THE SMALLEST REPRESENTATIONS OF NON-LINEAR COVERS OF ODD ORTHOGONAL GROUPS

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Abstract. We construct the smallest genuine representations of a non-linear covers of the group $\mathrm{SO}^0(p,q)$ where $p+q$ is odd. We determine correspondences of infinitesimal characters arising from restricting the smallest representations to dual pairs $\mathfrak{so}(p,a) \oplus \mathfrak{so}(b)$ where $a+b=q$.

1. Introduction

Let p and q be two positive integers ≥ 2, and let $G_{p,q}$, or simply G, be the central extension of $\mathrm{SO}^0(p,q)$ such that the maximal compact subgroup is $K = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$. This extension is the universal central extension if $p, q \neq 2$. Assume now that $p+q$ is odd. In particular, p and q have different parity. Without any loss of generality we shall assume that p is odd and q even. Let Z_G be the center of G. Then

$$Z_G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

However, there exists a unique element of order 2 in Z_G, denoted simply by -1, such that $G/\langle -1 \rangle$ is a linear group. An irreducible representation π of G is called genuine if $-1 \in Z_G$ acts as multiplication by -1 on π. The first part of this paper is devoted to constructing and establishing properties of one genuine representation V of the group G if $p-1<q$, two genuine representations V^+ and V^- of G if $p-1>q$, and four genuine representations V^+, V^-, V_o^+ and V_o^- if $p-1=q$. If $p-1=q$ then the group G is split with the absolute root system B_q. In the special case when $q=2$ and $p=3$ then G is isomorphic to the metaplectic group $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, and the four representations are irreducible components of two oscillator representations. Thus, our representations can be viewed as a natural generalization of the oscillator representation to odd orthogonal groups.

The first step in the construction is an explicit description of K-types of these representations. To this end, we need to recall a description of irreducible representations of $\mathrm{Spin}(n)$. Let $\Lambda(n)$ be the set of all highest weight of finite dimensional representations. We realize $\Lambda(n)$ as in Bourbaki [Bo], see Section 2 of this paper. In particular, any highest weight μ is given by

$$\mu = (x_1, \ldots, x_{[\frac{n}{2}]})$$

where x_i are either all integers or half-integers. Let -1_n be the unique element in $\mathrm{Spin}(n)$ such that $\mathrm{Spin}(n)/\langle -1_n \rangle \cong \mathrm{SO}(n)$. We can divide all irreducible representations of

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Spin(n) into two classes, depending whether -1_n acts as 1 or -1. This corresponds to writing

$$\Lambda(n) = \Lambda(n, 0) \cup \Lambda(n, \frac{1}{2})$$

where $\Lambda(n,0)$ consists of integral and $\Lambda(n, \frac{1}{2})$ of half-integral highest weights. Let τ_n^μ denote the irreducible representation of Spin(n) with the highest weight μ. Let $\mathcal{Z}_n$ be the center of the enveloping algebra of Spin(n). Recall that the infinitesimal character χ defines a ring homomorphism $\chi : \mathcal{Z}_n \to \mathbb{C}$.

We are now ready to describe the K-types of our representations. Assume that $p-1 < q$. In Section 2, we define a surjective homomorphism

$$j : \mathcal{Z}_q \to \mathcal{Z}_p.$$

Let χ be the infinitesimal character of an irreducible representation τ of Spin(p). Then the composition $\chi \circ j$ is an infinitesimal character of Spin(q), but not necessarily corresponding to a finite dimensional representation of Spin(q). More precisely, for every λ in $\Lambda(p, 0)$ define

$$\begin{cases} A(\lambda) = \lambda + \frac{(q-p)}{2}(1, \ldots, 1) \in \Lambda(p, \frac{1}{2}) \\ B(\lambda) = (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}, 0, \ldots, 0) \in \Lambda(q, 0). \end{cases}$$

Then $\chi \circ j$ is an infinitesimal character of a finite dimensional representation of Spin(q) if and only if the highest weight of the irreducible representation τ of Spin(p) is $A(\lambda)$ for some λ in $\Lambda(p, 0)$. If that is the case, then $\chi \circ j$ is the infinitesimal character of the irreducible representation of Spin(q) with the highest weight $B(\lambda)$. We now set V- a potential $(\mathfrak{g}, K)$-module - to be the K-module

$$V = \bigoplus_{\lambda \in \Lambda(p,0)} \tau_p^{A(\lambda)} \otimes \tau_q^{B(\lambda)}. \tag{1}$$

Since $-1 \in Z_G$ is given by $(-1_p, -1_q) \in K$ we see that V must correspond to a genuine representation of G, once we have defined an action of $\mathfrak{so}(p, q)$, the Lie algebra of G, on V.

If $p - 1 \geq q$ then the map j goes in the opposite direction. The main difference here lies in the fact that j is not surjective anymore. In particular, two different infinitesimal characters of Spin(q) pull back to the same infinitesimal character of Spin(p). As a consequence, we can construct two potential $(\mathfrak{g}, K)$-modules denoted by V^+ and V^-. The story has an additional twist if $p-1 = q$ enabling us to write down four potential modules in all, here. We refer the reader to Section 2 for details.

The structure of K-types is similar to the structure of K-types of representations of $\mathrm{SO}^0(p, q)$, where $p + q$ is even, that are local theta lifts of one dimensional unitary characters of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ (see [KO], [HL] and [ZH]).

From the explicit description of K-types it is not too difficult to determine the associated variety of V. Indeed, consider the nilpotent orbit of $\mathrm{SO}_{p+q}(\mathbb{C})$ corresponding to the partition $(2^{p-1}, 1^{q-p+2})$. It has a (unique) real form $\mathcal{O}_{2^{p-1}}$ for the group $\mathrm{SO}^0(p, q)$. Let

$\mathcal{O}^K_{2^{p-1}}$ be the $K_{\mathbb{C}}$-orbit corresponding to $\mathcal{O}_{2^{p-1}}$ via the Kostant-Sekiguchi correspondence. We have the following:

Theorem 1.1. *Recall that* $K = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$ *with* p *odd and* q *even. Suppose* $p-1 < q$. *The* K*-module* V *extends to an irreducible and unitarizable* $(\mathfrak{so}(p,q), K)$*-module. Moreover:*

(i) The infinitesimal character of V *is*

$$\mu_{p,q} = \left(\frac{p-1}{2}, \frac{p-3}{2}, \dots, 1, \frac{q-1}{2}, \frac{q-3}{2}, \dots, \frac{1}{2}\right).$$

(ii) The associated variety of V *is the closure of* $\mathcal{O}^K_{2^{p-1}}$.

(iii) The annihilator of V *in the enveloping algebra is the unique maximal ideal* $J_{\max}$ *with the infinitesimal central character* $\mu_{p,q}$.

(iv) The module V *is the unique* $(\mathfrak{so}(p,q), K)$*-module with the* K*-types as in* (1).

Here we remark that $\mathfrak{so}(p,q)$ *is the real Lie algebra of* $G_{p,q}$. *The complexification of* $\mathfrak{so}(p,q)$ *will be denoted by* $\mathfrak{so}_{p+q}(\mathbb{C})$.

It follows, from a result of Schimd and Vilonen [SV], that the wave front set of V is the real orbit $\mathcal{O}_{2^{p-1}}$. This, combined with a result of Huang and Li [HL], justifies the use of the attribute *smallest* in the title of our paper.

Analogous results hold for V^+, V^- and V_o^+, V_o^-. It must be noted, however, that our results overlap with some already existing in the literature. For example, if $p = 3$ then V is the minimal representation of $G_{3,q}$ constructed in [Sa], [To] and [BKo]. If $p-1 \geq q$ then representations V^+ and V^- were constructed by Knapp [Kn] and further studied by Trapa [T] by methods of cohomological induction. Our method is based on a simple observation that V is admissible for $\mathrm{Spin}(p)$. Such phenomenon is called *discretely decomposable restriction* in [Ko2]. In particular, the restriction of V to $\mathfrak{so}(p,1)$ decomposes as a direct sum of irreducible representations. We exploit this observation to define an explicit action of $\mathfrak{so}(p,1)$ on V. This then defines an action of $\mathfrak{so}(p,q)$ on V because $\mathfrak{so}(p,q)$ is generated by $\mathfrak{so}(p,1)$ and $\mathfrak{so}(q)$.

We then extend V to a $(\mathfrak{so}(p,q), \mathrm{Spin}(p) \times \mathrm{O}(q))$-module. This extension is needed for the second part of this paper which is devoted to dual pair correspondences arising from restricting V to dual pairs

$$\mathfrak{so}(p,a) \times \mathrm{O}(b), \quad a+b=q.$$

Using our explicit description of V we can show that the Theta-lift of any finite dimensional irreducible representation of $\mathrm{O}(b)$ is irreducible. See Theorem 9.1 and Remark 9.2. We build on this to establish a correspondence of infinitesimal characters. Of course, if $a = 0$ and $b = p$ then the correspondence of infinitesimal characters is given by j. In order to describe a general result let

$$\rho_n = \frac{1}{2}(n-2, n-4, n-6, \ldots) \in \Lambda(n) \tag{2}$$

denote the half sum of positive roots of $\mathfrak{so}(n)$. Given $\beta = (\beta_1, \ldots, \beta_r)$ and $\gamma = (\gamma_1, \ldots, \gamma_s)$, we will denote $(\beta_1, \ldots, \beta_r, \gamma_1, \ldots, \gamma_s)$ by (β, γ) if there is no fear of confusion. Let

$$\delta_{p,q-1} = (\rho_p, \rho_q).$$

We remark here that $\delta_{p,q-1}$ is the infinitesimal character of two (smallest) genuine representations of $G_{p,q-1}$ obtained as Theta-lifts of the trivial and the sign representations of O(1). (Compare this with (7.2.1) in [KO] for the ladder representations of the even orthogonal groups.) This statement is in essence a special case ($b = 1$) of the following theorem.

Theorem 1.2. *Assume that $a + b = q$. The representation V establishes the following correspondence of infinitesimal characters for the dual pair $\mathfrak{so}(p, a) \times \mathfrak{so}(b)$:*

$$\begin{cases} (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}, \rho_{a-1}) \longleftrightarrow (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}, \rho_{b-p+1}) \text{ if } b \geq p. \\ (\lambda_1, \ldots, \lambda_{\frac{b}{2}}, \mu_{p-b,q-b}) \longleftrightarrow (\lambda_1, \ldots, \lambda_{\frac{b}{2}}) \text{ if } b < p \text{ and } b \text{ is even.} \\ (\lambda_1, \ldots, \lambda_{\frac{b-1}{2}}, \delta_{p-b+1,q-b}) \longleftrightarrow (\lambda_1, \ldots, \lambda_{\frac{b-1}{2}}) \text{ if } b < p \text{ and } b \text{ is odd.} \end{cases}$$

Warning: We are not claiming here that the correspondence of infinitesimal characters is one to one. For example, if b is even and $b < p$ then the infinitesimal characters $(\lambda_1, \ldots, \lambda_{\frac{b}{2}})$ and $(\lambda_1, \ldots, -\lambda_{\frac{b}{2}})$ of $\mathfrak{so}(b)$ correspond to the same infinitesimal character of $\mathfrak{so}(p, a)$.

This paper is motivated by two beautiful papers of Bump, Friedberg and Ginzburg [BFG1] and [BFG2] where, for split groups, a p-adic version of V is constructed. In particular, the construction of K-types using the correspondence of infinitesimal characters induced by j is a real analogue of the correspondence of Satake parameters obtained in [BFG2].

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2. Potential $(\mathfrak{g}, K)$-modules

We use the standard realization of root systems of classical groups as in Bourbaki [Bo].

Representations of Spin(p) **where** p **is odd.** Let -1_p be the unique element in Spin(p) such that Spin(p)$/\langle -1_p \rangle \cong$ SO(p). Recall that the highest weight of an irreducible finite-dimensional representation of Spin(p) is given by

$$\lambda = (x_1, \ldots, x_{\frac{p-1}{2}})$$

where $x_i \in \frac{1}{2}\mathbb{Z}$, $x_1 \geq \ldots \geq x_{\frac{p-1}{2}} \geq 0$ and $x_1 \equiv \ldots \equiv x_{\frac{p-1}{2}} \pmod{\mathbb{Z}}$. The corresponding irreducible representation is denoted by τ_p^λ. Let $\Lambda(p)$ be the set of all highest weights. We can write

$$\Lambda(p) = \Lambda(p,0) \cup \Lambda(p,\frac{1}{2})$$

where if $\lambda \in \Lambda(p,e)$ then $\lambda_i \in e + \mathbb{Z}$. Note that -1_p acts as 1 on τ_p^λ if and only if λ is in $\Lambda(p,0)$. The center $\mathcal{Z}_p$ of the enveloping algebra is equal to

$$\mathcal{Z}_p \cong \mathbb{C}[\omega_1, \omega_2, \ldots, \omega_{\frac{p-1}{2}}]$$

where ω_k is the k-th symmetric function in $x_1^2, \ldots, x_{\frac{p-1}{2}}^2$. The value of an element ω in $\mathcal{Z}_p$ on the irreducible representation τ_p^λ is given by evaluating the polynomial ω on $\lambda + \rho_p$ where where ρ_p is the half sum of positive roots as defined in (2).

Representations of $\mathrm{Spin}(q)$ **where** q **is even.** Let -1_q be the unique element in $\mathrm{Spin}(q)$ such that $\mathrm{Spin}(q)/\langle -1_q \rangle \cong \mathrm{SO}(q)$. Recall that the highest weight of an irreducible finite-dimensional representation of $\mathrm{Spin}(q)$ is given by

$$\lambda = (x_1, \ldots, x_{\frac{q}{2}})$$

where $x_i \in \frac{1}{2}\mathbb{Z}$, $x_1 \geq \ldots \geq x_{\frac{q-2}{2}} \geq |x_{\frac{q}{2}}|$ and $x_1 \equiv \ldots \equiv x_{\frac{q}{2}} \pmod{\mathbb{Z}}$. The corresponding irreducible representation is denoted by τ_q^λ. Let $\Lambda(q)$ be the set of all highest weights. We can write

$$\Lambda(q) = \Lambda(q,0) \cup \Lambda(q,\frac{1}{2})$$

where if $\lambda \in \Lambda(q,e)$ then $\lambda_i \in e + \mathbb{Z}$. Note that -1_q acts as 1 on τ_q^λ if and only if λ is in $\Lambda(q,0)$.

The center $\mathcal{Z}_q$ of the enveloping algebra is equal to

$$\mathcal{Z}_q \cong \mathbb{C}[\omega_1, \ldots, \omega_{\frac{q-2}{2}}, \upsilon_{\frac{q}{2}}]$$

where ω_k is the k-th symmetric function in $x_1^2, \ldots, x_{\frac{q}{2}}^2$, and $\upsilon_{\frac{q}{2}} = x_1 \cdot \ldots \cdot x_{\frac{q}{2}}$. The value of the element ω in $\mathcal{Z}_q$ on the irreducible representation τ_q^λ of $\mathfrak{so}(q)$ with the highest weight λ is equal to $\omega(\lambda + \rho_q)$ where ρ_q is the half sum of positive roots as defined in (2).

We are now ready to define K-modules in the three cases, as follows:

Case 1: $p - 1 < q$. We have a surjective map $j : \mathcal{Z}_q \to \mathcal{Z}_p$ given by $j(\omega_k) = \omega_k$ for $k = 1, \ldots, \frac{p-1}{2}$ and $j = 0$ on remaining generators of $\mathcal{Z}_q$. For every $\lambda = (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}})$ in $\Lambda(p,0)$, we define

$$\begin{cases} A(\lambda) = \lambda + \frac{q-p}{2}(1, \ldots, 1) \in \Lambda(p, \frac{1}{2}) \text{ and} \\ B(\lambda) = (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}, 0, \ldots, 0) \in \Lambda(q,0) \end{cases} \tag{3}$$

where there are $\frac{q-p+1}{2}$ copies of 0 in $B(\lambda)$. Notice that the infinitesimal characters of $\tau_p^{A(\lambda)}$ and $\tau_q^{B(\lambda)}$ are matched by j, that is, for every ω in $\mathcal{Z}_q$

$$j(\omega)(A(\lambda)+\rho_p)=\omega(B(\lambda)+\rho_q).$$

Moreover, using the explicit description of j, one easily checks that there are no other pairs of representations with matching infinitesimal characters. Thus it is natural to consider

$$V=\bigoplus_{\lambda\in\Lambda(p,0)}\tau_p^{A(\lambda)}\otimes\tau_q^{B(\lambda)}$$

which is a representation of the compact Lie group $\mathrm{Spin}(p)\times\mathrm{SO}(q)$. In this way we have constructed K-types of V.

Case 2: $p-1\geq q$. We have a surjective map $j:\mathcal{Z}_p\to\mathcal{Z}_q$ given by $j(\omega_k)=\omega_k$ for $k=1,\ldots,\frac{q-2}{2}$, $j(\omega_{\frac{q}{2}})=v_{\frac{q}{2}}^2$ and $j=0$ on remaining generators of $\mathcal{Z}_q$, if any. Note that j is not surjective here. Indeed, consider the involution σ of $\Lambda(q)$ (and of $\mathcal{Z}_q$) defined by defined by

$$\sigma(x_1,\ldots,x_{\frac{q-2}{2}},x_{\frac{q}{2}})=(x_1,\ldots,x_{\frac{q-2}{2}},-x_{\frac{q}{2}}).$$

Then the image of $\mathcal{Z}_p$ is equal to the subalgebra of σ-invariant polynomials in $\mathcal{Z}_q$. In particular, two representations of $\mathrm{Spin}(q)$ will be matched with one representation of $\mathrm{Spin}(q)$. More precisely, for every λ in $\Lambda(q+1,0)$ define

$$\begin{cases}A(\lambda)=(\lambda,0,\ldots,0)\in\Lambda(p,0)\\ B^+(\lambda)=\lambda+\frac{p-q}{2}(1,\ldots,1)\in\Lambda(q,\frac{1}{2})\\ B^-(\lambda)=\sigma(B^+(\lambda))\in\Lambda(q,\frac{1}{2}).\end{cases}$$

Then the infinitesimal characters of $\tau_q^{B^+(\lambda)}$ and $\tau_q^{B^-(\lambda)}$ are matched with the infinitesimal character of $\tau_p^{A(\lambda)}$. Moreover, if $p-1>q$, there are no other matching pairs of representations of $\mathrm{Spin}(p)$ and $\mathrm{Spin}(q)$. We can now define V^+ and V^-, two representations of the compact Lie group $\mathrm{SO}(p)\times\mathrm{Spin}(q)$, by

$$V^{\pm}=\bigoplus_{\lambda\in\Lambda(q+1,0)}\tau_p^{A(\lambda)}\otimes\tau_q^{B^{\pm}(\lambda)}. \tag{4}$$

The separation of K-types into V^+ and V^- is natural since $V^+\otimes\mathfrak{p}$ and V^- contain no K-type in common if $p-1>q$. (Here $\mathfrak{p}=\mathfrak{p}_0\otimes\mathbb{C}$ and $\mathfrak{so}(p,q)=\mathfrak{k}_0\oplus\mathfrak{p}_0$ is the Cartan decomposition.)

Case 3: $p-1=q$. Finally, if $p-1=q$, there are additional two families of matching pairs of representations of $\mathrm{Spin}(p)$ and $\mathrm{SO}(p-1)$. For every $\lambda=(\lambda_1,\ldots,\lambda_{\frac{p-1}{2}})$ in $\Lambda(p,\frac{1}{2})$ define

$$\begin{cases}B^+(\lambda)=\lambda+\frac{1}{2}(1,\ldots,1)\in\Lambda(p-1,0)\\ B^-(\lambda)=\sigma(B^+(\lambda))\in\Lambda(p-1,0).\end{cases}$$

Then

$$V_o^{\pm} = \bigoplus_{\lambda \in \Lambda(p,\frac{1}{2})} \tau_p^{\lambda} \otimes \tau_{p-1}^{B^{\pm}(\lambda)}.$$

3. Representations of $\mathrm{Spin}(n,1)$

In this section we review some facts about representations of the group $\mathrm{Spin}(n,1)$. The maximal compact subgroup is $\mathrm{Spin}(n)$. We identify the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{Spin}(n)$ with the set of real skew-symmetric n by n matrices. Let E_{kl} be the elementary n by n matrix which is 1 at the kl-th entry and zero elsewhere. Let

$$I_{kl} = E_{lk} - E_{kl}.$$

We need to fix some additional notation. As in the previous section, let $\Lambda(n,e)$, where $e = 0, \frac{1}{2}$, denote the set of highest weights $\lambda = (\lambda_1, \ldots, \lambda_{[n/2]})$ of $\mathfrak{so}(n)$ where $\lambda_i \in \mathbb{Z} + e$. Hence the set of highest weights is $\Lambda(n) = \Lambda(n,0) \cup \Lambda(n,\frac{1}{2})$. If n is even, then we define $\sigma(\lambda) = (\lambda_1, \ldots, \lambda_{n/2-1}, -\lambda_{n/2})$. Let

$$\mathbf{1}_{[n/2]} = (1, \ldots, 1) \text{ and } \mathbf{0}_{[n/2]} = (0, \ldots, 0)$$

where there are $[\frac{n}{2}]$ copies of 1's and 0's respectively. We set $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 appears at the i-th position.

Gelfand-Zetlin basis. We now define the Gelfand-Zetlin basis of a finite dimensional representation of $\mathfrak{so}(n)$. Our main references are [VK] and [Zhe].

Given $\lambda = (\lambda_1, \ldots, \lambda_{[n/2]}) \in \Lambda(n,e)$, let τ_n^{λ} denote the irreducible representation of $\mathfrak{so}(n)$ with highest weight λ as in the introduction. We will equipped it with an $\mathfrak{so}(n)$-invariant Hermitian inner product. It is well known that the restriction of τ_n^{λ} to $\mathfrak{so}(n-1)$ is multiplicity free and

$$\tau_n^{\lambda} = \bigoplus_{\lambda \succ \lambda^1} \tau_{n-1}^{\lambda^1}$$

where $\lambda^1 = (\lambda^1_1, \ldots, \lambda^1_{[(n-1)/2]}) \in \Lambda(n-1,e)$ and $\lambda \succ \lambda^1$ is defined as follows. There are two cases:

- If n is odd then $[n/2] = (n-1)/2$ and $\lambda \succ \lambda^1$ is defined by
$$\lambda_1 \geq \lambda^1_1 \geq \lambda_2 \geq \lambda^1_2 \geq \ldots \geq \lambda_{(n-1)/2]} \geq \lambda^1_{(n-1)/2} \geq -\lambda_{(n-1)/2}.$$
- If n is even, then $\lambda \succ \lambda^1$ is defined by
$$\lambda_1 \geq \lambda^1_1 \geq \lambda_2 \geq \lambda^1_2 \geq \ldots \geq \lambda_{n/2-1} \geq \lambda^1_{n/2-1} \geq |\lambda_{n/2}|.$$

Let $\mathfrak{so}(i)$ be the subalgebra of $\mathfrak{so}(n)$ spanned by I_{kl} for all $k, l \leq i$. Consider the chain of decreasing Lie subalgebras $\mathfrak{so}(n) \supset \mathfrak{so}(n-1) \supset \ldots \supset \mathfrak{so}(2)$. By restricting τ_n^{λ} successively to these subalgebras, τ_n^{λ} becomes a direct sum of 1 dimensional irreducible representations of $\mathfrak{so}(2)$. By choosing a unit vector from each of these 1 dimensional subspaces, we have an orthonormal basis of τ_n^λ consisting of unit vectors called the *Gelfand-Zetlin* basis. Thus, each basis vector is represented by an array

$$M = (\lambda, \lambda^1, \ldots, \lambda^{n-2}) \tag{5}$$

where $\lambda^i \in \Lambda(n-i, e)$ and $\lambda_i \succ \lambda_{i+1}$. The corresponding unit vector will be denoted by $\mathbf{v}_M$. Note that this vector generates the irreducible representation $\tau_{n-i}^{\lambda^i}$ under the action of $\mathfrak{so}(n-i)$.

An explicit formula of $I_{i+1,i} \cdot \mathbf{v}_M$ as a linear combination of vectors in the Gelfand-Zetlin basis is given on page 364 in [VK].

Induced representations of $\mathrm{Spin}(n,1)$**.** Let $\mu \in \Lambda(n-1)$ and let $\mathrm{Ind}_{n,1}(\mu, c)$ denote the Harish-Chandra module of the normalized induced representation

$$\mathrm{Ind}_{\mathrm{Spin}(n-1)\cdot\mathbb{R}^+\cdot N}^{\mathrm{Spin}(n,1)}(\tau_{n-1}^\mu \otimes \exp(c) \otimes 1).$$

We will equip $\mathrm{Ind}_{n,1}(\mu, c)$ with an Hermitian form given by

$$\int_{\mathrm{Spin}(n)} f_1(k)\overline{f_2(k)}\; dk \tag{6}$$

for $f_1, f_2 \in \mathrm{Ind}_{n,1}(\mu, c)$ and dk is the Haar measure on $\mathrm{Spin}(n)$ with with the total volume 1.

We note that $\mathrm{Ind}_{n,1}(\mu, c)$ is multiplicity free as a representation of $\mathrm{Spin}(n)$. Indeed

$$\mathrm{Ind}_{n,1}(\mu, c) = \bigoplus_{\lambda \succ \mu} \tau_n^\lambda.$$

By the above discussion and (5), we can assign an orthonormal Gelfand-Zetlin basis to each τ_n^λ. Thus, $\mathrm{Ind}_{n,1}(\mu, c)$ has a basis consisting of vectors $\mathbf{v}_M$ one for every array M as in (5) starting with $\lambda \succ \mu$. This basis is denoted by B_μ. We shall assume that this basis is orthonormal under the Hermitian form in (6).

Hirai [Hi] and Klimyk [KG] determined all the irreducible subquotients of $\mathrm{Ind}_{n,1}(\mu, c)$ with the help of the orthonormal basis above. They also obtained a classification of the unitary dual of $\mathrm{Spin}(n, 1)$. In the rest of this section, we will reproduce some of their results below which we will need later. These results are entirely due to or easy consequences of [Hi] and [KG].

First, we shall give an explicit action of the Lie algebra $\mathfrak{so}(n, 1)$ on the induced representations. To that end, we identify the Lie algebra $\mathfrak{so}(n, 1)$ with the subspace of $(n + 1) \times (n + 1)$ real matrices spanned by I_{kl} for $l < k \le n$ and

$$J_{n+1,i} = E_{i,n+1} + E_{n+1,i}.$$

The Lie algebra $\mathfrak{so}(n, 1)$ is generated by $I_{i+1,i}$ $(i < n)$ and $J_{n+1,n}$. The elements $I_{i+1,i}$ $(i < n)$ preserve $\mathrm{Spin}(n)$-types, and the action is given on page 364 in [VK]. The most interesting, of course, is the action of $J_{n+1,n}$ which is given as follows. Let $\mathbf{v}_M$ be in B_μ.

Let M^{+s} (resp. M^{-s}) denote the array obtained from M by increasing (resp. decreasing) λ_s by 1. Then

$$J_{n+1,n}\mathbf{v}_M = \sum_{s=1}^{[\frac{n}{2}]} \omega_s(c,\mu,M)\mathbf{v}_{M^{+s}} - \sum_{s=1}^{[\frac{n}{2}]} \omega_s(-c,\mu,M^{-s})\mathbf{v}_{M^{-s}} + c\omega_0(\mu,M)\mathbf{v}_M \tag{7}$$

where

$$\omega_s(c,\mu,M) = (\lambda_s + \frac{n+1}{2} - s + c)\omega_s(\mu,M)$$

and $\omega_s(\mu,M)$ is defined on pages 418-419 in [VK]. It depends on μ and the first two rows $\lambda \succ \lambda^1$ of M only. Also $\omega_0(\mu,M) = 0$ if n is even. If n is odd then

$$\omega_0 = \prod_{i=1}^{\frac{n-1}{2}} \frac{(\mu_i + \frac{n-1}{2} - i)(\lambda_i^1 + \frac{n-1}{2} - i)}{(\lambda_i + \frac{n+1}{2} - i)(\lambda_i + \frac{n-1}{2} - i)}.$$

Note that $\omega_0(\mu,M) \neq 0$ if μ is half integral. This observation is crucial in the proof of uniqueness of V. More precisely, consider the Cartan decomposition $\mathfrak{so}(n)\oplus\mathfrak{p}_0$ of $\mathfrak{so}(n,1)$. Let Π be an $(\mathfrak{so}(n,1),\mathrm{Spin}(n))$-module with multiplicity free $\mathrm{Spin}(n)$-types. This situation occurs for all induced representations and, therefore, all irreducible $(\mathfrak{so}(n,1),\mathrm{Spin}(n))$-modules. Let τ^λ a $\mathrm{Spin}(n)$-type in Π. Since $\mathfrak{p}_0\otimes\mathbb{C}\cong\mathbb{C}^n$, the action of $\mathfrak{p}_0$ on τ^λ followed by the projection on τ^λ defines a $\mathrm{Spin}(n)$-equivariant map

$$p_\lambda : \mathbb{C}^n \otimes \tau^\lambda \to \tau^\lambda.$$

If n is even then $\mathbb{C}^n$, the standard representation of $\mathfrak{so}(n)$, does not contain a trivial weight. Therefore the tensor product $\mathbb{C}^n\otimes\tau^\lambda$ does not contain τ^λ. This shows that p_λ is always zero if n is even. In general, p_λ depends on the action of $J_{n+1,n}$. Uniqueness of V is based on the following.

Proposition 3.1. *Assume that n is odd. Let Π be an irreducible $(\mathfrak{so}(n,1),\mathrm{Spin}(n))$-module with half-integral types. Assume that $p_\lambda = 0$ for every $\mathrm{Spin}(n)$-type τ^λ of Π. Then Π is isomorphic to $\mathrm{Ind}_{n,1}(\mu,0)$ for some μ in $\Lambda(\frac{n-1}{2},\frac{1}{2})$. In particular, Π is determined by its minimal $\mathrm{Spin}(n)$-type.*

Proof. Suppose Π is a subquotient of $\mathrm{Ind}(\mu,c)$ for some μ in $\Lambda([\frac{n-1}{2}],\frac{1}{2})$. The map p_λ is determined by the action of the operator $J_{n+1,n}$. Since μ is half-integral, as we remarked above, $\omega_0(\mu,M)\neq 0$ for all M. This shows that $p_\lambda \neq 0$ for all types of the induced representation unless $c = 0$. This shows that Π is contained in $\mathrm{Ind}(\mu,0)$ for some μ. Since $\mathrm{Ind}(\mu,0)$ is irreducible and determined by its minimal type among all induced representations with $c = 0$, the proposition follows. $\square$

Discrete series. The group $\mathrm{Spin}(n,1)$ has square integrable representations if and only if n is even. Let $\mu \in \Lambda(n-1,e)$. Suppose that $c + e + \frac{1}{2}$ is a positive integer. Then

$\mathrm{Ind}_{n,1}(\mu, c)$ contains two discrete series representations (or limits of discrete series if $c = 0$ and $e = \frac{1}{2}$) with $\mathrm{Spin}(n)$-types

$$D^+ = \bigoplus_{\lambda_{\frac{n}{2}} \geq c'} \tau_n^\lambda \text{ and } D^- = \bigoplus_{\lambda_{\frac{n}{2}} \leq -c'} \tau_n^\lambda$$

where $c' = c + \frac{1}{2}$.

Remark. We would like to point out three errors in [VK]. The right hand side of Eq. (6) on page 418 should be divided by 2. The factor $(4l_{s,2k+1} - 1)$ in the denominator on the right hand side of Eq. (8) is incorrect. It should be $(4l_{s,2k+1}^2 - 1)$. The first line of page 419 should be $l_i = r_i + [\frac{n-1}{2}] - i$. Also see page 86 in [Hi] for the correct formulas.

4. Uniqueness

In this section we show that the K-modules introduced in Section 2 can be extended to $(\mathfrak{g}, K)$-modules in at most one way. Moreover, the extension is necessarily irreducible and unitarizable.

Let U denote the real vector space with basis $\{u_1, \ldots, u_{p+q}\}$. We equip U with a symmetric bilinear form B of signature (p, q) such that

$$\mathrm{B}(u_i, u_j) = \delta_{ij}\epsilon_i$$

where $\epsilon_i = 1$ if $i \leq p$ and $\epsilon_i = -1$ if $i > p$. We realize $\mathfrak{so}(p, q)$ as $(p+q) \times (p+q)$-matrices skew symmetric with respect to the bilinear form B. Let E_{ij} denote the $(p+q) \times (p+q)$ square matrix whose (i, j)-th entry is 1 and 0 elsewhere. Of special interest will be the following elements in $\mathfrak{so}(p, q)$:

$$\begin{cases} I_{i+1,i} = E_{i,i+1} - E_{i+1,i} \text{ for } i \neq p, \\ J_{p+1,p} = E_{p,p+1} + E_{p+1,p}. \end{cases}$$

For $i = 1, 2 \ldots$ let U_i denote the subspace of U spanned by $\{u_1, \ldots, u_i\}$. Let $q = a + b$ for some non-negative integers a and b. Then the stabilizer of U_{p+a} is

$$\mathfrak{so}(p, a) \oplus \mathfrak{so}(b).$$

These algebras are of special interest to us. We pick K so that its Lie algebra is the stabilizer of U_p. We also note that the subalgebra $\mathfrak{so}(p, 1)$ (case $a = 1$) is generated by $\mathfrak{so}(p)$ and $J_{p+1,p}$.

<u>Case of V.</u>

Proposition 4.1. *Assume that the K-module V extends to a $\mathfrak{so}(p, q)$-module. Then this extension is unique, irreducible and unitarizable. Moreover, the restriction of V to $\mathfrak{so}(p, 1) \times \mathrm{SO}_{q-1}$ decomposes discretely as a direct sum*

$$\bigoplus_{\lambda \in \Lambda(p,0)} \mathrm{Ind}_{p,1}\left(\lambda + \frac{q-p}{2}\mathbf{1}_{\frac{p-1}{2}}, 0\right) \otimes \tau_{q-1}^{(\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}, 0, \ldots, 0)}.$$

Proof. Recall that

$$V = \bigoplus_{\lambda \in \Lambda(p,0)} \tau_p^{A(\lambda)} \otimes \tau_q^{B(\lambda)}$$

where, for $\lambda = (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}) \in \Lambda(p,0)$, $A(\lambda) = \lambda + \frac{q-p}{2}\mathbf{1}_{\frac{p-1}{2}}$ and $B(\lambda) = (\lambda, \mathbf{0}_{\frac{q-p+1}{2}})$.

As in the previous section, let $\{\mathbf{v}_M\}$ be an orthonormal Gelfand-Zetlin basis of $\tau_p^{A(\lambda)}$ defined with respect to the chain of subgroups $\mathfrak{so}(U_p) \supseteq \mathfrak{so}(U_{p-1}) \supseteq \ldots$. Similarly let U_p^c denote the orthogonal complement of U_p and let $\{\mathbf{v}_N\}$ be an orthonormal Gelfand-Zetlin basis of $\tau_q^{B(\lambda)}$ defined with respect to the chain of subgroups $\mathfrak{so}(U_p^c) \supseteq \mathfrak{so}(U_{p+1}^c) \supseteq \ldots$. It follows that

$$\mathbf{v}_M \otimes \mathbf{v}_N \tag{8}$$

where M and N are arrays with the first row $A(\lambda)$ and $B(\lambda)$ respectively, is a basis of V. Since $\mathfrak{so}(p,q)$ is generated by $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ and $J_{p+1,p}$ it suffices to show that $J_{p+1,p}$ acts in only one possible way on the basis $\mathbf{v}_M \otimes \mathbf{v}_N$.

Note that $J_{p+1,p}$ and $\mathfrak{so}(p)$ generate $\mathfrak{so}(p,1)$. Define an equivalence relation on the basis vectors of V by $\mathbf{v}_M \otimes \mathbf{v}_N \sim \mathbf{v}_{M'} \otimes \mathbf{v}_{N'}$ if the arrays N and N' have all rows equal except, perhaps, the first row. Fix an equivalence class C. Let Π be the subspace of V spanned by all basis vectors in C. If V extends to a $\mathfrak{so}(p,q)$-module, then Π is an $(\mathfrak{so}(p,1), \mathrm{Spin}(p))$-module. Its $\mathrm{Spin}(p)$-types are described as follows. Let $(y_1, \ldots, y_{\frac{p-1}{2}}, 0 \ldots, 0)$ be the second row of the array N for all basis vectors in the equivalence class C. Define

$$\mu = (y_1, \ldots, y_{\frac{p-1}{2}}) + \frac{1}{2}(q-p)\mathbf{1}_{\frac{p-1}{2}} \in \Lambda(p-1, \frac{1}{2}). \tag{9}$$

Notice that $A(\lambda) \succ \mu$. In fact, the $\mathrm{Spin}(p)$-types of Π are the same as the types of the induced representation $\mathrm{Ind}_{p,1}(\mu, c)$. We claim that

$$\Pi \cong \mathrm{Ind}_{p,1}(\mu, 0). \tag{10}$$

In view of Proposition 3.1 we must show that $p_{A(\lambda)} = 0$ for any type $\tau^{A(\lambda)}$ of Π. In other words, we must show that the action of $J_{p+1,p}$ on $\tau^{A(\lambda)}$ composed with the projection on $\tau^{A(\lambda)}$ is zero. But this is easy. Indeed, the maximal $\tau^{A(\lambda)}$-isotypic summand of V is given by

$$\tau^{A(\lambda)} \otimes \tau^{B(\lambda)}.$$

The action of $J_{p+1,p}$ is a part of the action of $\mathfrak{p}_0 \otimes \mathbb{C} \cong \mathbb{C}^p \otimes \mathbb{C}^q$ (of the Cartan decomposition of $\mathfrak{so}(p,q)$). Since q is even, $\mathbb{C}^q \otimes \tau^{B(\lambda)}$ does not contain $\tau^{B(\lambda)}$ as a summand. This shows that $p_{A(\lambda)} = 0$ and Π must be isomorphic to $\mathrm{Ind}_{p,1}(\mu, 0)$. Let $T : \mathrm{Ind}_{p,1}(\mu, 0) \to \Pi$ be an isomorphism. In particular, the action of $J_{p+1,p}$ must be equal to $T \circ \pi(J_{p+1,p}) \circ T^{-1}$ where $\pi(J_{p+1,p})$ is the action of $J_{p+1,p}$ on $\mathrm{Ind}_{p,1}(\mu, 0)$. Since T is unique up to a non-zero scalar the action of $\mathfrak{so}(p,q)$ on V is unique.

Next we show that V is unitarizable. Let $\bar{V}$ denote the Hermitian dual of V. Since it has the same K-type as V, it follows that $\bar{V}$ and V are isomorphic $(\mathfrak{so}(p,q), K)$-modules. This isomorphism induces a non-degenerate $(\mathfrak{so}(p,q), K)$-invariant Hermitian form on V.

We may assume that it is positive definite on the minimal K-type. We claim that the Hermitian form is positive definite so V is unitarizable. Let $\tau_{p,q}^{\lambda} := \tau_p^{A(\lambda)} \otimes \tau_q^{B(\lambda)}$ where $\lambda = (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}})$. Suppose $\tau_{p,q}^{\lambda-\varepsilon_s}$ is nonzero and the signatures on $\tau_{p,q}^{\lambda}$ and $\tau_{p,q}^{\lambda-\varepsilon_s}$ are different. Define

$$\mu' = (\lambda_1, \ldots, \lambda_{\frac{p-1}{2}}) - \varepsilon_s + \frac{q-p}{2}\mathbf{1}_{\frac{p-1}{2}} \in \Lambda(p-1) \tag{11}$$

and $\Pi' = \mathrm{Ind}_{p,1}(\mu', 0)$. Then Π' intersects the two K-types $\tau_{p,q}^{\lambda}$ and $\tau_{p,q}^{\lambda-\varepsilon_s}$ non-trivially. The restriction of the invariant Hermitian form of V to Π' is positive definite since Π' is unitarizable. This contradicts the fact that the signatures are different on $\tau_{p,q}^{\lambda}$ and $\tau_{p,q}^{\lambda-\varepsilon_s}$ and proves our claim.

Finally we show that V is irreducible. Suppose W is a proper submodule of V. By taking its orthogonal complement if necessary, we assume that W does not contain the minimal K-type. Let $\tau_{p,q}^{\lambda} = \tau_p^{A(\lambda)} \otimes \tau_q^{B(\lambda)}$ be a K-type in W such that $\sum_i \lambda_i$ is minimal. Since $\tau_{p,q}^{\lambda}$ is not the minimal K-type, $\tau_{p,q}^{\lambda-\varepsilon_s}$ is nonzero for some $s \leq \frac{p-1}{2}$. We define μ' and $\Pi' = \mathrm{Ind}_{p,1}(\mu', 0)$ as in (11). Then Π' intersects $\tau_{p,q}^{\lambda}$ and $\tau_{p,q}^{\lambda-\varepsilon_s}$ non-trivially. Hence W contains $\tau_{p,q}^{\lambda-\varepsilon_s}$. This contradicts that fact that $\sum_i \lambda_i$ is minimal in W. Therefore V is irreducible and this completes the proof of the proposition. $\square$

By scaling the basis vectors $\mathbf{v}_M \otimes \mathbf{v}_N$ in (8) if necessary, we may assume that the basis vectors they form an orthonormal basis of V and, the action of $J_{p+1,p}$ on Π in (10) is the same as the action of $J_{p+1,p}$ on the basis vectors B_μ of the induced representation in (7):

$$\begin{aligned} J_{p+1,p}\mathbf{v}_M \otimes \mathbf{v}_N &= \sum_{s=1}^{\frac{p-1}{2}} (\lambda_s + \frac{q+1}{2} - s)\omega_s(\mu, M)\mathbf{v}_{M^{+s}} \otimes \mathbf{v}_{N^{+s}} \\ &- \sum_{s=1}^{\frac{p-1}{2}} (\lambda_s + \frac{q-1}{2} - s)\omega_s(\mu, M^{-s})\mathbf{v}_{M^{-s}} \otimes \mathbf{v}_{N^{-s}}. \end{aligned} \tag{12}$$

<u>Case of $V_o^{\pm}$.</u> We remind the reader that $q = p-1$ holds here.

Proposition 4.2. *Assume that V_o^+ (respectively V_o^-) extends to a $\mathfrak{so}(p, p-1)$-module. Then this extension is unique, irreducible and unitarizable. Moreover, the restriction of $V_o^{\pm}$ to $\mathfrak{so}(p,1) \times \mathrm{SO}(p-2)$ decomposes discretely as a direct sum*

$$\bigoplus_{\lambda \in \Lambda(p-2,0), \lambda_{\frac{p-3}{2}} \geq 1} \mathrm{Ind}_{p,1}\left((\lambda, 1) - \frac{1}{2}\mathbf{1}_{\frac{p-1}{2}}, 0\right) \otimes \tau_{p-2}^{\lambda}.$$

The proof is identical to one of Proposition 4.1, so it is omitted.

<u>Case of $V^{\pm}$.</u> Since V^+ and V^-, as $(\mathfrak{so}(p,q), K)$-modules have been constructed by Knapp in [Kn], here we describe a somewhat different result needed to construct V, V_o^+ and V_o^- via the Wallach transfer of V^+ and V^-.

Proposition 4.3. *The representation V^+ (resp. V^-) is the unique representation with K-types as in* (4)*. The restriction of V^+ (respectively V^-) to $\mathrm{SO}(p-1)\times\mathfrak{so}(q,1)$ decomposes discretely as a direct sum*

$$\bigoplus_{\lambda\in\Lambda(p-1,0),\lambda_{\frac{q}{2}+1}=0} \tau^{\lambda}_{p-1}\otimes\Theta(\lambda)$$

where $\Theta(\lambda)$ is the discrete series representation of $\mathrm{Spin}(q,1)$ (or limit of discrete series if $p-1=q$ and $\lambda_{\frac{q}{2}}=0$) whose minimal $\mathrm{Spin}(q)$-type has highest weight $(\lambda_1,\ldots,\lambda_{\frac{q}{2}})+\frac{p-q}{2}\mathbf{1}_{\frac{q}{2}}$ (respectively $\sigma((\lambda_1,\ldots,\lambda_{\frac{q}{2}})+\frac{p-q}{2}\mathbf{1}_{\frac{q}{2}})$) and the infinitesimal character

$$(\lambda_1,\ldots,\lambda_{\frac{q}{2}})+\frac{p-q}{2}\mathbf{1}_{\frac{q}{2}}+\rho_{q+1}.$$

5. Existence

Since $\mathfrak{so}(p,q)$ is generated by $\mathfrak{so}(p,1)$ and $\mathfrak{so}(q)$, the formula for $J_{p+1,p}$ in (12) gives a representation of $\mathfrak{so}(p,q)$ on V, provided that certain relations have been verified. The same also applies to $V^{\pm}$ and $V_o^{\pm}$. While the verification is straightforward, it is also rather cumbersome. It is much quicker to construct V and $V_o^{\pm}$ from $V^{\pm}$ using the Wallach transfer.

If $p-1\geq q$ then Knapp [Kn] has constructed two representations with the same K-types as V^+ and V^-. Trapa [T] has further established some properties of V^+ and V^-. In order to state their results, define

$$\mu_{p,q}=\begin{cases}\left(\frac{p-1}{2},\frac{p-3}{2},\ldots,1,\frac{q-1}{2},\frac{q-3}{2},\ldots,\frac{1}{2}\right) & \text{if } p<q-1\\ \left(\frac{q}{2},\frac{q-2}{2},\ldots,1,\frac{p-2}{2},\frac{p-4}{2},\ldots,\frac{1}{2}\right) & \text{if } p\geq q-1.\end{cases}$$

Theorem 5.1. *If $p-1\geq q$ then V^+ and V^- can be extended to unitarizable and irreducible $(\mathfrak{so}(p,q),K)$-modules. The infinitesimal character of V^+ and V^- is $\mu_{p,q}$ and the annihilator in the universal algebra is the unique maximal two sided ideal with the infinitesimal character $\mu_{p,q}$.*

We can now use the Wallach transfer to construct V from $V^{\pm}$. Take first $p-1<q$. Consider $V^{\pm}$ for $\mathfrak{so}(q+1,p-1)$. The restriction of $V^{\pm}$ to $\mathrm{SO}(q)\times\mathfrak{so}(p-1,1)$ is given by Proposition 4.3. Let $\Gamma^i_{\mathfrak{so}(p,1)}$ denote the i-th derivative of the Zuckerman functor with respect to $\mathfrak{so}(p,1)$. Then, by [Wa1], $\Gamma^{\frac{p-1}{2}}_{\mathfrak{so}(p,1)}$ applied to a discrete series representation produces an irreducible finite dimensional representation with the same infinitesimal character. It follows that the $\mathfrak{so}(p,q)$-modules

$$\Gamma^{\frac{p-1}{2}}_{\mathfrak{so}(p-1,1)}(V^+)\text{ and }\Gamma^{\frac{p-1}{2}}_{\mathfrak{so}(p-1,1)}(V^-)$$

have the same types as V. By uniqueness of V in Proposition 4.1, these two representations must be isomorphic. Since Zuckerman's functor can only increase the annihilator, we have the following theorem:

Theorem 5.2. *If $p-1<q$ then V can be extended to unitarizable and irreducible $(\mathfrak{so}(p,q),K)$-module. The infinitesimal character of V is $\mu_{p,q}$ and the annihilator in the universal algebra is the unique maximal two sided ideal with the infinitesimal character $\mu_{p,q}$.*

Finally, assume that $p-1=q$. Consider $V^{\pm}$ for $\mathfrak{so}(p,p-1)$. The restriction of $V^{\pm}$ to $\mathrm{SO}(p-1)\times\mathfrak{so}(p-1,1)$ is given by Proposition 4.3. Then, by [Wa1], $\mathfrak{so}(p,p-1)$-modules

$$\Gamma^{\frac{p-1}{2}}_{\mathfrak{so}(p-1,1)}(V^{+}) \text{ and } \Gamma^{\frac{p-1}{2}}_{\mathfrak{so}(p-1,1)}(V^{-})$$

have the same types as V_o^+ and V_o^-, respectively. Again, since Zuckerman's functor can only increase the annihilator, we have the following theorem:

Theorem 5.3. *If $p-1=q$ then $V_o^{\pm}$ can be extended to unitarizable and irreducible $(\mathfrak{so}(p,p-1),K)$-modules. The infinitesimal character of $V_o^{\pm}$ is $\mu_{p,p-1}$ and the annihilator in the universal algebra is the unique maximal two sided ideal with the infinitesimal character $\mu_{p,p-1}$.*

6. Associated variety

In this section we will compute the associated varieties of V, $V^{\pm}$ and $V_o^{\pm}$. A definition and basic properties of associated varieties could be found in [Vo2]. In order to simplify notation, define

$$m=\min\left(\frac{p-1}{2},\frac{q}{2}\right).$$

Recall that nilpotent orbits of the complex group $\mathrm{O}_n(\mathbb{C})$ are parameterized by partitions of n such that every even part has an even multiplicity. The classification of (real) nilpotent $\mathrm{O}(p,q)$-orbits on $\mathfrak{so}(p,q)$ is refined as follows: To every partition we attach the Young diagram as usual. Then we insert signs $+$ and $-$ into the boxes corresponding to odd (length) rows such that the signs alternate. Then this signed partition parameterizes an orbit of $\mathrm{O}(p,q)$ if and only if the difference of the number of positive and negative signs is equal to the signature $p-q$. Two signed partitions correspond to the same real orbit if and only if one signed partition can be obtained form another by permuting the rows of the same length.

Case $p-1<q$. Consider the partition $(2^{p-1},1^{q-p+2})$. The number of odd rows is $q-p+2$. On the other hand, the signature is equal to $p-q$. Thus we can mark the Young diagram by putting $+$ in the first row of length 1 and $-$ in all other. Let $\mathcal{O}_{2^{p-1}}$ be the corresponding nilpotent $\mathrm{O}(p,q)$-orbit. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ denote the complexified Cartan decomposition of $\mathfrak{so}(p,q)$. Let $\mathcal{O}^K_{2^{p-1}}$ be the $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$ corresponding to $\mathcal{O}_{2^{p-1}}$ by the Kostant-Sekiguchi correspondence. We shall now describe this orbit in more details. Let $\bar{K}_{\mathbb{C}}=\mathrm{SO}_p(\mathbb{C})\times\mathrm{SO}_q(\mathbb{C})$. Recall that $\mathfrak{p}\cong\mathbb{C}^p\otimes\mathbb{C}^q$ under the action of $\bar{K}_{\mathbb{C}}$. Let $(\ ,\ )$ denote a $\mathrm{SO}_p(\mathbb{C})$-invariant symmetric bilinear form on $\mathbb{C}^p$. Pick a basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_p\}$ of $\mathbb{C}^p$ so that $(\mathbf{e}_i,\mathbf{e}_j)=0$ except $(\mathbf{e}_i,\mathbf{e}_{p+1-i})=1$. We do likewise for $\mathbb{C}^q$ to get a basis $\{\mathbf{f}_1,\ldots,\mathbf{f}_q\}$.

Using these two bases, we can identify $\mathfrak{p}$ with the set of p by q matrices so that the $\mathbf{e}_i \otimes \mathbf{f}_j$ corresponds to the elementary matrix $E_{i,j}$. Define

$$P = E_{1,1} + E_{2,2} + \cdots + E_{m,m} \in \mathfrak{p}.$$

Let ι be the standard representation of $\mathfrak{so}_{p+q}(\mathbb{C})$. Then it is a simple exercise to show that

- the null space of $\iota(P)$ has the dimension $q+1$.
- $\iota(P)^2 = 0$, that is, $\iota(P)$ is a nilpotent element.

This implies that P belongs to the complex nilpotent $\mathrm{O}_{p+q}(\mathbb{C})$-orbit corresponding to the partition $(2^{p-1}, 1^{q-p+2})$. Thus, the orbit $\mathcal{O}^K_{2^{p-1}}$ is generated by P.

For the rest of this section, we identify $\mathfrak{p}$ with $\mathfrak{p}^*$ using the Killing form, and we identify adjoint orbits with co-adjoint orbits.

Let $U_n(\mathfrak{g})$ denote the standard filtration on the universal enveloping algebra of $\mathfrak{so}(p,q)$. Let $F_n = U_n(\mathfrak{g}) \cdot \tau_{\min}$ be the subspace in V where $\tau_{\min} = \tau_p^{\frac{q-p}{2}\mathbf{1}_m} \otimes \mathbb{C}$ is the minimal K-type of V. The graded module

$$\mathrm{Gr}(V) = \bigoplus_{n=0}^{\infty} F_n/F_{n-1}$$

is a $(\mathrm{Sym}(\mathfrak{p}), K_{\mathbb{C}})$-module generated by $\tau_{\min} = F_0$. Note that $\mathfrak{k} \cdot F_n \subseteq F_n$ so $\mathfrak{k}$ acts trivially on the graded module. By induction we have $\mathfrak{p} \cdot F_n = F_{n+1}$. By the formula $J_{p+1,p} \in \mathfrak{p}$ on the K-types of V in (12), we see that

$$\mathrm{Gr}_n(V) = F_n/F_{n-1} = \bigoplus_{\lambda} \tau_p^{\lambda + \frac{q-p}{2}\mathbf{1}_m} \otimes \tau_q^{(\lambda,\mathbf{0})} \tag{13}$$

where the sum is taken over $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(p,0)$ such that $\sum_i \lambda_i = n$. Let I be the annihilator ideal of $\mathrm{Gr}(V)$ in $\mathrm{Sym}(\mathfrak{p})$. It is also the annihilator ideal of $\tau_{\min} = \mathrm{Gr}_0(V)$. The variety in $\mathfrak{p}^* \cong \mathfrak{p}$ cut out by I is called the *associated variety* of V. We now state the main theorem of this section.

Theorem 6.1. *The associated variety of V is $\bar{\mathcal{O}}^K_{2^{p-1}}$, the algebraic closure of the $K_{\mathbb{C}}$-orbit. More precisely,*

$$\mathrm{Sym}(\mathfrak{p})/I = \Gamma(\bar{\mathcal{O}}^K_{2^{p-1}}),$$

the ring of regular functions on $\bar{\mathcal{O}}^K_{2^{p-1}}$.

Proof. The first step in the proof is a description of the space of regular functions on $\bar{\mathcal{O}}^K_{2^{p-1}}$ as a $K_{\mathbb{C}}$-module.

Lemma 6.2. *Let $\Gamma(\bar{\mathcal{O}}^K_{2^{p-1}})$ be the algebra of regular functions on the closure $\bar{\mathcal{O}}^K_{2^{p-1}}$ of $\mathcal{O}^K_{2^{p-1}}$. We have the following isomorphism of $\bar{K}_{\mathbb{C}} = \mathrm{SO}_p \times \mathrm{SO}_q$-modules*

$$\Gamma(\bar{\mathcal{O}}^K_{2^{p-1}}) = \bigoplus_{\lambda \in \Lambda(p,0)} \tau_p^{\lambda} \otimes \tau_q^{(\lambda,\mathbf{0})}. \tag{14}$$

Proof. Recall that we have identified $\mathfrak{p}$ with the set of complex $p \times q$ matrices. Consider the $m \times m$ block located in the upper left corner. The stabilizer of this block in $\mathrm{SO}_p \times \mathrm{SO}_q$ is a product of two maximal parabolic subgroups $Q_1 = L_1 N_1$ and $Q_2 = L_2 N_2$ with Levi factors

$$L_1 = \mathrm{GL}_m \text{ and } L_2 = \mathrm{GL}_m \times \mathrm{SO}_{q-2m}.$$

Since P is the identity matrix located in the $m \times m$ block, it is now clear that the stabilizer of P in $\bar{K}_{\mathbb{C}}$ is

$$\bar{K}_{\mathbb{C}}(P) = \triangle \mathrm{GL}_m \times N_1 \times \mathrm{SO}_{q-2m} \times N_2 \tag{15}$$

where $\triangle \mathrm{GL}_m = \mathrm{GL}_m$ is diagonally embedded in $\mathrm{GL}_m \times \mathrm{GL}_m \subseteq L_1 \times L_2$. It follows that every regular function on $\bar{\mathcal{O}}^K_{2^{p-1}}$ gives a right $\bar{K}_{\mathbb{C}}(P)$-invariant regular function on $\bar{K}_{\mathbb{C}}$. The Peter-Weyl Theorem implies that the subspace of $\bar{K}_{\mathbb{C}}(P)$-invariant functions on $\bar{K}_{\mathbb{C}}$ has precisely the types given by the right hand side of (14). This implies that $\Gamma(\bar{\mathcal{O}}^K_{2^{p-1}})$ is contained in the right hand side of (14)

To prove the opposite inclusion, we consider $\mathrm{Sym}(\mathfrak{p})$ as a $\mathrm{GL}_p \times \mathrm{GL}_q$-module, since $\mathfrak{p}$ has been identified with the set of $p \times q$-matrices. Then

$$\mathrm{Sym}(\mathfrak{p}) = \bigoplus_Y \tau^Y_{\mathrm{GL}_p} \otimes \tau^Y_{\mathrm{GL}_q} \tag{16}$$

where the sum is taken over Young diagrams Y with at most $\min(p,q)$ rows (see Thm 2.1.2 in [Ho2]). Recall that a joint highest weight vector of $\tau^Y_{\mathrm{GL}_p} \otimes \tau^Y_{\mathrm{GL}_q}$ is given as a product of determinants of $r \times r$ square blocks located in the upper left hand corner of $\mathfrak{p}$. If Y has at most m rows then the blocks needed are of size $r \leq m$ and the highest weight vector does not vanish on the matrix P. Hence it generates a representation of $\bar{K}_{\mathbb{C}}$ isomorphic to $\tau^\lambda_p \otimes \tau^{(\lambda,\mathbf{0})}_q$ in $\mathrm{Sym}(\mathfrak{p})$ as well as in $\Gamma(\bar{\mathcal{O}}^K_{2^{p-1}})$. The lemma is proved. □

Let J be the prime ideal in $\mathrm{Sym}(\mathfrak{p})$ corresponding to $\bar{\mathcal{O}}^K_{2^{p-1}}$. Since I and J have the same Krull dimension and J is prime, in order to show that $J = I$, it suffices to show that $J \subseteq I$. We need the following lemma.

Lemma 6.3. *Let $\tau = \tau^\gamma_p \otimes \tau^{\gamma'}_q$ be a $K_{\mathbb{C}}$-type in $\mathrm{Sym}^n(\mathfrak{p})$ which does not lie in the ideal I. Then $\gamma' = (\gamma, 0, \dots, 0)$ and τ is generated by a joint highest weight vector of the representation $\tau^{(\gamma,\mathbf{0}_{p-m})}_{\mathrm{GL}_p} \otimes \tau^{(\gamma,\mathbf{0}_{q-m})}_{\mathrm{GL}_q}$ in $\mathrm{Sym}^n(\mathfrak{p})$ in (16).*

Proof. Note that any irreducible summand of $\tau \otimes \tau_{\min}$ is isomorphic to $\tau^{\gamma+\nu}_p \otimes \tau^{\gamma'}_q$ for some weight ν of $\tau_p^{\frac{q-p}{2}\mathbf{1}}$. Since τ is not contained in I, $\tau \cdot \tau_{\min} \neq 0$ in $\mathrm{Gr}_n(V)$. By (13), $\gamma' = (\lambda, \mathbf{0})$ for some λ and $\sum_i \lambda_i = n$ and

$$\gamma + \nu = \lambda + \frac{q-p}{2}\mathbf{1}_{\frac{p-1}{2}}.$$

By the theory of $\mathrm{SO}_p \times \mathrm{GL}_q$ harmonics on $\mathfrak{p}^*$ (for example see [Ho1]), the representation τ_p^γ first appears in degree

$$\sum \gamma_i = \sum_i \lambda_i + \frac{q-p}{2}m - \sum_i \nu_i = n + \frac{q-p}{2}m - \sum_i \nu_i \geq n.$$

The last equality holds if and only if ν is the highest weight vector of $\tau_p^{\frac{q-p}{2}\mathbf{1}}$ ie $\nu = \frac{q-p}{2}\mathbf{1}$. Since, by the assumption, τ_p^γ occurs in the degree n we must have $\lambda = \gamma$. On the other hand, the first occurrence of τ_p^γ happens in $\mathrm{Sym}^n(\mathfrak{p})$ within the harmonics. It follows that τ occurs in the harmonics $\tau_p^\gamma \otimes \tau_{\mathrm{GL}_q}^{(\gamma,\mathbf{0}_{q-m})}$. By the branching rule from GL_q to SO_q, $\tau_{\mathrm{GL}_q}^{(\gamma,\mathbf{0}_{q-m})}$ contains $\tau_q^{(\gamma,\mathbf{0})}$ with multiplicity one and it is generated by the highest weight vector. This joint highest weight vector also generates $\tau_{\mathrm{GL}_p}^{(\gamma,\mathbf{0}_{p-m})} \otimes \tau_{\mathrm{GL}_q}^{(\gamma,\mathbf{0}_{q-m})}$ in $\mathrm{Sym}^n(\mathfrak{p})$. This proves the lemma. □

Since the highest vector of $\tau_{\mathrm{GL}_p}^{(\gamma,\mathbf{0}_{p-m})} \otimes \tau_{\mathrm{GL}_q}^{(\gamma,\mathbf{0}_{q-m})}$ does not vanish on P, we have shown that any K-type outside I is also outside J. This shows that $J \subseteq I$, as desired. Theorem 6.1 is proved. □

Case $p-1 \geq q$. In this case we only state the results. Consider the partition $(2^q, 1^{p-q})$. There is only one real form of this orbit for $\mathrm{O}(p,q)$. Indeed, since the number of odd rows is $p-q$ and the signature is $p-q$ we have to enter $+$ in all rows of length one. Let $\mathcal{O}_{2^q}$ be the corresponding nilpotent $\mathrm{O}(p,q)$-orbit. By Theorem 9.3.4 [CM], $\mathcal{O}_{2^q}$ is a union of two $\mathrm{SO}(p,q)^0$-orbits, denoted by $\mathcal{O}_{2^q}^+$ and $\mathcal{O}_{2^q}^-$ respectively. Let $\mathcal{O}_{2^q}^{K,+}$ and $\mathcal{O}_{2^q}^{K,-}$ be the $K_\mathbb{C}$-orbits in $\mathfrak{p}$ corresponding to $\mathcal{O}_{2^q}^+$ and $\mathcal{O}_{2^q}^-$, respectively, via the Kostant-Sekiguchi correspondence. If we identify $\mathfrak{p}$ with the space of complex $p \times q$ matrices then the two $K_\mathbb{C}$-orbits are generated by elements

$$P^+ = E_{1,1} + \cdots + E_{m,m} \text{ and } P^- = E_{1,1} + \cdots + E_{m-1,m-1} + E_{m,m+1}.$$

Theorem 6.4. *The associated variety of V^+ and of V_o^+ is the closure of the $K_\mathbb{C}$-orbit $\mathcal{O}_{2^q}^{K,+}$. The associated variety of V^- and of V_o^- is the closure of the $K_\mathbb{C}$-orbit $\mathcal{O}_{2^q}^{K,-}$.*

One also could prove the above theorem for $V^\pm$ using the results of Knapp [Kn] and Trapa [T].

7. Langlands parameters

In this section we will compute Langlands' parameters of V, $V^\pm$ and $V_o^\pm$. In order to simplify some notation, let

$$m = \min\left(\frac{p-1}{2}, \frac{q}{2}\right).$$

Case $p \leq q-1$. Fix a minimal parabolic subgroup $P_{\min} = M_{\min}A_{\min}N_{\min}$. The root system of $G_{p,q}$ relative to $A_{\min}$ is B_p. We shall realize this root system in a standard fashion, so that

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ \ldots \ , \alpha_{p-1} = \varepsilon_{p-1} - \varepsilon_p \text{ and } \ \beta = \varepsilon_p$$

are simple roots. Long root spaces are one-dimensional and for every long root α we have an embedding

$$\varphi_\alpha : \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to G_{p,q}$$

where $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is the metaplectic cover $\mathrm{SL}_2(\mathbb{R})$. Let Z_α be the image, under φ_α, of the center of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Note that Z_α is a cyclic group of order 4. On the other hand, for every short root and β in particular we have an embedding

$$\varphi_\beta : \mathrm{Spin}(q-p+1,1) \to G_{p,q}.$$

The existence of this embedding is a combination of two facts. First, the rank one Lie subalgebra of $\mathfrak{so}(p,q)$ corresponding to β is $\mathfrak{so}(q-p+1,1)$ which gives a map from $\mathrm{Spin}(q-p+1,1)$ into $\mathrm{Spin}(p,q)$. Second, $\mathrm{Spin}(q-p+1,1)$ is topologically simply connected, so the map lifts to $G_{p,q}$. Let Z_β be the image of the center of $\mathrm{Spin}(q-p+1,1)$. Then Z_β is a cyclic group of order 2.

Pseudo-spherical principal series. Here we define the principal series representations associated to the minimal parabolic subgroup. Note that the connected component of $M_{\min}$ is $\mathrm{Spin}(q-p)$, and

$$M_{\min} = \mathrm{Spin}(q-p) \times Z_\beta \times Z_{\alpha_1} \times_{\mu_2} \ldots \times_{\mu_2} Z_{\alpha_{p-1}}$$

where $\mu_2 = \{\pm 1\}$ is the subgroup of $G_{p,q}$ such that $G_{p,q}/\mu_2$ is linear. Define

$$M_{\min,\beta} = \mathrm{Spin}(q-p) \times Z_\beta.$$

Then $M_{\min}/M_{\min,\beta}$ is a Heisenberg group of order 2^p. Let S be the unique representation of this group such that the center μ_2 acts via the unique non-trivial character. The dimension of S is 2^m.

Let $\chi \in \mathfrak{a}^*_{\min}$ be such that $\mathrm{Re}(\chi)$ is dominant. Then we have a pseudo-spherical, standard module (normalized induction)

$$\mathrm{Ind}_{P_{\min}}[S \otimes \exp(-\chi)]. \tag{17}$$

Generalized principal series. For every $k = 1, \ldots, m$, the group $G_{p,q}$ has a parabolic subgroup $P = MAN$ - depending on k - such that $A = (\mathbb{R}^+_{\mathrm{long}})^m \times \mathbb{R}^+_{\mathrm{short}}$ and the connected component of M is

$$M_0 = \mathrm{Spin}(q-p) \times \widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\mu_2} \ldots \times_{\mu_2} \widetilde{\mathrm{SL}}_2(\mathbb{R}). \tag{18}$$

where there are m factors of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. More precisely,

(i) The factors of $A = (\mathbb{R}^+_{\mathrm{long}})^m \times \mathbb{R}^+_{\mathrm{short}}$ correspond to m long roots $\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \ldots, \varepsilon_{2k-3} + \varepsilon_{2k-2}, \varepsilon_{2k} + \varepsilon_{2k+1}, \ldots, \varepsilon_{p-1} + \varepsilon_p$ and the short root ε_{2k-1}.

(ii) The factors $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ in (18) correspond to the following m long roots: α_1, α_3, $\ldots$, α_{2k-3}, α_{2k}, α_{2k+2}, $\ldots$, α_{p-1}.

Define

$$M_\beta = M_0 \times Z_\beta.$$

The quotient M/M_β is a $\mathbb{Z}/2\mathbb{Z}$-vector space of rank m. It is spanned by h_α, where h_α is an element of order 4 in M_α as α runs through the following m long roots:

$$\alpha_2, \alpha_4, \ldots, \alpha_{2k-2}, \alpha_{2k-1}, \alpha_{2k+1}, \ldots, \alpha_{p-2}.$$

Assume now that $p < q-1$ so that $\frac{q-p}{2} \geq 3/2$. Let $D(\frac{q-p}{2})^+$ and $D(\frac{q-p}{2})^-$ be the holomorphic and anti-holomorphic discrete series representations of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ such that the lowest weight of $D(\frac{q-p}{2})^+$ is $\frac{q-p}{2}$ and the highest weight of $D(\frac{q-p}{2})^-$ is $-\frac{q-p}{2}$. The infinitesimal character of these two representation is $\frac{q-p-2}{2}$. Define a representation W of M by

$$W = \mathrm{Ind}_{M_\beta}^{M}[(D(\frac{q-p}{2})^+)^{\otimes m}],$$

where $\mathrm{Spin}(q-p) \times Z_\beta$ acts trivially. We claim that W is irreducible. Indeed, the restriction of W to M_β is a sum of 2^m terms of type

$$D(\frac{q-p}{2})^{\epsilon_1} \otimes \ldots \otimes D(\frac{q-p}{2})^{\epsilon_m}$$

where $\epsilon_1, \ldots, \epsilon_m = \pm$ (all possible choices). Since these summands are mutually non-isomorphic, irreducibility of W follows at once from Mackey's criterion.

Define a standard module for $G_{p,q}$ by

$$\mathrm{Ind}_P^{G_{p,q}}[W \otimes \exp(-\eta)] \tag{19}$$

where

$$\begin{aligned} \eta &= \eta_1(\varepsilon_1+\varepsilon_2) + \ldots + \eta_k\varepsilon_k + \ldots + \eta_{m+1}(\varepsilon_{p-1}+\varepsilon_p) \\ &= (\eta_1, \eta_1 \ldots, \eta_{k-1}, \eta_{k-1}, \eta_k, \eta_{k+1}, \eta_{k+1}, \ldots, \eta_{m+1}, \eta_{m+1}), \end{aligned} \tag{20}$$

and $\eta_1 > \ldots > \eta_{m+1} > 0$. This representation has a unique submodule with the minimal K-type $\tau_p^{\frac{q-p}{2}\mathbf{1}} \otimes \mathbb{C}$.

Theorem 7.1. *We have the following:*

(i) Suppose $q = p+1$. Then V is the Langlands submodule of a normalized induced (pseudo-spherical) principal series representation (17) *with*

$$\chi = \frac{1}{2}(p, p-1, p-2, \ldots, 1).$$

(ii) Suppose $q > p+1$. Then V is the unique Langlands submodule of a normalized induced principal series representation in (19) *where $k = 1$ and*

$$\eta = (\eta_1, \ldots, \eta_{\frac{p+1}{2}}) = (\frac{p+q}{4}, \frac{p-1}{2}, \ldots, 2, 1) + \frac{q-p-2}{4}\mathbf{1}_{\frac{p+1}{2}}.$$

The submodule V in each case is generated by the minimal K-type $\tau_p^{\frac{q-p}{2}\mathbf{1}} \otimes \mathbb{C}$ of the principal series representation.

Proof. The first case, when $q = p+1$, is trivial since V is a pseudospherical representation of the split group $G_{p,p+1}$. See also [A-V] and [Wa2].

Lemma 7.2. *Assume that $q > p+1$. Then V is a submodule of* $\mathrm{Ind}_P^{G_{p,q}}[W \otimes \exp(-\eta)]$ *for some k.*

Proof. This is a standard procedure so we only give a sketch. We use Proposition 4.1 in [Vo1]. There one constructs λ in $i\mathfrak{t}_0^*$ from the minimal K-type. Its stabilizer in $\mathfrak{so}_{p+q}(\mathbb{C})$ is a theta stable parabolic $\mathfrak{l} + \mathfrak{n}$ such that $\mathfrak{l}_0 = \mathfrak{su}_{m,m} + \mathfrak{sl}_2 + \mathfrak{u}_1^{\frac{q-p-1}{2}}$. We may choose a split torus $\mathfrak{a}_0$ in $\mathfrak{l}_0$ such that $\mathfrak{a}_0$ is the Lie algebra of A in $P = MAN$ in (18). The discrete series parameter on M could be read off from λ. $\square$

Since the discrete series representation $D(\frac{q-p}{2})^+$ of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ embeds into the pseudo-spherical principal series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ with $\exp(\frac{q-p-2}{2})$ on the "A" part of the minimal parabolic subgroup, it follows from induction in stages that

$$\mathrm{Ind}_P[W \otimes \exp(-\eta)] \subseteq \mathrm{Ind}_{P_{\min}}[S \otimes \exp(-\chi)], \tag{21}$$

where $\chi = (\chi_1, \dots, \chi_p) = \eta + \xi$,

$$\begin{cases} \eta = (\eta_1, \eta_1 \dots, \eta_{k-1}, \eta_{k-1}, \eta_k, \eta_{k+1}, \eta_{k+1}, \dots, \eta_{m+1}, \eta_{m+1}) \\ \xi = \frac{q-p-2}{4}(1, -1, \dots, 1, -1, 0, 1, -1, \dots, 1, -1) \end{cases}$$

and 0 is at the position $2k-1$. Since the infinitesimal character of the pseudo-spherical principal series is $\mathrm{Ind}_{P_{\min}}[S \otimes \exp(-\chi)]$ is (χ, ρ_{q-p}) we see that $(\eta + \xi, \rho_{q-p})$ is equal to $\mu_{p,q}$ up to a Weyl group element. Let $a = \frac{q-p-2}{4}$. If we remove ρ_{q-p} from $(\eta+\xi, \rho_{q-p})$ and $\mu_{p,q}$ then, up to a permutation of entries, we have

$$(\eta_1 + a, \eta_1 - a, \dots, \eta_{k-1} + a, \eta_{k-1} - a, \eta_k, \eta_{k+1} + a, \eta_{k+1} - a, \dots, \eta_{m+1} + a, \eta_{m+1} - a)$$

equals $(1, 2, \dots, \frac{p-1}{2}, \frac{q-p}{2}, \frac{q-p+2}{2}, \dots, \frac{q-1}{2})$.

If $k \neq 1$ then, by comparing the largest entries, $\eta_1 + a = \frac{q-1}{2}$. This implies that the second entry $\eta_1 - a = \frac{p+1}{2}$ which is not an entry of $\mu_{p,q}$. Hence $k = 1$ and $\eta_1 = \frac{q-1}{2}$. Since $q > p+1$, the next largest entry is $\frac{q-3}{2}$ and we conclude that $\eta_2 + a = \frac{q-3}{2}$ and $\eta_2 - a = \frac{p-1}{2}$. We can apply this argument repeatedly to conclude that η has the desired form. Theorem 7.1 is proved. $\square$

In the above computation of the Langlands parameter, we only use the minimal K-type and infinitesimal character of V. This gives the following corollary.

Corollary 7.3. *The module V is the unique irreducible $(\mathfrak{so}(p,q), K)$-module with the minimal K-type $\tau_p^{\frac{p-q}{2}\mathbf{1}} \otimes \mathbb{C}$ and infinitesimal character $\mu_{p,q}$.*

Next we consider $p-1 \geq q$ so $m = \frac{q}{2}$. Let $P_{\min} = M_{\min}A_{\min}N_{\min}$ be a minimal parabolic subgroup of $G_{p,q}$. The restricted root system for $A_{\min}$ is of type B_q. We realize B_q so that the simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{q-1} = \varepsilon_{q-1} - \varepsilon_q, \text{ and } \beta = \varepsilon_q.$$

We assume that $N_{\min}$ is "spanned" by positive roots. We discuss the split $(p-1 = q)$ and non-split $(p-1 > q)$ cases separately.

Case $p-1 > q$. Let $P = MAN$ be a parabolic subgroup of $G_{p,q}$ containing $P_{\min}$ such that the connected component M_0 of M is

$$M_0 = \mathrm{Spin}(p-q) \times \widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\mu_2} \ldots \times_{\mu_2} \widetilde{\mathrm{SL}}_2(\mathbb{R}).$$

Here there are m copies of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ corresponding to the restricted roots $\varepsilon_{2i-1} - \varepsilon_{2i}$ for $i = 1, \ldots, m$. Define

$$M_\beta = M_0 \times Z_\beta.$$

Since $p-1 > q$ we have $\frac{1}{2}(p-q) \geq \frac{3}{2}$. We recall that $D(\frac{p-q}{2})^+$ denotes the discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ with the lowest weight $\frac{p-q}{2}$ and $D(\frac{p-q}{2})^-$ is its dual module. Let $E = (\epsilon_1, \ldots, \epsilon_m)$ be an m-tuple of signs where $\epsilon_i = \pm$. We define

$$W^E = \mathrm{Ind}_{M_\beta}^M [D\left(\frac{p-q}{2}\right)^{\epsilon_1} \otimes \ldots \otimes D\left(\frac{p-q}{2}\right)^{\epsilon_m}]$$

where the subgroup $\mathrm{Spin}(p-q) \times Z_\beta$ acts trivially. The representations W^E are irreducible by the Mackey irreducibility criterion. Indeed, the restriction of W^E back to M_β consists of summands

$$D\left(\frac{p-q}{2}\right)^{\epsilon'_1} \otimes \ldots \otimes D\left(\frac{p-q}{2}\right)^{\epsilon'_m}$$

for all possible combinations of signs $(\epsilon'_1, \ldots, \epsilon'_m)$ such that $\prod_{i=1}^m \epsilon'_i = \prod_{i=1}^m \epsilon_i$. (Here we identify $\epsilon = +$ and $-$ with $\epsilon = 1$ and -1 respectively. There are 2^{m-1} such combinations which is precisely the index of M_β in M.) Thus, not only are representations W^E irreducible, but two such representations W^E and $W^{E'}$ are isomorphic if and only if the above product condition is satisfied. Thus, we have two isomorphism classes which we denoted by W^+ and W^-, where the sign is $\prod_{i=1}^m \epsilon_i$.

Next, $A \cong (\mathbb{R}^+)^m$, with the coordinates given by the long roots $\varepsilon_{2i-1} + \varepsilon_{2i}$, $i = 1, \ldots, m$. Let $\exp \eta$ denote a character of A where $\eta = \sum_{i=1}^m \eta_i(\varepsilon_{2i-1} + \varepsilon_{2i})$. If $\eta_1 > \ldots > \eta_{m-1} > |\eta_m|$ then the induced representation

$$\mathrm{Ind}_P^{G_{p,q}}(W^\pm \otimes \exp(-\eta))$$

contains a unique irreducible submodule. Its minimal K-type is the minimal type of $V^\pm$.

Theorem 7.4. *If $p > q+1$ then $V^{\pm}$ is the irreducible Langlands submodule of $\mathrm{Ind}_P^{G_{p,q}}[W^{\pm} \otimes \exp(-\eta)]$ where*

$$(\eta_1, \dots, \eta_{\frac{q}{2}}) = (\frac{q}{2}, \frac{q-2}{2}, \dots, 1) + \frac{1}{4}(p-q-2)\mathbf{1}_{\frac{q}{2}}.$$

Case $p-1=q$. Let $P = MAN$ be the parabolic subgroup in the standard position such that M contains the following as a subgroup of finite index:

$$M_\beta = Z_\beta \times \widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\mu_2} \dots \times_{\mu_2} \widetilde{\mathrm{SL}}_2(\mathbb{R}).$$

Here β is the unique short simple root and, as before, Z_β the image of of the center of $\mathrm{Spin}(2,1) \cong \mathrm{SL}_2(\mathbb{R})$ under φ_β. There are m copies of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, corresponding to the long simple roots $\alpha_1, \alpha_3, \dots, \alpha_{2m-1}$. Define an irreducible representation of M - trivial on Z_β - by

$$W_o^{\pm} = \mathrm{Ind}_{M_\beta}^{M}[D(\frac{3}{2})^{\epsilon_1} \otimes \dots \otimes D(\frac{3}{2})^{\epsilon_m}]$$

which, in an analogy with the case $p > q+1$, depends only on the sign of $\prod_{i=1}^m \epsilon_i$. Then the Langlands submodule of

$$\mathrm{Ind}_P^{G_{p,q}}[W_o^{\pm} \otimes \exp(-\eta)]$$

contains the minimal K-type of $V_o^{\pm}$.

The minimal parabolic $P_{\min} = M_{\min}A_{\min}N_{\min}$ of $M_{p,p-1}$ is the Borel subgroup. Let $\exp(\eta)$ be a dominant character of A where $\eta = \sum_{i=1}^q \eta_i \varepsilon_i$. Let $S^{\pm}$ be the irreducible representation of $M_{\min}$ such that the Langlands submodule of the pseudo-spherical principal series representation

$$\mathrm{Ind}_{P_{\min}}^{G_{p,p-1}}[S^{\pm} \otimes \exp(-\eta)]$$

contains the minimal K-type of $V^{\pm}$.

Theorem 7.5. *Here $p-1=q$ and $m = \frac{p-1}{2} = \frac{q}{2}$.*

(i) $V^{\pm}$ is a pseudo-spherical submodule of $\mathrm{Ind}_{P_{\min}}^{G_{p,p-1}}[S^{\pm} \otimes \exp(-\eta)]$ where

$$(\eta_1, \dots, \eta_{p-1}) = \frac{1}{2}(p-1, p-2, \dots, 2, 1).$$

(ii) $V_o^{\pm}$ is the irreducible Langlands submodule of $\mathrm{Ind}_P^{G_{p,p-1}}[W_o^{\pm} \otimes \exp(-\eta)]$ where

$$(\eta_1, \dots, \eta_m) = (m, m-1, \dots, 1) - \frac{1}{4}\mathbf{1}_m.$$

8. An extension to disconnected group

The main purpose of this section is to extend V to a $(\mathfrak{so}(p,q), \mathrm{Spin}(p) \times \mathrm{O}(q))$-module. This is necessary to obtain a one to one correspondence when we next restrict V to $\mathfrak{so}(p,a) \times \mathrm{O}(b)$ where $a+b=q$, and $\mathrm{O}(b) \subseteq \mathrm{O}(q)$.

Representations of $\mathrm{O}(n)$**.** We first describe a classification of irreducible representations of $\mathrm{O}(n)$. Let $\Lambda(\mathrm{O}(n))$ denote the subset of elements in $\mathbb{Z}^n$ such of the form

$$(\lambda_1, \ldots, \lambda_k, \mathbf{0}_{n-k}) \ \text{ or } (\lambda_1, \ldots, \lambda_k, \mathbf{1}_{n-2k}, \mathbf{0}_k)$$

where the λ_i's are positive integers, and $k \leq [\frac{n}{2}]$. Irreducible representations of $\mathrm{O}(n)$ are parameterized by $\Lambda(\mathrm{O}(n))$ as follows (see [GoW] or [Ho2]). Roughly speaking, for every λ in $\Lambda(\mathrm{O}(n))$, $\tau^{\lambda}_{\mathrm{O}(n)}$ is the irreducible finite dimensional representation of $\mathrm{O}(n)$ generated by a highest weight vector of the finite dimensional representation of GL_n with the highest weight λ. In particular, note that

$$\tau^{(\lambda_1,\ldots,\lambda_k,\mathbf{1}_{n-2k},\mathbf{0}_k)}_{\mathrm{O}(n)} = \det{}_n \otimes \tau^{(\lambda_1,\ldots,\lambda_k,\mathbf{0}_{n-k})}_{\mathrm{O}(n)}.$$

Elements of $\Lambda(\mathrm{O}(n))$ are called *highest weights* of $\mathrm{O}(n)$. Given a highest weight λ in $\Lambda(\mathrm{O}(n))$, we define

$$c(\lambda) = (\lambda_1, \ldots, \lambda_k, \mathbf{0}_{[n/2]-k}) \in \Lambda(n). \tag{22}$$

The restriction of $\tau^{\lambda}_{\mathrm{O}(n)}$ to $\mathrm{SO}(n)$ is irreducible and isomorphic to $\tau_n^{c(\lambda)}$ unless $n = 2k$ and $\lambda_k > 1$. In this case $\tau^{\lambda}_{\mathrm{O}(n)}$ is isomorphic to a direct sum $\tau_n^{c(\lambda)} \oplus \tau_n^{\sigma(c(\lambda))}$. In any case the infinitesimal character of $\tau^{\lambda}_{\mathrm{O}(n)}$ is $c(\lambda) + \rho_n$.

Next we discuss branching rule and tensor which are well known. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(\mathrm{O}(n))$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_{n-1}) \in \Lambda(\mathrm{O}(n-1))$. We write $\lambda \succ_{\mathrm{O}} \lambda'$ if

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \ldots \geq \lambda'_{n-1} \geq \lambda_n.$$

The restriction of $\tau^{\lambda}_{\mathrm{O}(n)}$ to $\mathrm{O}(n-1)$ is given by

$$\tau^{\lambda}_{\mathrm{O}(n)} = \oplus_{\lambda \succ_{\mathrm{O}} \lambda'} \tau^{\lambda'}_{\mathrm{O}(n-1)}.$$

This branching rule can be used to define another Gelfand-Zetlin basis of unit vectors of $\tau^{\lambda}_{\mathrm{O}(n)}$ by successive restrictions. Then each vector is represented by an array $N = (\lambda, \lambda^1, \ldots, \lambda^{n-1})$ such that $\lambda^i \in \Lambda(\mathrm{O}(n-i))$ and $\lambda^{i-1} \succ_{\mathrm{O}} \lambda^i$. We will denote the corresponding basis vector by $\mathbf{v}^{\mathrm{O}}_N$. We warn that this basis is closely related but different from the Gelfand-Zetlin basis of $\mathfrak{so}(n)$ introduced in Section 3. Let N^{+si} (respectively N^{-si}) denote the array obtained from N by adding (respectively subtracting) 1 from the s-th entry of λ^i.

Lemma 8.1. *Let $I_{i+1,i}$ be the element in $\mathfrak{so}(n)$ introduced in the beginning of Section 3. Then*

$$I_{i+1,i}\mathbf{v}^{\mathrm{O}}_N = \sum_{s=1}^{i} \alpha_s \mathbf{v}^{\mathrm{O}}_{N^{+s,n-i}} + \sum_{s=1}^{i} \beta_s \mathbf{v}^{\mathrm{O}}_{N^{-s,n-i}}$$

where the number α_s (respectively β_s) is non-zero as long as the array $N^{+s,n-i}$ (respectively $N^{-s,n-i}$) satisfies the Gelfand-Zetlin pattern.

Proof. We only give a sketch. First, note that it suffices to prove the statement in the case $i+1=n$. In this case the proof involves writing out the basis vectors $\mathbf{v}_N^{\mathrm{O}}$ in terms of the basis vectors $\mathbf{v}_N$ and then applying the formula for the action of $I_{n,n-1}$ on the basis $\mathbf{v}_N$ given on the page 364 in [VK]. □

Extension. We now extend V to a $(\mathfrak{so}(p,q), \mathrm{Spin}(p)\times \mathrm{O}(q))$-module. This extension will be very convenient for investigating dual pairs correspondences. There are two possible extensions and one differs from the other by the determinant character of $\mathrm{O}(p,q)$. Let ς be the diagonal matrix $\mathrm{diag}(1,\ldots,1,-1)$ in $\mathrm{O}(q)$. We define an action of ς on the basis vector $\mathbf{v}_M\otimes\mathbf{v}_N$ by $(-1)^{n_1-n_2}$ where n_1 and n_2 are the sums of the entries of the top and second top rows of the Gelfand-Zetlin array N respectively. With respect to this extension V has $\mathrm{Spin}(p)\times\mathrm{O}(q)$-types

$$\bigoplus_{\lambda\in\Lambda(p,0)} \tau_p^{\lambda+\frac{p-q}{2}\mathbf{1}}\otimes\tau_{\mathrm{O}(q)}^{(\lambda,\mathbf{0})}$$

where, we abbreviated, $\mathbf{1}=\mathbf{1}_{\frac{p-1}{2}}$ and $\mathbf{0}=\mathbf{0}_{q-\frac{p-1}{2}}$. Note that V has a basis consisting of vectors $\mathbf{v}_M\otimes\mathbf{v}_N^{\mathrm{O}}$.

9. Compact dual pair correspondences

In this section we restrict the $(\mathfrak{so}(p,q), \mathrm{Spin}(p)\times\mathrm{O}(q))$-module V to $\mathfrak{so}(p,a)\times\mathrm{O}(b)$ where $a+b=q$ and $\mathrm{O}(b)$ is included in $\mathrm{O}(q)$ in a standard way. Since V is $\mathrm{Spin}(p)$-admissible, we have a direct sum

$$V=\bigoplus_{\lambda'\in\Lambda(\mathrm{O}(b))}\Theta(\lambda')\otimes\tau_{\mathrm{O}(b)}^{\lambda'} \tag{23}$$

where $\Theta(\lambda')$ are admissible and unitarizable Harish-Chandra modules of $G_{p,a}$. Every summand on the right hand side of (23) is spanned by basis vectors $\mathbf{v}_M\otimes\mathbf{v}_N^{\mathrm{O}}$ of V where $\lambda^{q-b}=\lambda'$ in N. We now state the main theorem of this section.

Theorem 9.1. *Let $\lambda'\in\Lambda(\mathrm{O}(b))$ and $\Theta(\lambda')$ defined in (23). If $\Theta(\lambda')$ is nonzero, then it is an irreducible unitarizable Harish-Chandra module of $G_{p,a}$. Moreover, if $\lambda'\neq\gamma'$ then $\Theta(\lambda')$ and $\Theta(\gamma')$ are not isomorphic.*

Remark 9.2. Theorem A in Part II of [KO] proves a similar result for the ladder representation of $\mathrm{O}(p,q)$ where $p+q$ is even. Also compare with Theorem 3 in [Ko1] and, [GW].

Proof. We first describe the minimal $\mathrm{Spin}(p)\times\mathrm{SO}(a)$-type of $\Theta(\lambda')$. To this end, for any n-tuple $x=(x_1,\ldots,x_n)$ of real numbers define its height to be

$$|x|=\sum_{i=1}^{n}|x_i|.$$

If $\Theta(\lambda') \neq 0$ then $\tau_{\mathrm{O}(b)}^{\lambda'}$ is contained in $\tau_{\mathrm{O}(q)}^{(\lambda,\mathbf{0})}$ for some λ in $\Lambda(p,0)$. It follows, from branching rules from $\mathrm{O}(q)$ to $\mathrm{O}(b)$, that the number of non-zero entries in λ' is less than or equal to

$$u = \min\left(b, \frac{p-1}{2}\right).$$

In other words, we can write $\lambda' = (\lambda'_1, \ldots, \lambda'_u, 0, \ldots, 0)$. Furthermore, the smallest height λ in $\Lambda(p,0)$ such that $\tau_{\mathrm{O}(q)}^{(\lambda,\mathbf{0})}$ contains $\tau_{\mathrm{O}(b)}^{\lambda'}$ is

$$\lambda = (\lambda'_1, \ldots, \lambda'_u, 0, \ldots, 0) \tag{24}$$

and, in this case, $\tau_{\mathrm{O}(b)}^{\lambda'}$ is contained in $\tau_{\mathrm{O}(q)}^{(\lambda,\mathbf{0})}$ with multiplicity one. It follows that $\mathrm{SO}(a)$ acts on this summand trivially. Summarizing, we have shown that the $\mathrm{Spin}(p) \times \mathrm{SO}(a)$-type

$$\tau_p^{\lambda + \frac{q-p}{2}\mathbf{1}} \otimes \mathbb{C} \tag{25}$$

appears with multiplicity one in $\Theta(\lambda')$ where λ is given in terms of λ' as in (24). It is the minimal type of $\Theta(\lambda')$. Clearly, if $\lambda' \neq \gamma'$, then the minimal $\mathrm{Spin}(p) \times \mathrm{SO}(a)$-types of $\Theta(\lambda')$ and $\Theta(\gamma')$ are distinct.

It remains to show that $\Theta(\lambda')$ is irreducible. We will prove this by induction on a. When $a = 1$, this is Proposition 4.1. By a see-saw pair argument, the restriction of $\Theta(\lambda')$ to $\mathfrak{so}(p, a-1)$ decomposes as a direct sum

$$\Theta(\lambda') = \oplus_{\lambda''} \Theta(\lambda'') \tag{26}$$

where the sum is taken over all $\lambda'' \in \Lambda(\mathrm{O}(b+1))$ such that $\lambda'' \succ_{\mathrm{O}} \lambda'$.

Suppose $\Theta(\lambda')$ is reducible. Then it decomposes completely since it is admissible and unitarizable. Let Π be a proper submodule of $\Theta(\lambda')$ which does not contain the minimal $\mathrm{Spin}(p) \times \mathrm{SO}(a)$-type (25). By the induction assumption, the summands on the right hand side of (26) are irreducible and mutually non-isomorphic. Hence Π is a direct sum of some summands $\Theta(\lambda'')$ in (26). Let $\Theta(\lambda'')$ be a summand of Π such that the height of λ'' is minimal. Since Π does not contain the minimal type (25),

$$|\lambda''| > |\lambda'|.$$

This shows that for some s we have $\lambda'' - \varepsilon_s \succ_{\mathrm{O}} \lambda'$. Notice that the intersection

$$(\Theta(\lambda'') \otimes \tau_{\mathrm{O}(b+1)}^{\lambda''}) \cap (\Theta(\lambda') \otimes \tau_{\mathrm{O}(b)}^{\lambda'})$$

contains a vector of the form $\mathbf{v}_M \otimes \mathbf{v}_N^{\mathrm{O}}$ where the array N contains $\lambda'' \succ_{\mathrm{O}} \lambda'$. By Lemma 8.1 $I_{p+a,p+a-1}$, acting on this vector, gives a non-zero summand involving $\mathbf{v}_M \otimes \mathbf{v}_{N'}^{\mathrm{O}}$ where N' is obtained from N by replacing λ'' by $\lambda'' - \varepsilon_s$. In other words Π contains $\Theta(\lambda'' - \varepsilon_s)$. However, this contradicts the assumption that the height $|\lambda''|$ is minimal for $\Theta(\lambda'')$ contained in Π. Hence $\Theta(\lambda')$ is irreducible. The theorem is proved. $\square$

10. Action of Casimir operators

Let a and b be a pair of non-negative integers such that $a+b=q$. Then $\mathfrak{so}(p,a)\oplus\mathfrak{so}(b)$ is a dual pair in $\mathfrak{so}(p,q)$. In this section we shall compute a matching of the Casimir operators of the two Lie algebras acting on V.

Let Ω_n denote the Casimir operator of $\mathfrak{so}(n)$. We remind the reader that Ω_n acts by the scalar $||\lambda||^2-||\rho_n||^2$ on a representation of $\mathfrak{so}(n)$ with the infinitesimal character λ, and that the infinitesimal character of $\tau_n^{\lambda'}$ is $\lambda'+\rho_n$. Likewise, for $\lambda'\in\Lambda(\mathrm{O}(n))$, $\tau_{\mathrm{O}(n)}^{\lambda'}$ has infinitesimal character $c(\lambda')+\rho_n$ where $c(\lambda')$ was defined in (22).

Proposition 10.1. *Let Ω_{p+a} and Ω_b be the Casimir elements of $\mathfrak{so}(p,a)$ and $\mathfrak{so}(b)$, respectively. Then*

$$\Omega_{p+a}-\Omega_b+\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)\left(\frac{p+q}{2}-b\right)$$

annihilates V.

Proof. In order to simplify notation we shall work with $\mathfrak{so}_{p+q}(\mathbb{C})$ instead of $\mathfrak{so}(p,q)$. (The Casimir operator is invariant of the choice of real form.) Recall that the algebra $\mathfrak{so}_{p+q}(\mathbb{C})$ can be identified with the set of skew-symmetric matrices. The Casimir operator of $\mathfrak{so}_{p+q}(\mathbb{C})$ is equal to

$$\Omega_{p+q}=-\sum_{1\leq i<j\leq p+q} I_{ij}^2.$$

Clearly, in order to prove the lemma, it suffices to show that the operator annihilates every summand $\Theta(\lambda')\otimes\tau_{\mathrm{O}(b)}^{\lambda'}$ in (23). Since the factors of any summand are irreducible representations, we know that is that Ω_{p+a} and Ω_b act as scalars on the summand. Thus, in order to evaluate Ω_{p+a} on $\Theta(\lambda')$, it suffices to do so on a carefully chosen vector. We define, as in the previous section,

$$\lambda=(\lambda_1',\ldots,\lambda_u',0,\ldots,0)\in\Lambda(p,0).$$

Then the restriction of $\tau_{\mathrm{O}(q)}^{(\lambda,\mathbf{0})}$ to $\mathrm{O}(b)$ contains $\tau_{\mathrm{O}(b)}^{\lambda'}$ with multiplicity one. It follows that we have a $\mathrm{Spin}(p)\times\mathrm{SO}(a)\times\mathrm{O}(b)$-type

$$\delta=\tau_p^{\lambda+\frac{q-p}{2}\mathbf{1}}\otimes\mathbb{C}\otimes\tau_{\mathrm{O}(b)}^{\lambda'}$$

in $\Theta(\lambda')\otimes\tau_{\mathrm{O}(b)}^{\lambda'}$. Since $\tau_{\mathrm{O}(b)}^{\lambda'}$ is contained in $\tau_{\mathrm{O}(q-1)}^{(\lambda',\mathbf{0}_{q-b-1})}\subseteq\tau_{\mathrm{O}(q)}^{(\lambda',\mathbf{0}_{q-b})}$ it follows from Proposition 4.1 that δ is contained in

$$\mathrm{Ind}_{p,1}\left(\lambda+\frac{q-p}{2}\mathbf{1},0\right)\otimes\tau_{\mathrm{O}(q-1)}^{(\lambda',\mathbf{0})}.$$

Since the infinitesimal character of a principal series $\mathrm{Ind}_{p,1}(\mu,0)$ is $(\mu+\rho_{p-1},0)$, a simple calculation shows that

$$\Omega_{p+1}-\Omega_p=-\sum_{i=1}^{p} I_{p+1,i}^2=-\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)-|\lambda'|$$

on δ. Let j be an integer such that $p+1 < j \leq p+a$, and w a Weyl group element which permutes ε_{p+1} and ε_j. We can pick a representative of w in $\mathrm{SO}(a)$. Since $w(-\sum_{i=1}^{p} I_{p+1,i}^2) = -\sum_{i=1}^{p} I_{ji}^2$ and $\mathrm{SO}(a)$ acts trivially on δ, the sums $-\sum_{i=1}^{p} I_{ji}^2$ for $p+1 < j \leq p+a$ act by the same scalar on δ. Summing over all j, we get that

$$-\sum_{j=p+1}^{p+a}\sum_{i=1}^{p} I_{ji}^2 = -a\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) - a|\lambda'|$$

on δ. Since

$$\Omega_{p+a} = \Omega_p + \Omega_a - \sum_{j=p+1}^{p+a}\sum_{i=1}^{p} I_{ji}^2$$

and the values of Ω_p, Ω_a and Ω_b on δ are easily calculated, lemma is reduced to a straightforward check. $\square$

Proposition 10.2. *Let λ' in $\Lambda(b,0)$ such that $\Theta(\lambda') \neq 0$. Recall that the infinitesimal character of $\tau_{\mathrm{O}(b)}^{\lambda'}$ is*

$$\lambda = c(\lambda') + \rho_b = (\lambda_1, \ldots, \lambda_{[\frac{b}{2}]}).$$

Let $r = \min([\frac{b}{2}], \frac{p-1}{2})$. Then there exists numbers $\nu_{r+1}, \ldots, \nu_{[\frac{p+a}{2}]}$ independent of λ' such that the infinitesimal character of $\Theta(\lambda')$ is $(\lambda_1, \ldots, \lambda_r, \nu_{r+1}, \ldots, \pm\nu_{[\frac{p+a}{2}]})$. In particular, V establishes a correspondence

$$(\lambda_1, \ldots, \lambda_r, \nu_{r+1}, \ldots, \pm\nu_{[\frac{p+a}{2}]}) \longleftrightarrow (\lambda_1, \ldots, \lambda_r, \rho_{b-2r})$$

for the dual pair $\mathfrak{so}(p,a) \oplus \mathfrak{so}(b)$.

Proof. Let $\nu = (\nu_1, \ldots, \nu_{[\frac{p+a}{2}]})$ be the infinitesimal character of $\Theta(\lambda')$. Assume that $\lambda'' = \lambda' \pm \varepsilon_i$ is a highest weight for some $i \leq \min(b, \frac{p-1}{2})$. Then Lemma 8.1 shows that the action of $\mathfrak{so}(p,q)$ on $\Theta(\lambda') \otimes \tau_{\mathrm{O}(b)}^{\lambda'}$ followed by the projection on $\Theta(\lambda'') \otimes \tau_{\mathrm{O}(b)}^{\lambda''}$ is non-zero. Since

$$\mathfrak{so}(p,q) = \mathfrak{so}(p,a) \oplus \mathfrak{so}(b) \oplus \mathfrak{p}_0$$

where $\mathfrak{p}_0 \otimes \mathbb{C} = \mathbb{C}^{p+a} \otimes \mathbb{C}^b$, it follows that $\Theta(\lambda'')$ is a subquotient of $\mathbb{C}^{p+a} \otimes \Theta(\lambda')$. This shows that the infinitesimal character of $\Theta(\lambda'')$ is $\nu \pm \varepsilon_j$ for some j or it is equal to ν. The last possibility might happen only for $p+a$ odd, ie b is even.

Before we state the next lemma we note that the infinitesimal character of $\tau_{\mathrm{O}(b)}^{\lambda''}$ is

$$c(\lambda'') + \rho_b = \begin{cases} \lambda - \varepsilon_i \text{ if } \lambda'' = \lambda' - \varepsilon_i \text{ or } \lambda'' = \lambda' + \varepsilon_{b-i+1} \text{ for some } i < \frac{b+1}{2} \\ \lambda + \varepsilon_i \text{ if } \lambda'' = \lambda' + \varepsilon_i \text{ or } \lambda'' = \lambda' - \varepsilon_{b-i+1} \text{ for some } i < \frac{b+1}{2} \\ \lambda \text{ if } \lambda'' = \lambda' \pm \varepsilon_{\frac{b+1}{2}}. \end{cases}$$

The last case occurs only for b odd.

Lemma 10.3. *Let λ' and λ'' as above. Assume that for some λ' the infinitesimal character of $\Theta(\lambda')$ is given by $\nu = (\lambda_1, \ldots, \lambda_r, \nu_{r+1}, \ldots, \nu_{[\frac{p+a}{2}]})$ for some ν_i.*

(i) *Suppose that the infinitesimal character of* $\tau^{\lambda''}_{\mathrm{O}(b)}$ *is* $\lambda + \varepsilon_i$ *for some* $i \leq r$*. Then the infinitesimal character of* $\Theta(\lambda'')$ *is* $\nu + \varepsilon_i$*.*
(ii) *Suppose that the infinitesimal character of* $\tau^{\lambda''}_{\mathrm{O}(b)}$ *is* $\lambda - \varepsilon_i$ *for some* $i \leq r$*. Then the infinitesimal character of* $\Theta(\lambda'')$ *is* $\nu - \varepsilon_i$*.*
(iii) *Suppose that the infinitesimal character of* $\tau^{\lambda''}_{\mathrm{O}(b)}$ *is* λ*. This happens only if* b *is odd and* $\lambda'' = \lambda' \pm \varepsilon_{\frac{b+1}{2}}$*. Then the infinitesimal character of* $\Theta(\lambda'')$ *is* $\sigma(\nu)$*.*

Proof. First, we claim that the infinitesimal character of $\Theta(\lambda'')$ is not equal to ν if b is even. If the infinitesimal character of $\Theta(\lambda')$ is ν then the matching of Casimir operators in Proposition 10.1 implies that $||\lambda \pm \varepsilon_i||^2 = ||\lambda||^2$ and $\lambda_i = \mp 1/2$. However, λ_i is an integer since b is even. This is a contradiction and it proves our claim.

Let us prove (i). The infinitesimal character of $\Theta(\lambda'')$ is $\nu \pm \varepsilon_j$ for some j. Assume first that it is $\nu + \varepsilon_j$. Then Proposition 10.1 implies that

$$||\nu + \varepsilon_j||^2 - ||\nu||^2 = ||\lambda + \varepsilon_i||^2 - ||\lambda||^2,$$

that is, $\nu_j = \lambda_i$. Permutation of i-th and j-th places - as an element of the absolute Weyl group of $\mathfrak{so}(p,a)$ - replaces $\nu + \varepsilon_j$ by $\nu + \varepsilon_i$, as desired. Similarly, if the infinitesimal character is $\nu - \varepsilon_i$, then $\nu_j = -\lambda_i$. If $i \neq j$ then we can replace $\nu - \varepsilon_j$ by $\nu + \varepsilon_i$ by permuting the two places and changing the signs of both of them. If $i = j$ then $\nu_i = -\lambda_i$ implies that $\lambda_i = 0$. It follows that b is even and $p + a$ odd. Hence the absolute Weyl group of $\mathfrak{so}(p,a)$ is a B type and $\nu + \epsilon_i$ can be replaced by $\nu - \epsilon_i$. The case (ii) is proved analogously. For the last case, the infinitesimal character of $\Theta(\lambda'')$ is $\nu \pm \varepsilon_j$ for some j. Then Proposition 10.1 implies that $||\nu \pm \varepsilon_j||^2 = ||\nu||^2$, that is, $\nu_j = \mp\frac{1}{2}$. It follows that $\nu \pm \varepsilon_j$ is Weyl group equivalent to $\sigma(\nu)$. The lemma is proved. □

It remains to show that the infinitesimal character of $\Theta(\lambda')$ is of the desired form for one λ'. Pick λ' so that $\lambda'_1 > \ldots > \lambda'_r$. Then $\lambda' + \varepsilon_i$ is a highest weight for all $i = 1, \ldots, r$. Since the infinitesimal character of $\Theta(\lambda' + \varepsilon_i)$ is equal to $\nu \pm \varepsilon_j$ for some j it follows, from Proposition 10.1 that $\lambda_i = \pm\nu_j$. This shows that every λ_i is up to a sign equal to an entry of ν. Since the absolute rank of $\mathfrak{so}(p,a)$ is bigger then r, the Weyl group can in any case rearrange the entries of ν so that it begins with $\lambda_1, \ldots, \lambda_r$. □

Remark 10.4. The correspondence is independent of the real form of the complex dual pair $\mathfrak{so}_{p+a}(\mathbb{C}) \oplus \mathfrak{so}_b(\mathbb{C})$ in $\mathfrak{so}_{p+q}(\mathbb{C})$.

Based on this observation, we can now give a proof of the first correspondence of infinitesimal characters in Theorem 1.2, that is, when $p \leq b \leq q$. Indeed, the above mentioned correspondence of infinitesimal characters is also equal to that of the dual pair $\mathfrak{so}(p, b-p) \oplus \mathfrak{so}(p+a)$. This uniquely determines the ν_i's in Proposition 10.2 and proves the first correspondence in Theorem 1.2.

11. Correspondence of infinitesimal characters

In order to determine the correspondence of infinitesimal characters for the dual pair $\mathfrak{so}_{p+a}(\mathbb{C}) \oplus \mathfrak{so}_b(\mathbb{C})$ (Theorem 1.2) it remains to determine the ν_i's in Proposition 10.2 if $b < p$. Recall that for every $p \leq q-1$ we have an embedding

$$V \subseteq \mathrm{Ind}_{P_{\min}}[S \otimes \exp(-\chi)],$$

where χ is given by Theorem 7.1 if $p = q-1$ and is constructed by means of η as explained in (21), otherwise. Next, for every $r = 1, \ldots, p$ consider

$$G_{r,r} \times_{\mu_2} G_{p-r,q-r} \subseteq G_{p,q}$$

where the simple roots of $G_{r,r}$ are $\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \varepsilon_{r-1} + \varepsilon_r$.

Let $P' = M'A'N'$ and $P'' = M''A''N''$ be the minimal parabolic subgroups of $G_{r,r}$ and $G_{p-r,q-r}$, respectively, in standard position with respect to our choices of simple roots. Let

$$\begin{aligned} \chi' &= -(\chi_1, \ldots, \chi_r) + \left(\frac{p+q}{2} - r\right)\mathbf{1}_r, \\ \chi'' &= -(\chi_{r+1}, \ldots, \chi_p). \end{aligned}$$

Lemma 11.1. *For every $r = 1, \ldots, p$ there exists a nonzero homomorphism of $G_{r,r} \times G_{p-r,q-r}$-modules*

$$V \to \mathrm{Ind}_{P'}^{G_{r,r}}[S' \otimes \chi'] \otimes \mathrm{Ind}_{P''}^{G_{p-r,q-r}}[S'' \otimes \chi'']$$

for some $M' \times M''$-summand $S' \otimes S''$ of S. In particular - for this χ' and χ'' - we have a correspondence of infinitesimal characters

$$\chi' \longleftrightarrow (\chi'', \rho_{q-p}). \tag{27}$$

Proof. Since V is a submodule of $\mathrm{Ind}_{P_{\min}}[S \otimes (-\exp\chi)]$, the Frobenius reciprocity implies that there exists a non-trivial homomorphism of $P_{\min}$-modules $V \to S \otimes \exp(\rho - \chi)$. Restricting to $P' \times P''$ and using the Frobenius reciprocity again, proves the lemma. $\square$

As we shall see in a moment, the correspondence (27) gives ν_i's if $q - p \leq b < q$. In order to deal with $b < q - p$ we need one more statement. If $b < q - p$ then $a > p$. For such a let $P_a = M_a A_a N_a$ be the minimal parabolic subgroup of $G_{p,a} \subseteq G_{p,q}$. In particular, we have the following obvious lemma:

Lemma 11.2. *Assume that $a + b = q$ and $a > p$. There exists a non-zero homomorphism of $G_{p,a} \times \mathrm{Spin}(b)$-modules*

$$V \to \mathrm{Ind}_{P_a}^{G_{p,a}}[S \otimes \chi] \otimes \mathbb{C}.$$

In particular we have a correspondence of two the infinitesimal characters

$$\left(-\chi + \frac{b}{2}\mathbf{1}_p, \rho_{a-p}\right) \longleftrightarrow \rho_b. \tag{28}$$

Proof of Theorem 1.2. The first correspondence in the theorem was established in Remark 10.4. It remains to deal with $b < p$.

Case 1: $q = p + 1$. In this case $\chi = \frac{1}{2}(p, p-1, \ldots, 1)$ and, up to a Weyl group action, (27) becomes

$$(\frac{p-r}{2}, \frac{p-r-1}{2}, \ldots, \frac{p-2r+1}{2}) \longleftrightarrow (\frac{p-r}{2}, \frac{p-r-1}{2}, \ldots, \frac{1}{2}).$$

If we cancel out the same numbers from both sides of the above correspondences until one side is empty, then the remaining numbers are ν_i's. If b is even, we use $r = b/2$. Then the remaining terms are on the left side. They are

$$(\frac{p-2r}{2}, \frac{p-2r-1}{2}, \ldots, \frac{1}{2}) = (\frac{p-b}{2}, \frac{p-b-1}{2} \ldots, \frac{1}{2})$$

and this is $\mu_{p-b,p-b+1}$, as desired. If b is odd, then we use $r = (a+p)/2$, and the remaining terms are on the right side. They are

$$(0, \frac{-1}{2}, \ldots \frac{p-2r+1}{2}) = (0, -\frac{1}{2}, \ldots, -\frac{p-b}{2})$$

and this is $\delta_{p-b+1,p-b+1}$, up to a Weyl group action. This proves the third correspondence.

Case 2: $q > p + 1$. We set $e = 0$ if r is even and $e = \frac{1}{2}$ if r is odd. Let $m = \frac{p-1}{2}$ and $m' = \frac{q}{2}$. Then up to a Weyl group action, (27) becomes

$$\begin{aligned}&(m-r+1, m-r+2, \ldots, m-\frac{r}{2}+e; m'-r+\frac{1}{2}, m'-r+\frac{3}{2}, \ldots, m'-\frac{r+1}{2}-e)\\ &\longleftrightarrow (1, 2, \ldots, m-\frac{r}{2}+e; \frac{1}{2}, \frac{3}{2}, \ldots, m'-\frac{r+1}{2}-e).\end{aligned}$$

Likewise, we remove the same set of numbers from both sides of the above correspondences until one side is empty. Then the numbers that are left behind would be the ν_i's. Set $b = 2r$ and assume that $b < p$. Then $r < m$, $r < m'$ and $(\nu_i) = \mu_{p-b,q-b}$. This gives the second correspondence of Theorem 1.2. If we set $p + a = 2r$ and assume that $b = p + q - 2r < p$ then, since $r \leq p$, we must also have that $b \geq q - p$. In this case we get $(\nu_i) = \delta_{p-b+1,q-b}$. This gives the third correspondence of Theorem 1.2 for the case $p > b \geq q - p$.

Finally if b is odd, $b \leq q - p$ and $b < p$, then we refer to Lemma 11.2. Then up to a Weyl group action, (28) becomes

$$(\rho_b, \rho_{p-b+1}, \rho_{q-b+1}) \longleftrightarrow \rho_b$$

This gives $(\nu_i) = \delta_{p-b,q-b+1}$ and proves the third correspondence. With this, we complete the proof of Case 2 and also the proof of Theorem 1.2. $\square$

Since the annihilator of $V^{\pm}$ in $\mathfrak{so}(p,q)$ is the same as the annihilator of V in $\mathfrak{so}(q+1,p-1)$ Theorem 1.1 also gives a matching of infinitesimal characters for $V^{\pm}$ except when $q = p-1$. In this case we have the following:

Theorem 11.3. *Assume $q = p-1$. Assume that a and b are positive integers such that $a+b = p-1$. Then V^+, V^-, V_o^+ and V_o^- establish the following correspondence of infinitesimal characters for the dual pair $\mathfrak{so}(p,a)\times\mathfrak{so}(b)$:*

$$\begin{cases}(\lambda_1,\dots,\lambda_{\frac{b}{2}},\mu_{p-b,p-b-1}) \longleftrightarrow (\lambda_1,\dots,\lambda_{\frac{b}{2}}) \text{ if } b \text{ is even}\\ (\lambda_1,\dots,\lambda_{\frac{b-1}{2}},\delta_{p-b,p-b}) \longleftrightarrow (\lambda_1,\dots,\lambda_{\frac{b-1}{2}}) \text{ if } b \text{ is odd}\end{cases}$$

where we recall $\delta_{p-b,p-b} = (\rho_{p-b},\rho_{p-b+1})$ or $\left(\frac{p-b-1}{2},\frac{p-b-2}{2},\dots,\frac{1}{2},0\right)$, up to a Weyl group action.

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