shows that under the assumptions in Lemma 4(ii), (3) is impossible. This implies that α is a long simple root, that is $\alpha = \alpha_1$ or α_2.

A similar check shows that (3) holds if and only if $\alpha' = w^{-1}\alpha$ is either α_1, α_2 or $\alpha_1 + \alpha_2$. Note that M^0 cannot be $SL_3(\mathbb{R})$ because it does not have discrete series representation. Hence $M^0 = SL_2(\mathbb{R})$ and this proves (i). The restriction of λ_1 to the diagonal $H_\alpha \in \mathfrak{sl}_2$ corresponding to α is either $0, \pm 1$ or ± 2. We can remove the negative signs since $SL_2(\mathbb{R})^\pm \subseteq M$. This proves (ii). □

Next we list all the possible minimal K_1-types of (2) satisfying the last lemma. If the parabolic subgroup is the Borel subgroup, then the minimal K_1-types are given in the last column of Table 1. If the parabolic is not the Borel subgroup, then the K_1-types are given in the following table.

Table 2

	Minimal K_1-types
$\lambda = 1$	$S^4(\mathbb{C}^2) \boxtimes 1_{\mathrm{Sp}_6}$ and $S^3(\mathbb{C}^2) \boxtimes \mathbb{C}^6$
$\lambda = 2$	$S^6(\mathbb{C}^2) \boxtimes 1_{\mathrm{Sp}_6}$ and $S^5(\mathbb{C}^2) \boxtimes \mathbb{C}^6$

The proof is a little long but not hard so we will leave it to the reader.

A degenerate principal series representation of F_4**.** Let $P_{12} = M_{12} \cdot (\mathbb{R}^+)^2 \cdot N_{12}$ be the (non-cuspidal) standard parabolic subgroup whose Levi factor M_{12} has simple long roots $\{\alpha_1, \alpha_2\}$. Let $\phi_i : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{F}_4$ be the homomorphism induced by the simple real root α_i. We have

$$M_{12} = \mathrm{SL}_3(\mathbb{R}) \times L_4$$

where L_4 is the Klien four group generated by $\phi_3(-1)$ and $\phi_4(-1)$.

Let $I_{12}(x\varpi_4 + y\varpi_3)$ denote the Harish-Chandra module of the normalized induced spherical degenerate principal series representation

$$\mathrm{Ind}_{M_{12}\cdot(\mathbb{R}^+)^2\cdot N_{12}}^{\mathrm{F}_4}(1_{M_{12}} \boxtimes a^{x\varpi_4 + y\varpi_3} \boxtimes 1). \tag{4}$$

It has infinitesimal character $x\varpi_3 + y\varpi_4 + \rho(\mathfrak{sl}_3)$.

Lemma 8. *Suppose x and y satisfy Lemma 4(ii). Then the spherical degenerate principal series representation $I_{12}(x\varpi_4 + y\varpi_3)$ is irreducible.*

Proof. Suppose $I_{12}(x\varpi_4 + y\varpi_3)$ is reducible. Hence it contains a non-spherical irreducible subquotient, say π_1'. Now π_1' will also satisfy Lemma 7 and it will contain one of non-trivial minimal K_1-types given in Table 2 or the last column of Table 1. It is straightforward to check that none of these non-trivial K_1-types is a K_1-type of the degenerate principal series representation. □

Now we can state the main result for E_8. Let π_1 and π_2 be irreducible Harish-Chandra modules of F_4 and G_2 respectively.

Theorem 9. *Suppose $\pi_1 \boxtimes \pi_2$ is a quotient of the minimal representation of E_8. Then:*

(i) *Suppose that π_2 is a generic spherical principal series representation with infinitesimal character $\lambda_2 = x\varpi_1 + y\varpi_2$. Then π_1 is the irreducible degenerate principal series representation $I_{12}(x\varpi_4 + y\varpi_3)$.*

(ii) *Suppose $\pi_1 = I_{12}(x\varpi_4 + y\varpi_3)$ such that Lemma 4(ii) holds for these x and y. Then π_2 is a generic spherical principal series representation with infinitesimal character λ_2.*

Proof. We will first prove (ii). By the correspondence of infinitesimal characters, the infinitesimal character of π_2 satisfies Lemma 4. Hence π_2 is a generic principal series representation. By Lemma 2, it cannot be the non-spherical representation and hence it is the spherical principal series. This proves (ii).

We will now prove (i). By the correspondence of infinitesimal character, the infinitesimal character of π_1 satisfies Lemma 7. Hence π_2 contains one of the minimal K_1-types in Table 2 or the last column of Table 1. By Lemma 2, the minimal K_1-type must be the trivial K_1-type so π_1 is the unique spherical representation with infinitesimal character $x\varpi_4 + y\varpi_3 + \rho(\mathfrak{sl}_3)$.

Now $I_{12} := I_{12}(x\varpi_4 + y\varpi_3)$ is also an irreducible spherical representation with the same infinitesimal character as π_1. Hence $\pi_1 = I_{12}$ because irreducible spherical representations are uniquely determined by their infinitesimal characters. This completes the proof of Theorem 9. □

3. Littlewood-Richardson (LR) Rule

The rest of this paper is devoted to the proof of Lemma 2. First we recall the famous Littlewood-Richardson branching rule (LR rule for short) for the restriction of representations from $\mathfrak{gl}_{n+m}$ to $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ which we will use many times later. Recall that a partition ν of n parameterizes an irreducible representation V_ν of $\mathfrak{gl}_n$.

Theorem 10. *(Littlewood-Richardson rule) Let λ, μ and ν be a partition of $m+n$, m and n, respectively. The multiplicity $c_{\mu\nu\lambda}$ of $V_\mu \boxtimes V_\nu$ in V_λ is equal to the number of way the Young diagram for μ can be expanded to the Young diagram of λ by a strict ν-expansion. More precisely, if $\nu = (\nu_1, \ldots, \nu_k)$, a ν-expansion is obtained by first adding μ_1 boxes, with no two boxes in the same column, and putting the integer 1 in each of these boxes. We then add μ_2 boxes with a 2 in the same fashion, and so on. An expansion is called strict if, when integers in the boxes are listed from right to left, starting with the top row and working down, for every t between 1 and $\mu_1 + \ldots + \mu_k$ the first t integers on this list contain each integer z between 1 and $k-1$ at least as many times as the next integer $z+1$.*

Remark. In order to calculate LR coefficients efficiently, we note the following two properties necessarily satisfied by any strict ν-expansion:

- The integers in boxes are strictly increasing in each column, and are increasing (but not necessarily strictly) in each row.
- The first row can contain only boxes with a 1, the second row can only contain boxes with 1 and 2, and so on.

4. Proof of Lemma 2(i) for E_7

We will work exclusively with complexified Lie algebras.

Let $\mathfrak{k}_2 = \mathfrak{sl}_{2,l} + \mathfrak{sl}_{2,s}$ be the Lie algebra of K_2 where $\mathfrak{sl}_{2,l}$ corresponds to a long root, and $\mathfrak{sl}_{2,s}$ to a short root. We will identify the Lie algebra $\mathfrak{k} = \mathfrak{sl}_8$ of K with the set of 8 by 8 traceless matrices. Then $\mathfrak{sl}_{2,l} \subseteq \mathfrak{sl}_8$ can be arranged to occupy upper-left 2×2 block in $\mathfrak{sl}_8$. The centralizer of $\mathfrak{sl}_{2,l}$ in $\mathfrak{sl}_8$ is $\mathfrak{gl}_6$ where the center of $\mathfrak{gl}_6$ consists of the diagonal matrices

$$\{d(z) := \mathrm{diag}(-3z, -3z, z, z, z, z, z, z) : z \in \mathbb{C}\}.$$

The identification with $\mathfrak{gl}_6$ is done so that the central elements $d(z)$ act by z on the one dimensional representation with the highest weight $(1, 0, 0, 0, 0, 0)$. It follows easily from LR rule that

$$V_{\mathfrak{sl}_8}(n\varpi_4)^{\mathfrak{sl}_{2,l}} = \sum_{k=0}^{n} V_{\mathfrak{gl}_6}(k,k,2k-n,2k-n,k-n,k-n). \tag{5}$$

Next, using $\mathbb{C}^6 = \mathbb{C}^2 \otimes \mathbb{C}^3$ we can embed $\mathfrak{sl}_{2,s} + \mathfrak{gl}_3$ into $\mathfrak{gl}_6$. In this way we have completely described an embedding of

$$\mathfrak{k}_1 + \mathfrak{k}_2 = \mathfrak{gl}_3 + (\mathfrak{sl}_{2,l} + \mathfrak{sl}_{2,s})$$

into $\mathfrak{k}$. In order to prove Lemma, we need to analyze the $\mathfrak{gl}_3$-invariants of representations appearing on the right hand side of (5). First of all, if the center of $\mathfrak{gl}_3$ acts trivially, then $n = 2k$ in (5). We will now denote

$$V_k := V_{\mathfrak{gl}_6}(k,k,0,0,-k,-k).$$

Next, we consider

$$\mathfrak{k}_1 = \mathfrak{gl}_3 \subset \mathfrak{gl}_3 + \mathfrak{gl}_3 \subset \mathfrak{gl}_6$$

where $\mathfrak{gl}_3$ is embedded diagonally. Suppose $W \otimes W'$ is a representation of $\mathfrak{gl}_3 + \mathfrak{gl}_3$ appearing in the restriction of V_k. Then $W \otimes W'$ gives rise to a 1 dimensional invariant subspace of $\mathfrak{f}_1$ if and only if $W' = W^*$, the dual representation of W. If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the highest weight of W, then $(W \otimes W^*)^{\mathfrak{k}_1}$ is contained in the weight $2\sum_i \lambda_i$ space for $\mathfrak{sl}_{2,s}$.

In order to prove Lemma 2(i), we need to show that the representation $S^2(\mathbb{C}^2)$ of $\mathfrak{sl}_{2,s}$ does not appear in $V_k^{\mathfrak{k}_1}$. To that end, it suffices to show that the dimensions of the weight 2 space and the weight (-4) space for $\mathfrak{sl}_{2,s}$ are equal. This follows immediately from the following proposition.

Proposition 11. *Let W be a representation of $\mathfrak{gl}_3$ of highest weight λ, and let $m(k,\lambda)$ denote the multiplicity of $W \otimes W^*$ in V_k. If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$, then $m(\lambda, k) = m(\lambda^-, k)$ where $\lambda^- = (\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1)$.*

Proof. We need to calculate $m(\lambda)$. This is accomplished using LR rule in the following Lemma.

Lemma 12. *Suppose $W = V_{\mathfrak{gl}_3}(\lambda_1, \lambda_2, \lambda_3)$. Assume first that $\lambda_2 \leq 0$. Then the multiplicity of $W \boxtimes W^*$ in V_k zero unless $k \leq \lambda_1 + \lambda_2$. If it is nonzero then it is equal to*

$$\begin{cases} \min(\lambda_2 - \lambda_3, \lambda_1) + 1 \text{ if } k \geq \lambda_1 - \lambda_3 \\ \min(\lambda_2 - \lambda_1 + k, \lambda_3 + k) + 1 \text{ if } k < \lambda_1 - \lambda_3. \end{cases}$$

To find the multiplicity when $\lambda_2 \geq 0$, we use the symmetry by interchanging W with W^*. In this case the multiplicity is

$$\begin{cases} \min(\lambda_1 - \lambda_2, -\lambda_3) + 1 \text{ if } k \geq \lambda_1 - \lambda_3 \\ \min(\lambda_3 - \lambda_2 + k, k - \lambda_1) + 1 \text{ if } k < \lambda_1 - \lambda_3. \end{cases}$$

Proof. In order apply the LR rule, we consider $U = \det^k \otimes W$, $U' = \det^k \otimes W^*$ and $E = \det^k \otimes V_k$. Hence U has highest weight $(a, b, c) = (k, k, k) + \lambda$, U' has highest weight $(a', b', c') = (k, k, k) + \lambda^*$ and E has highest weight $(2k, 2k, k, k, 0, 0)$. The multiplicity of $U \boxtimes U'$ in E is equal to $m(k, \lambda)$.

From the LR rule, $0 \leq c \leq k$ and $0 \leq b \leq a \leq 2k$. The same is true for a', b', c'. Since $b + b' = 2k$, by interchanging the role of U and U', we may assume that $b \leq k \leq b'$.

Let Y, Y' and Z denote the Young diagrams of U, U' and E respectively. We place Y inside Z and fill in the remaining spaces in Z with the boxes from U'.

By the remark after Theorem 10 there is a unique way of filling the first 2 rows of Z, namely, c' 1-boxes on the first row and, $a - b = b' - c'$ 1-boxes on the second row and c' 2-boxes on the second row. All the 3-boxes (there are c' of them) appear on the 4-th row of Z. The Young diagram is given in the figure below.

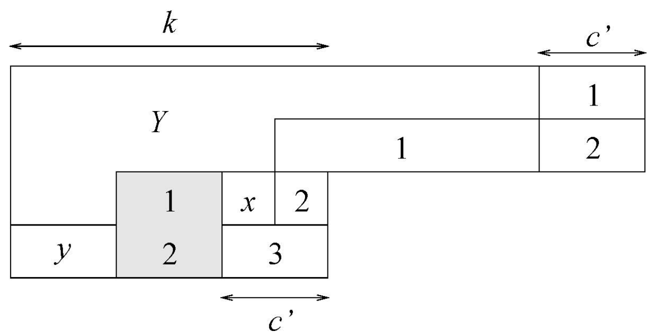


Figure 1

It shows that $c' \geq b' - k = \lambda_2$.

It remains to fill the last 2 rows of Z with 1-boxes and 2 boxes. Suppose $c + c' < k$ as shown in the Young diagram above, then some of the boxes are uniquely determined as shown in the shaded area. It remains to fill x with 1-boxes and 2-boxes. Let

$$z = \max(0, k - c - c') = \max(0, a - c - k).$$

Now x has $b - c - z$ boxes which is the same number as the remaining 1-boxes to be filled. Hence the number of ways of filling is equal to the number of ways of putting 1-boxes into $y = \min(c, k - c')$. This is equal to

$$\min(b - c - z, c, k - c') + 1 = \min(b - c - z, c, a - k) + 1 = \min(b - c, b - a + k, c, a - k) + 1.$$

If we substitute λ_i back into a, b, c above, and the condition $k \geq \lambda_1 - \lambda_3$ (or $k < \lambda_1 - \lambda_3$), we would recover the multiplicity stated in the lemma. The lemma is proved. $\square$

We will now prove Proposition 11. We first assume that $\lambda_2 \leq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$, or $\lambda_2 < 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = -2$. One easily checks that the multiplicity is

$$\begin{cases} \lambda_2 - \lambda_3 + 1 \text{ if } k \geq \lambda_1 - \lambda_3 \\ \lambda_2 - \lambda_1 + k + 1 \text{ if } k < \lambda_1 - \lambda_3. \end{cases}$$

Assume now that $\lambda_2 > 0$ and $\lambda_1+\lambda_2+\lambda_3 = 1$, or $\lambda_2 \geq 0$ and $\lambda_1+\lambda_2+\lambda_3 = -2$. One easily checks that the multiplicity is

$$\begin{cases} \lambda_1 - \lambda_2 + 1 \text{ if } k \geq \lambda_1 - \lambda_3 \\ \lambda_3 - \lambda_2 + k + 1 \text{ if } k < \lambda_1 - \lambda_3. \end{cases}$$

Proposition 11 follows. □

5. Proof of Lemma 2(ii) for E_7

Recall that in Section 4 we defined the following sequence of embeddings:

$$\mathfrak{k}_1 = \mathfrak{gl}_3 \to \mathfrak{gl}_3 \times \mathfrak{gl}_3 \to \mathfrak{gl}_6 \to \mathfrak{sl}_8 = \mathfrak{k}.$$

Hence we get an embedding of the complex groups GL_3 into SL_8. We will denote this GL_3 by GL_3^c. Since $K(\mathbb{C}) = \mathrm{SL}_8/\mu_2$ this gives an embeding of GL_3^c/μ_2 into $K(\mathbb{C})$. The group GL_3^c/μ_2 can be identified with $K_1(\mathbb{C}) = \mathrm{GL}_3$ as follows. Let $zs \in \mathrm{GL}_3^c$ where z is a scalar matrix and $s \in \mathrm{SL}_3$. Then

$$\phi(zs) = z^{-2}s \tag{6}$$

defines a map from GL_3^c onto GL_3 with kernel μ_2. When pulled back by ϕ the minimal $K_1(\mathbb{C})$-type with the highest weight representation $(1,1,0)$ becomes the representation $V_{\mathfrak{k}_1}(-1,-1,-2)$ of GL_3^c.

5.1. We will begin the proof of Lemma 2(ii) which states that the subrepresentation $V_{\mathfrak{k}_1}(-1,-1,-2)$ does not occur in the space of $\mathfrak{sl}_{2,s}$-invariants on right hand side of (5). First of all, notice that not all summands in (5) will contain $V_{\mathfrak{k}_1}(-1,-1,-2)$. Indeed the central character of $V_{\mathfrak{k}_1}(-1,-1,-2)$ is -4. This implies that $4(-n+2k) = -4$ which is equivalent to $2k = n+1$. Thus, we only need to consider

$$V'_k := V_{\mathfrak{gl}_6}(k,k,-1,-1,-k-1,-k-1).$$

Let $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ and $\lambda' = (\lambda'_1,\lambda'_2,\lambda'_3)$ denote two highest weights of $\mathfrak{gl}_3$. We would like to know the multiplicities of

$$V_{\mathfrak{k}_1}(-1,-1,-2) \subset V_{\mathfrak{gl}_3}(\lambda) \otimes V_{\mathfrak{gl}_3}(\lambda') \subset V'_k \tag{7}$$

such that $S = \sum_i \lambda_i - \lambda'_i \in \{0,2\}$. For technical reason, we allow $S = -2$ as well. Note that S is the weight for the torus of $\mathfrak{sl}_{2,s}$ acting on $W \otimes W'$. Since the central character, with respect to $\mathfrak{k}_1$, is $-4 = \sum_i \lambda_i + \lambda'_i$, we can rewrite

$$S = \sum_i \lambda_i - \lambda'_i = \sum_i 2\lambda_i - (\lambda_i + \lambda'_i) = 2(\sum_i \lambda_i + 2).$$

Hence $S \in \{-2,0,2\}$ is equivalent to $\sum_i \lambda_i \in \{-3,-2,-1\}$.

We consider the first inclusion in (7).

Lemma 13. *The multiplicity of $V_{\mathfrak{k}_1}(-1,-1,-2)$ in $V_{\mathfrak{gl}_3}(\lambda_1,\lambda_2,\lambda_3) \otimes V_{\mathfrak{gl}_3}(\lambda'_1,\lambda'_2,\lambda'_3)$ is either 0 or 1. It is 1 if and only if one of the following situations holds:*

(i) $\lambda_1 + \lambda_3' = -2,\ \lambda_2 + \lambda_2' = \lambda_3 + \lambda_1' = -1,$
(ii) $\lambda_1 > \lambda_2,\ \lambda_1 + \lambda_3' = -1,\ \lambda_2 + \lambda_2' = -2,\ \ \lambda_3 + \lambda_1' = -1,$
(iii) $\lambda_2 > \lambda_3,\ \lambda_1 + \lambda_3' = \lambda_2 + \lambda_2' = -1,\ \ \lambda_3 + \lambda_1' = -2.$ □

Note that (iii) is obtained from (i) by interchanging the role of λ and λ'. The proof is just another exercise in LR rule and we will leave it to the reader.

We will see later that in the proof of Lemma 14 that in order for V_k' to contain $V_{\mathfrak{gl}_3}(\lambda_1, \lambda_2, \lambda_3) \otimes V_{\mathfrak{gl}_3}(\lambda_1', \lambda_2', \lambda_3')$ it is necessary that we have

$$k \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq -k-1,\quad k \geq \lambda_1' \geq \lambda_2' \geq \lambda_3' \geq -k-1. \tag{8}$$

From now on, we will refer to the three cases in Lemma 13 satisfying (8) as Cases (i), (ii) and (iii) respectively. For Case (i), one can show that $\lambda_1 \leq k-1$

Let $m(\lambda, \lambda', k)$ denote the multiplicity of $V_{\mathfrak{gl}_3}(\lambda_1, \lambda_2, \lambda_3) \otimes V_{\mathfrak{gl}_3}(\lambda_1', \lambda_2', \lambda_3')$ in V_k' such that tensor product also contains $V_{\mathfrak{k}_1}(-1, -1, -2)$.

Lemma 14. *Suppose $S = \sum_i \lambda_i - \lambda_i' = 0$ or 2. Then the multiplicity $m(\lambda, \lambda', k)$ in the three cases in Lemma 13 are given in the table below.*

			$m(\lambda, \lambda', k)$
Case (i)	*if* $k \leq \lambda_1 - \lambda_3$	*if* $\lambda_2 \geq 0$	$k - \lambda_2 + \lambda_3 + 1$
		if $\lambda_2 \leq -1$	$k - \lambda_1 + \lambda_2 + 1$
	if $k > \lambda_1 - \lambda_3$	*if* $\lambda_2 \geq 0$	$\lambda_1 - \lambda_2 + 1$
		if $\lambda_2 \leq -1$	$\lambda_2 - \lambda_3 + 1$
Case (ii)	*if* $k \leq \lambda_1 - \lambda_3 - 1$	*if* $\lambda_2 \geq 0$	$k - \lambda_2 + \lambda_3 + 1$
		if $\lambda_2 \leq -1$	$k - \lambda_1 + \lambda_2 + 2$
	if $k > \lambda_1 - \lambda_3 - 1$	*if* $\lambda_2 \geq 0$	$\lambda_1 - \lambda_2$
		if $\lambda_2 \leq -1$	$\lambda_2 - \lambda_3 + 1$
Case (iii)	*if* $k \leq \lambda_1 - \lambda_3 - 1$	*if* $\lambda_2 \geq 0$	$k - \lambda_2 + \lambda_3 + 2$
		if $\lambda_2 \leq -1$	$k - \lambda_1 + \lambda_2 + 1$
	if $k > \lambda_1 - \lambda_3 - 1$	*if* $\lambda_2 \geq 0$	$\lambda_1 - \lambda_2 + 1$
		if $\lambda_2 \leq -1$	$\lambda_2 - \lambda_3$

Proof. We would like to apply the LR rule so we set $U = \det^{k+1} \otimes V_{\mathfrak{gl}_3}(\lambda_1, \lambda_2, \lambda_3)$, $U' = \det^{k+1} \otimes V_{\mathfrak{gl}_3}(\lambda_1', \lambda_2', \lambda_3')$ and $E' = \det^{k+1} \otimes V_k'$. The highest weight of U is $(a, b, c) = (k+1, k+1, k+1) + \lambda$, the highest weight of U' is $(a', b', c') = (k+1, k+1, k+1) + \lambda'$, and the highest weight of E' is $(2k+1, 2k+1, k, k, 0, 0)$. Then

$$V_{\mathfrak{k}_1}(2k+1, 2k+1, 2k) \subset U \otimes U' \subset E'.$$

and the multiplicities are not affected. By interchanging the U and U' if necessary, we may assume that $b \leq k$, that is, $\lambda_2 \leq -1$. Let Y, Y' and Z be the Young diagrams of U, U' and E'. We embed Y into Z and we fill Z with the boxes of Y'. An almost identical argument as before gives the same figure as Figure 1. Here $z = \max(0, k - c - c')$. The number of 2-boxes in the second row is not less than the number c' of 3-boxes.

In all cases the number of remaining 1-boxes is not greater than the number of boxes in x. There are $b-c-z-e$ remaining 1-boxes where $e=0$ in Cases (i) and (ii) and $e=1$ in Case (iii). Hence the multiplicity of $U \boxtimes U'$ in E' is equal to the number of ways of filling the remaining 1-boxes in $y=\min(c,k-c')$ which is

$$\min(c,k-c',b-c-z-e)+1=\min(c,k-c',b-c-e,b-k+c'-e)+1.$$

More explicitly, the multiplicity in the three cases are:

$$\begin{aligned}
\text{Case (i):}\quad & \min(c,a-k,b-c,k-a+b)+1\\
& =\min(\lambda_3+k+1,\lambda_1+1,\lambda_2-\lambda_3,k-\lambda_1+\lambda_2)+1\\
\text{Case (ii):}\quad & \min(c,a-k-1,b-c,k+1-a+b)+1\\
& =\min(\lambda_3+k+1,\lambda_1,\lambda_2-\lambda_3,k+1-\lambda_1+\lambda_2)+1\\
\text{Case (iii):}\quad & \min(c,a-k-1,b-c-1,k-a+b)+1\\
& =\min(\lambda_3+k+1,\lambda_1,\lambda_2-\lambda_3-1,k-\lambda_1+\lambda_2)+1.
\end{aligned}$$

The fact that $\sum_i \lambda_i \in \{-3,-2,-1\}$ and $\lambda_2 \le -1$ implies that $\lambda_1+\lambda_3+1 \ge \lambda_2$. Then the above multiplicity simplifies to

$$\begin{aligned}
\text{Case (i):}\quad & \min(\lambda_2-\lambda_3,k-\lambda_1+\lambda_2)+1\\
\text{Case (ii)':}\quad & \min(\lambda_2-\lambda_3,k+1-\lambda_1+\lambda_2)+1\\
\text{Case (iii):}\quad & \min(\lambda_2-\lambda_3-1,k-\lambda_1+\lambda_2)+1.
\end{aligned}$$

Here Case (ii)' refers all of Case (ii) except the situation where $\sum_i \lambda_i=-3$ and $\lambda_2=-1$, which is of no use to us.

By interchange the role of λ and λ', we obtain the cases where $\lambda_2 \ge 0$. This is where we need the fact that $\sum_i \lambda=-3$ so that $\sum_i \lambda'_i=-1$.

The table in Lemma 14 follows immediately by comparing the calculations made above. This proves Lemma 14. □

Proof of Lemma 2(ii) for E_7**.** In order to prove the lemma, it suffices to show that the dimensions of the weight 2 space and weight 0 space for $\mathfrak{sl}_{2,s}$ are equal. This is equivalent to

$$\sum m(\lambda,\lambda',k)=\sum m(\lambda,\lambda',k) \tag{9}$$

where the first (resp. second) sum is taken over all (λ,λ') satisfying Lemma 13 and such that $S=\sum_i \lambda_i-\lambda'_i=2$ (resp. $S=0$). We have seen before that $S=2$ (resp. $S=0$) is equivalent to $\sum_i \lambda_i=-1$ (resp. $\sum_i \lambda_i=-2$).

We refer to Lemma 14. Suppose $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ satisfies $\sum_i \lambda=-1$. We define

$$\tilde\lambda=(\tilde\lambda_1,\tilde\lambda_2,\tilde\lambda_3)=(-\lambda_3-1,-\lambda_2-1,-\lambda_1-1).$$

We gather some properties of the transformation $\lambda \mapsto \tilde\lambda$.

(a) Since $\sum_i \lambda=-1$, we have $\sum_i \tilde\lambda=-2$.
(b) We have $\lambda_1-\lambda_3=\tilde\lambda_1-\tilde\lambda_3$, $\lambda_1-\lambda_2=\tilde\lambda_2-\tilde\lambda_3$. $\lambda_2-\lambda_3=\tilde\lambda_1-\tilde\lambda_2$.
(c) If $\lambda_2 \ge 0$, then $\tilde\lambda_2 \le -1$. Conversely if $\lambda_2 \le -1$, then $\tilde\lambda_2 \ge 0$.

(d) If λ belongs to Case (i) (resp. Case (ii), Case (iii)), then $\tilde{\lambda}$ also belongs to Case (i) (resp. Case (iii), Case (ii)).

We will say more about (d). Indeed if we refer to the table in Lemma 14, the transformation $\lambda \mapsto \tilde{\lambda}$ preserves multiplicities. For example, if we refer to the table in Lemma 14, then the transformation sends the first (resp. second) line of Case (i) for $(\lambda_1, \lambda_2, \lambda_3)$ to the second (resp. first) line of Case (i) for $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$. This proves (9) and Lemma 2(ii). □

6. Proof of Lemma 2(i) for E_8

6.1. Lie subalgebras of $\mathfrak{e}_8$. Let

$$\mathfrak{so}_{16},\ \mathfrak{sl}_{2,l}^{\mathrm{F}} + \mathfrak{sp}_6 \text{ and } \mathfrak{sl}_{2,l}^{\mathrm{G}} + \mathfrak{sl}_{2,s}$$

be the complexified Lie algebras of the maximal compact subgroups K, K_1, K_2 of $E_{8,8}$, $F_{4,4}$ and $G_{2,2}$ respectively. The Lie algebra $\mathfrak{so}_{16}$ contains

$$\mathfrak{so}_{12} + \mathfrak{so}_4 = \mathfrak{so}_{12} + (\mathfrak{sl}_{2,l}^{\mathrm{F}} + \mathfrak{sl}_{2,l}^{\mathrm{G}}).$$

The standard representation $\mathbb{C}^{12}$ of $\mathfrak{so}_{12}$ can be written as a product of two symplectic spaces $\mathbb{C}^6 \otimes \mathbb{C}^2$. This gives an embedding of $\mathfrak{sp}_6 + \mathfrak{sl}_{2,s}$ into $\mathfrak{so}_{12}$, and the embedding of $\mathfrak{k}_1 + \mathfrak{k}_2$ into $\mathfrak{k}$ is completely described. Using the notation of the root system of $\mathfrak{e}_8$ in [1], $\mathfrak{sl}_{2,l}^{\mathrm{F}}$ and $\mathfrak{sl}_{2,l}^{\mathrm{G}}$ correspond to the simple roots $\varepsilon_1 + \varepsilon_2$ and $\varepsilon_2 - \varepsilon_1$ respectively.

We need one additional subalgebra of $\mathfrak{so}_{12}$. If we decompose the standard representation $\mathbb{C}^{12} = \mathbb{C}^6 + (\mathbb{C}^6)^*$ of $\mathfrak{so}_{12}$ into two isotropic subspaces, then we get $\mathfrak{sp}_6 \subset \mathfrak{gl}_6 \subset \mathfrak{so}_{12}$. The center of $\mathfrak{gl}_1$ of $\mathfrak{gl}_6$ forms a torus of $\mathfrak{sl}_{2,s}$.

Recall that $\mathfrak{so}_{16}$ has two maximal parabolic subalgebras with Levi component isomorphic to $\mathfrak{gl}_8$. We set $\mathfrak{gl}_8^{\mathrm{G}}$ and $\mathfrak{gl}_2^{\mathrm{G}}$ to be the Lie subalgebras such that

$$\mathfrak{so}_{16} \supset \mathfrak{gl}_8^{\mathrm{G}} \supset \mathfrak{gl}_6 + \mathfrak{gl}_2^{\mathrm{G}} = \mathfrak{gl}_6 + (\mathfrak{gl}_1 + \mathfrak{sl}_{2,l}^{\mathrm{G}}).$$

Similarly we define $\mathfrak{gl}_8^{\mathrm{F}}$ and $\mathfrak{gl}_2^{\mathrm{F}}$. Note that the center of $\mathfrak{gl}_2^{\mathrm{G}}$ is equal to the split torus of $\mathfrak{sl}_{2,l}^{\mathrm{F}}$ and the center of of $\mathfrak{gl}_2^{\mathrm{F}}$ is equal to the split torus of $\mathfrak{sl}_{2,l}^{\mathrm{G}}$.

Two branching rules. We will state two branching rules that we will need later. The first branching rule is a special case of one due to T. Enright and M. Hunziker. One can also give a direct proof using Borel-Weil theorem.

Lemma 15. *Let ϖ_8 denote the fundamental weight corresponding to the half-spin representation of $\mathfrak{so}_{16}$ acting on $\mathfrak{p}$. Then ϖ_8 is perpendicular to the roots of $\mathfrak{gl}_8^{\mathrm{F}}$, and*

(i)
$$V_{\mathfrak{so}_{16}}(n\varpi_8)|_{\mathfrak{gl}_8^{\mathrm{F}}} = \sum V_{\mathfrak{gl}_8}(a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_4)$$
where the sum is taken over $\frac{n}{2} \geq a_1 \geq \ldots \geq a_4 \geq -\frac{n}{2}$ such that $a_i - \frac{n}{2} \in \mathbb{Z}$, and

(ii)
$$V_{\mathfrak{so}_{16}}(n\varpi_8)|_{\mathfrak{gl}_8^{\mathrm{G}}} = \sum V_{\mathfrak{gl}_8}(a_1, a_2, a_2, a_3, a_3, a_4, a_4, a_5)$$
where the sum is taken over $\frac{n}{2} \geq a_1 \geq \ldots \geq a_5 \geq -\frac{n}{2}$ such that $a_i - \frac{n}{2} \in \mathbb{Z}$. □

We now state the second branching rule.

Lemma 16. *Consider $\mathfrak{sp}_6$ in $\mathfrak{gl}_6$. Let $\lambda = (\lambda_1, \dots, \lambda_6)$ be a highest weight of $\mathfrak{gl}_6$.*

(i) *The dimension of $(V_{\mathfrak{gl}_6}(\lambda))^{\mathfrak{sp}_6}$ is either 0 or 1. It is one if and only if $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, and $\lambda_5 = \lambda_6$.*

(ii) *The representation $V_{\mathfrak{gl}_6}(\lambda)$ contains the representation $S^r(\mathbb{C}^6)$ of $\mathfrak{sp}_6$ with either multiplicity 0 or 1. It is 1 if and only if*

$$r = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6.$$

Proof. Part (i) follows from the Cartan-Helgason theorem (see page 535 in [6]). For (ii), we consider

$$V_{\mathfrak{gl}_6}(\lambda) \otimes S^r(\mathbb{C}^6) = \sum_{\lambda'} c^{\lambda'}_{\lambda, r\varepsilon_1} V_{\mathfrak{gl}_6}(\lambda')$$

as representations of $\mathfrak{gl}_6$. Here the LR number $c^{\lambda'}_{\lambda, r\varepsilon_1}$ is either 0 or 1. The representation $V_{\mathfrak{gl}_6}(\lambda)$ contains $S^r(\mathbb{C}^6)$ of $\mathfrak{sp}_6$ if and only if some $V_{\mathfrak{gl}_6}(\lambda')$ on the right hand side of the equation contains the trivial representation of $\mathfrak{sp}_6$. Now (ii) follows from (i). □

Lemma 17. *Suppose $V_{\mathfrak{so}_{16}}(n\varpi_8)$ contains the irreducible representation $S^s(\mathbb{C}^2) \boxtimes S^r(\mathbb{C}^6)$ of $\mathfrak{sl}^{\mathrm{F}}_{2,l} + \mathfrak{sp}_6$, then $r \geq s$ and $r \equiv s \pmod 2$.*

Proof. If $V_{\mathfrak{gl}_6}(\lambda)$ contains $S^r(\mathbb{C}^6)$ of $\mathfrak{sp}_6$, then by Lemma 16(i), $r = \sum_{i=1}^3 \lambda_{2i-1} - \lambda_{2i}$. By Lemma 15(ii), it enough to check if $S^s(\mathbb{C}^2) \boxtimes V_{\mathfrak{gl}_6}(\lambda)$ is a submodule of $V_{\mathfrak{gl}_8^{\mathrm{F}}}(a, a, b, b, c, c, d, d)$. In other words, we need to find the values of s such that the LR number $c^{(a,a,b,b,c,c,d,d)}_{(s+f,f),\lambda} \neq 0$ where f is arbitrary. The lemma follows from a direct calculation. □

Proposition 18. *The generic principal series representations of $\mathrm{F}_{4,4}$ are not quotients of the minimal representations of $\mathrm{E}_{8,8}$.*

Proof. Indeed there are three families of generic principal series representations and they contain the K_1-types $S^4(\mathbb{C}^2) \boxtimes \mathbb{C}$, $S^2(\mathbb{C}^2) \otimes \mathbb{C}$ and $S^5(\mathbb{C}^2) \boxtimes \mathbb{C}^6$ respectively. On the other by Lemma 17 these K_1-types do not appear in the the restriction of the minimal representation. □

We will begin the proof of Lemma 2(i). We need to show that the representation $S^2(\mathbb{C}^2)$ of $\mathfrak{sl}_{2,s}$ does not appear in $(V_{\mathfrak{so}_{16}}(n\varpi_8))^{\mathfrak{sl}^{\mathrm{G}}_{2,l} + \mathfrak{sl}^{\mathrm{F}}_{2,l} + \mathfrak{sp}_6}$. First of all, by the standard branching rules for $\mathfrak{so}_{16} \downarrow \mathfrak{so}_{15} \downarrow \dots \downarrow \mathfrak{so}_4 = \mathfrak{sl}^{\mathrm{G}}_{2,l} + \mathfrak{sl}^{\mathrm{F}}_{2,l}$, the space of invariants $(V_{\mathfrak{so}_{16}}(n\varpi_8))^{\mathfrak{sl}^{\mathrm{G}}_{2,l} + \mathfrak{sl}^{\mathrm{F}}_{2,l}}$ is zero if n is odd. Hence we will assume that n is even. Next, by Lemma 17,

$$(V_{\mathfrak{so}_{16}}(n\varpi_8))^{\mathfrak{sl}^{\mathrm{G}}_{2,l} + \mathfrak{sl}^{\mathrm{F}}_{2,l} + \mathfrak{sp}_6} = (V_{\mathfrak{so}_{16}}(n\varpi_8))^{\mathfrak{gl}^{\mathrm{G}}_2 + \mathfrak{sp}_6}.$$

Here $\mathfrak{gl}^{\mathrm{G}}_2 = \mathfrak{gl}_1 + \mathfrak{sl}^{\mathrm{G}}_{2,l}$ where $\mathfrak{gl}_1$ is the torus of $\mathfrak{sl}^{\mathrm{F}}_{2,l}$. By Lemma 16(i) $V_{\mathfrak{gl}_6}(\lambda)$ contains $1_{\mathfrak{sp}_6}$ if and only if $\lambda = (a, a, b, b, c, c)$. For such a λ, we have

$$1_{\mathfrak{sp}_6} \boxtimes 1_{\mathfrak{gl}^{\mathrm{G}}_2} \subset V_{\mathfrak{gl}_6}(\lambda) \boxtimes 1_{\mathfrak{gl}^{\mathrm{G}}_2} \subset V_{\mathfrak{gl}^{\mathrm{G}}_8}(a_1, a_2, a_2, a_3, a_3, a_4, a_4, a_5) \subset V_{\mathfrak{so}_{16}}(n\varpi_8). \tag{10}$$

Here $a, b, c, a_i \in \mathbb{Z}$. The first and the last containments are of multiplicity one due to Lemmas 16(i) and 15(ii) respectively.

The center $\mathfrak{gl}_1$ of $\mathfrak{gl}_6$ is the torus of $\mathfrak{sl}_{2,s}$ and it acts on $V_{\mathfrak{gl}_6}(\lambda)$ by $\sum_i \lambda_i$. We will show that the representation $S^2(\mathbb{C}^2)$ of $\mathfrak{sl}_{2,s}$ does not occur by showing that the dimensions of the weight 4 space and the weight (-2) space of $\mathfrak{sl}_{2,s}$ are the same on $V_{\mathfrak{so}_{16}}(n\varpi_8)^{\mathfrak{sl}^G_{2,l}+\mathfrak{sl}^F_{2,l}+\mathfrak{sp}_6}$. Now, $V_{\mathfrak{gl}_6}(\lambda)$ contributes to the weight 4 or -2 if $2a+2b+2c = 4$ or -2, respectively. Thus, in order to obtain the desired result it suffices to show the following.

Proposition 19. *Let $\lambda = (a, a, b, b, c, c)$ be such that $a+b+c = 2$ and let $m(a, b, c; a_1, a_2, \dots, a_5)$ be the multiplicity of the middle inclusion in* (10)*. Then*

$$m(a, b, c; a_1, a_2, \dots, a_5) = m(a-1, b-1, c-1; a_1, a_2, \dots, a_5).$$

Proof. In order to verify the proposition we need to calculate the multiplicities. This will be accomplished using the Littlewood-Richardson rule in the following Lemma.

Lemma 20. *Suppose $a+b+c = \pm1, \pm2$. Let $m = m(a, b, c; a_1, a_2, \dots, a_5)$ be the multiplicity of the middle inclusion in* (10)*. If $m \neq 0$, then it is necessary that $a_1 \geq a \geq b \geq c \geq -a_1$ and*

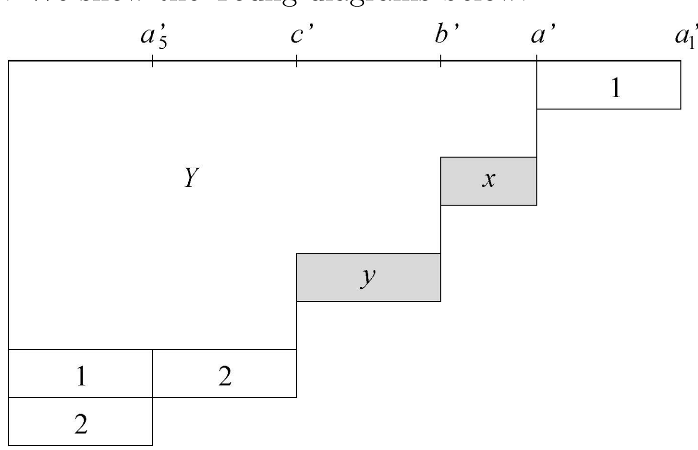
$$\lambda_2 = a_2,\ \lambda_4 = a_3,\ \lambda_6 = a_4,\ a_5 = -a_1. \tag{11}$$

Furthermore

		m
$b \geq 0$	$a-c \leq a_1$	$a-b+1$
	$a-c \geq a_1$	$a_1-b+c+1$
$b \leq 0$	$a-c \leq a_1$	$b-c+1$
	$a-c \geq a_1$	$a_1-a+b+1$

If m takes negative values in the above table, then we set $m = 0$.

Proof. In order to apply the LR rule we twist representations with $\det^{\frac{n}{2}}$. Let $a'_i = a_i + \frac{n}{2}$, $a' = a + \frac{n}{2}$, $b' = b + \frac{n}{2}$, $c' = c + \frac{n}{2}$. First we place the Young diagram Y of $V_{\mathfrak{gl}_6}(a', a', b', b', c', c')$ into that of $V_{\mathfrak{gl}_8}(a'_1, a'_2, a'_2, \dots, a'_5)$. Next we will fill the remaining spaces with $\frac{n}{2}$ copies of 1-boxes and 2-boxes. We show the Young diagrams below:



Without going into the details, LR rule shows that $a = a_2$, $b = a_3$, $c = a_4$, $a_5 = -a_1$. It remains to fill the shaded region x and y with $\frac{n}{2} - c' = -c$ 2-boxes.

First suppose $b \geq 1$. Since $a + b + c = \pm 1, \pm 2$, we have

$$\text{Length of } y \geq -c \geq \text{Length of } x$$

By the LR rule, the multiplicity m is

$$\min(a' - b', a'_1 - a', a'_1 - b' + c) + 1 = \min(a - b + 1, a_1 - a + 1, a_1 - b + c + 1).$$

Since $a + b + c \leq 2$, $a_1 - a + 1 \geq a_1 - b + c$. This proves the case $b \geq 1$.

Next by sending $(a, b, c) \mapsto (-c, -b, -a)$ and $a_i \mapsto a_{5-i+1}$, we pass from the case $b \geq 1$ to $b \leq -1$.

The proof of the case $b = 0$ is similar and easier. The lemma is proved. □

The proof of proposition is now identical to the proof of Proposition 11. We leave details to the reader. □

7. Proof of Lemma 2(ii) for E_8

We continue with the notations in §6.1. The proof is almost identical with the one in §6 but more tedious. We need show that the representations $\mathbb{C}^2 \boxtimes \mathbb{C}^6$ and $S^2(\mathbb{C}^2) \boxtimes 1_{\mathfrak{sp}_6}$ of $\mathfrak{sl}_{2,l}^{\mathrm{F}} + \mathfrak{sp}_6$ does not appear in $V_{\mathfrak{so}_{16}}(n\varpi_8)^{\mathfrak{sl}_{2,l}^{\mathrm{G}} + \mathfrak{sl}_{2,s}}$. The statement for $S^2(\mathbb{C}^2) \boxtimes 1_{\mathfrak{sp}_6}$ follows immediately from Lemma 17. We are now left with $\mathbb{C}^2 \boxtimes \mathbb{C}^6$.

Once again, by the standard branching rules for $\mathfrak{so}_{16} \downarrow \mathfrak{so}_{15} \downarrow \ldots \downarrow \mathfrak{so}_4 = \mathfrak{sl}_{2,l}^{\mathrm{G}} + \mathfrak{sl}_{2,l}^{\mathrm{F}}$, $\mathrm{Hom}_{\mathfrak{sl}_{2,l}^{\mathrm{G}} + \mathfrak{sl}_{2,l}^{\mathrm{F}}}(1_{\mathfrak{sl}_{2,l}^{\mathrm{G}}} \boxtimes \mathbb{C}^2, V_{\mathfrak{so}_{16}}(n\varpi_8))$ is zero unless n is odd. Hence we will assume that n is odd. Next, by Lemma 17,

$$\mathrm{Hom}_{\mathfrak{sp}_6 + \mathfrak{sl}_{2,l}^{\mathrm{G}} + \mathfrak{sl}_{2,l}^{\mathrm{F}}}(\mathbb{C}^6 \boxtimes 1_{\mathfrak{sl}_{2,l}^{\mathrm{G}}} \boxtimes \mathbb{C}^2, V_{\mathfrak{so}_{16}}(n\varpi_8)) = \mathrm{Hom}_{\mathfrak{sp}_6 + \mathfrak{gl}_2^{\mathrm{G}}}(\mathbb{C}^6 \boxtimes \mathbb{C}_{-\frac{1}{2}}, V_{\mathfrak{so}_{16}}(n\varpi_8))$$

where $\mathbb{C}_{-1/2} = V_{\mathfrak{gl}_2}(-1/2, -1/2)$ is a one-dimensional representation of $\mathfrak{gl}_2^{\mathrm{G}}$. We recall Lemma 16(ii) that $V_{\mathfrak{gl}_6}(\lambda)$ contains $\mathbb{C}^6$ of $\mathfrak{sp}_6$ if and only if λ is of the form

$$\begin{aligned} \lambda_I &= (a + \tfrac{1}{2}, a - \tfrac{1}{2}, b, b, c, c), \\ \lambda_{II} &= (a, a, b + \tfrac{1}{2}, b - \tfrac{1}{2}, c, c) \quad \text{or} \\ \lambda_{III} &= (a, a, b, b, c + \tfrac{1}{2}, c - \tfrac{1}{2}). \end{aligned}$$

We have inserted the '$\frac{1}{2}$'s so that there is more symmetry in our calculations. For $\lambda = \lambda_I$, λ_{II} or λ_{III}, we have by Lemma 15(ii),

$$\mathbb{C}^6 \boxtimes \mathbb{C}_{-1/2} \subset V_{\mathfrak{gl}_6}(\lambda) \boxtimes \mathbb{C}_{-1/2} \subset V_{\mathfrak{gl}_8^{\mathrm{G}}}(a_1, a_2, a_2, a_3, a_3, a_4, a_4, a_5) \subset V_{\mathfrak{so}_{16}}(n\varpi_8). \tag{12}$$

where λ and a_i lie in $\frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$. The first and the last containments are of multiplicity one due to Lemma 16(ii) and 15(ii) respectively.

The center $\mathfrak{gl}_1$ of $\mathfrak{gl}_6$ is the torus of $\mathfrak{sl}_{2,s}$ and it acts on $V_{\mathfrak{gl}_6}(\lambda)$ by $\sum_i \lambda_i$. We will prove that the trivial representation of $\mathfrak{sl}_{2,s}$ does not occur by showing that the dimensions of the 0-eigenspace and the 2-eigenspace of $\mathfrak{gl}_1$ are the same. The action of $\mathfrak{gl}_1$ implies that $2a + 2b + 2c = 0$ or 2, that is, $a + b + c = 0$ or 1.

Let $m = m(a, b, c; a_1, a_2, \ldots, a_5)$ denote multiplicity of the middle inclusion in (12). There is no ambiguity as to whether we are using $\lambda = \lambda_I$, λ_{II} or λ_{III} in the definition of m. This is because for λ_I (resp. λ_{II}, λ_{III}), the entry a (resp. b, c) is an integer while the rest of the entries are odd multiples of $\frac{1}{2}$.

Lemma 21. *Let $\lambda = \lambda_I, \lambda_{II}$ or λ_{III} and $m = m(a, b, c; a_1, a_2, \ldots, a_5)$ as above. Suppose $a + b + c = -1, 0$ or 1. If $m \neq 0$, then it is necessary that all the entries in λ lies in the closed interval $[-a_1, a_1]$ and*

$$\lambda_2 = a_2,\ \lambda_4 = a_3,\ \lambda_6 = a_4,\ a_5 = -a_1. \tag{13}$$

Furthermore

λ			m
λ_I	$b \geq \frac{1}{2}$	$a - c \leq a_1$	$a - b + \frac{1}{2}$
		$a - c \geq a_1$	$a_1 - b + c + \frac{1}{2}$
	$b \leq -\frac{1}{2}$	$a - c \leq a_1$	$b - c + 1$
		$a - c \geq a_1$	$a_1 - a + b + 1$
λ_{II}	$b \geq 1$ *or* ($b = 0$ *and* $a = -c$)	$a - c - \frac{1}{2} \leq a_1$	$a - b + \frac{1}{2}$
		$a - c - \frac{1}{2} \geq a_1$	$a_1 - b + c + 1$
	$b \leq 0$	$a - c - \frac{1}{2} \leq a_1$	$b - c + \frac{1}{2}$
		$a - c - \frac{1}{2} \geq a_1$	$a_1 - a + b + 1$
λ_{III}	$b \geq \frac{1}{2}$	$a - c \leq a_1$	$a - b + 1$
		$a - c \geq a_1$	$a_1 - b + c + 1$
	$b \leq -\frac{1}{2}$	$a - c \leq a_1$	$b - c + \frac{1}{2}$
		$a - c \geq a_1$	$a_1 - a + b + \frac{1}{2}$

If m takes negative values in the above table, then we set $m = 0$.

Proof. First we observe a symmetry. By sending $(a, b, c) \mapsto (-c, -b, -a)$ and $a_i \mapsto a_{5-i+1}$, we send λ_I to λ_{III}, and $(\lambda_{II}, b \geq 1)$ to $(\lambda_{II}, b \leq -1)$. The multiplicity m remains unchanged by these transformations.

We will now prove the lemma for λ_I which implies the lemma for λ_{III} by the symmetry.

Again, in order to apply the LR rule we twist representations with $\det^{\frac{n}{2}}$. Thus, let $a_i' = a_i + \frac{n}{2}$, $a' = a + \frac{n-1}{2}$, $b' = b + \frac{n}{2}$, $c' = c + \frac{n}{2}$ and $x' = x + \frac{n}{2} = \frac{n-1}{2}$. First we place the Young diagram Y of $V_{\mathfrak{gl}_6}(a'+1, a', b', b', c', c')$ into that of $V_{\mathfrak{gl}_8}(a_1', a_2', a_2', \ldots, a_5')$. Next we will fill the remaining spaces with $\frac{n-1}{2}$ copies of 1-boxes and 2-boxes. The Young diagram is almost identical to the one in the proof of Lemma 20 except that Y has one more box in the first row. A check on the diagram shows that $a = a_2$, $b = a_3$, $c = a_4$, $a_5 = -a_1$. It remains to fill the shaded region x and y with $\frac{n-1}{2} - c' = -\frac{1}{2} - c$ 2-boxes.

Suppose $b \geq \frac{1}{2}$. Since $a + b + c = 0, \pm 1$, we have

$$\text{Length of } y \geq -\frac{1}{2} - c \geq \text{Length of } x$$

The multiplicity is nonzero if and only if $a_1' - b' - 1 \geq -\frac{1}{2} - c$, that is, $b - c \leq a_1 - \frac{1}{2}$. If it is nonzero, then it equals

$$\min(a - b - \frac{1}{2}, a_1' - a - \frac{n+1}{2}, a_1' - \frac{n+1}{2} - b + c) + 1.$$

Finally one checks that the second term is greater or equal to the third term.

The proof is similar for $b \leq -\frac{1}{2}$. This proves the lemma for λ_I.

We will not prove the lemma for λ_{II} but we will give an outline. First we consider $b \leq 0$. Next we apply the symmetry to get $b \geq -1$. Note that the symmetry fails to produce the formula for $b = 0$. □

Proof of Lemma 2(ii) **for** E$_8$**.** Fix $a_1 \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$. Using (13) we set

$$\mu(a, b, c) := m(a, b, c; a_1, a_2 = \lambda_2, a_3 = \lambda_4, a_4 = \lambda_6, a_5 = -a_1).$$

We note that $a_1 \geq a \geq b \geq c \geq -a_1$. For any other values a, b, c where the inequalities does not hold, we set $\mu(a, b, c) = 0$. Let

$$S = \left\{(\alpha, \beta, \gamma) \in (\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z})^3 : a_1 \geq \alpha \geq \beta \geq \gamma \geq -a_1, \alpha + \beta + \gamma = \frac{1}{2}\right\}$$

For $(\alpha, \beta, \gamma) \in S$, we define three differences

$$\begin{aligned} d_1 &= \mu(\alpha + \tfrac{1}{2}, \beta, \gamma) - \mu(\alpha - \tfrac{1}{2}, \beta, \gamma), \\ d_2 &= \mu(\alpha, \beta + \tfrac{1}{2}, \gamma) - \mu(\alpha, \beta - \tfrac{1}{2}, \gamma), \\ d_3 &= \mu(\alpha, \beta, \gamma - \tfrac{1}{2}) - \mu(\alpha, \beta, \gamma - \tfrac{1}{2}). \end{aligned}$$

Lemma 22. $d_1 + d_2 + d_3 = 0$.

Proof. The lemma is an immediate consequence of the following table.

			d_1	d_2	d_3
$\beta \geq \frac{1}{2}$	$\alpha - \gamma \leq a_1 - \frac{1}{2}$		1	-1	0
	$\alpha - \gamma \geq a_1 + \frac{1}{2}$	$a_1 \leq \beta - \gamma - \frac{3}{2}$	0	0	0
		$a_1 \geq \beta - \gamma - \frac{1}{2}$	0	-1	1
$\beta \leq -\frac{1}{2}$	$\alpha - \gamma \leq a_1 - \frac{1}{2}$		0	1	-1
	$\alpha - \gamma \geq a_1 + \frac{1}{2}$	$a_1 \leq \alpha - \beta - \frac{3}{2}$	0	0	0
		$a_1 \geq \alpha - \beta - \frac{1}{2}$	-1	1	0

The proof of the table follows from the tedious case by case checking by hand. We will leave this to the reader. □

Lemma 22 implies that the 2-eigenspace and the 0-eigenspace of torus of $\mathfrak{sl}_{2,s}$ has the same dimension, that is, the trivial representation of $\mathfrak{sl}_{2,s}$ does not occur. This proves Lemma 2(ii). □

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