RESTRICTIONS OF QUATERNIONIC REPRESENTATIONS

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ABSTRACT

In [GW2] the K-types of the continuations of the quaternionic discrete series of a quaternionic Lie group G are associated with projective orbits $\mathcal{O}$ of certain subgroups in $G(\mathbb{C})$. In this paper, we will show that the restrictions of the representations to quaternionic subgroups are closely related with the intersection of the Zariski closure of $\mathcal{O}$ with hyperplanes. We apply this to the minimal representations of the exceptional groups of real rank 4 and investigate the correspondences of certain compact dual pairs.

1. Introduction

1.1. We refer to §3 of [GW2] and §2 of this paper for the definition of the double cover G of a quaternionic real form G_0 of a complex Lie group $G(\mathbb{C})$. G has maximal compact subgroup K of the form $K_1 \times M$ where $K_1 \simeq SU_2$. It has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{p} = \mathbb{C}^2 \otimes V_M$ where V_M is a self dual representation of $M(\mathbb{C})$. See Table 2 for examples of G_0.

Choose a positive root system Φ^+ with respect to a compact Cartan subgroup in K such that K_1 corresponds to the highest root $\tilde{\alpha}$. In this paper we will denote K_1 by $SU_2(\tilde{\alpha})$. There is a family of discrete series representations of G which corresponds to Φ^+ and they are called the quaternionic discrete series representations. In §2 we will investigate representations which are continuations of the quaternionic discrete series representations. We will call their Harish-Chandra modules *quaternionic* representations. We will abuse notation and continue to refer them as representations of G.

Let G' be a quaternionic Lie subgroup of G containing $SU_2(\tilde{\alpha})$. We will show in Theorem 3.4.1 that a unitary quaternionic representation of G decomposes discretely into quaternionic representations when restricted to G'. One explanation for such a result to hold is that quaternionic representations are admissible with respect to $SU_2(\tilde{\alpha})$ and they remain admissible when restricted to the Lie subgroup G'. In addition the theorem states that the spectrum of the restriction is determined by the cokernels of homomorphisms of finite dimensional representations of the compact subgroup $M \cap G'$.

1.2. Gross and Wallach [GW1] [GW2] construct certain unitary representations in the continuations of the quaternionic discrete series. Each representation is associated with an $M(\mathbb{C})$-orbit $\mathcal{O}$ in $\mathbb{P}V_M$ in the sense that it has K-types ($K = SU_2 \times M$)

$$\sum_{n=0}^{\infty} Sym^{n+k}(\mathbb{C}^2) \otimes A^n(\overline{\mathcal{O}}).$$

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Here $\bigoplus_n A^n(\overline{\mathcal{O}})$ is the coordinate ring of the Zariski closure of $\mathcal{O}$ in $\mathbb{P}V_M$. In §4 we will apply Theorem 3.4.1 and deduce Corollary 4.2.1 which states that the irreducible components of the restriction to G' of such a representation is almost determined by the coordinate ring of the intersection of $\overline{\mathcal{O}}$ with a subspace in $\mathbb{P}V_M$.

1.3. In §5 we will study exceptional compact dual pairs correspondences. Each of the four exceptional Lie algebras $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ has a unique quaternionic real form $\mathfrak{g}_0$, which has real root system of type F_4. Let G be the corresponding real quaternionic Lie group. Then $M(\mathbb{C})$ has a unique minimal closed orbit $\mathcal{O} = Z$ in $\mathbb{P}V_M$. There is a unitary representation in the continuation of the quaternionic discrete series which is associated with this orbit. We will follow [GW1] and denote this representation by σ_Z. It is annihilated precisely by the Joseph ideal and it is thus called the *minimal* representation.

A pair of subgroups $H_1 \times_C H_2$ (cf. (1)) in G is called a *dual pair* if the centralizer of H_i in G is H_{i+1}. The dual pair is called compact if either H_1 or H_2 is compact. The main motivation and objective of this paper is to investigate the restrictions of the minimal representations σ_Z to compact dual pairs.

Exceptional dual pairs correspondences are investigated by [HPS], [Li1], [Li2], [GS] and [L1]. So far the method of solving compact dual pairs correspondences is mainly done through the computations on K-types and branching rules. Theorem 3.4.1 provides an alternative approach to this problem. We will show in Proposition 5.4.2 that a large number of compact dual pairs correspondences exist and we tabulate the results in the appendix.

In §6 we work out the correspondences for the dual pairs $SU(2,1) \times H_2$ in the four exceptional groups.

1.4. Finally in §7 we apply the same method to the holomorphic discrete series representations and their continuations [Wa] [RV]. We will prove analogous results on the restrictions of the holomorphic representations to holomorphic subgroups. Restrictions of the holomorphic discrete series representations have been investigated in [Ma] [JV]. Compact dual pairs correspondences of the minimal representations, in particular the Weil representations are well known (see [Ho], [KaV] and many more).

1.5. We define some notations. $\pi_G(a_1\varpi_1 + \ldots + a_n\varpi_n)$ will denote the irreducible finite dimensional complex representation of a semisimple Lie group G with highest weight $a_1\varpi_1 + \ldots + a_n\varpi_n$ where ϖ_i are the fundamental weights given in Planches [Bou] . If V is a representation of G, then $S^n(V) = Sym^n V$ will denote its n-th symmetric product and V^* its dual representation. U_1^n will denote the n-th power of the fundamental character of the 1 dimensional compact torus U_1. μ_n will denote the cyclic group of order n. Suppose H_1 and H_2 are subgroups of G and C lies in the centers of both H_1 and H_2, then we denote

$$H_1 \times_C H_2 := (H_1 \times H_2)/\{(z,z) : z \in C\}. \tag{1}$$

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2. Quaternionic Groups and Representations

2.1. In this section we define some notations. In §2.2 we briefly recall the definition of quaternionic real form of an algebraic group. In §2.3 we will define quaternionic representations and review some of their properties. Finally we construct filtrations of the quaternionic representations which we need in §3.

2.2. Let $G(\mathbb{C})$ be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let G_c be a compact real form with Lie algebra $\mathfrak{g}_c$. Let τ be the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_c$. Let $\mathfrak{h}_c$ be a compact Cartan subalgebra (CSA) of $\mathfrak{g}_c$ and define $\mathfrak{h} = \mathfrak{h}_c \otimes \mathbb{C}$. Choose a positive root system Φ^+ with respect to $\mathfrak{h}$ and denote its highest weight by $\tilde{\alpha}$. Define

$$\mathfrak{g}_i = \sum_{\langle \alpha, \tilde{\alpha} \rangle = i} \mathfrak{g}_\alpha \text{ for } i \in \mathbb{Z}.$$

For $i = 0$ we will write $\mathfrak{g}_{(0)}$ so as to avoid confusion with $\mathfrak{g}_0$ defined in the next paragraph. Then $\mathfrak{g}_i = 0$ if $i \neq 0, \pm 1, \pm 2$ and $\mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm \tilde{\alpha}}$. Define

$$\begin{aligned}
\mathfrak{h}_0 &= [\mathfrak{g}_2, \mathfrak{g}_{-2}] \subset \mathfrak{h} \\
\mathfrak{su}_2(\tilde{\alpha}) &= \mathfrak{g}_2 \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_{-2} \\
\mathfrak{u} &= \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \overline{\mathfrak{u}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \\
\mathfrak{l} &= \mathfrak{h} \oplus \mathfrak{g}_{(0)} \\
\mathfrak{q} &= \mathfrak{l} \oplus \mathfrak{u}, \quad \overline{\mathfrak{q}} = \mathfrak{l} \oplus \overline{\mathfrak{u}}.
\end{aligned}$$

$\mathfrak{q}$ and $\overline{\mathfrak{q}}$ are opposite two-step nilpotent parabolic subalgebras with Levi factors $\mathfrak{l}$. $\mathfrak{l} = \mathfrak{h}_0 \oplus \mathfrak{m}$ for some reductive subalgebra $\mathfrak{m}$. Denote $V_M = \mathfrak{g}_1$ as the representation of $\mathfrak{m}$. It is a self dual representation of $\mathfrak{m}$.

We recall the definition of the quaternionic real form G_0 of $G(\mathbb{C})$ with Lie algebra $\mathfrak{g}_0$ in $\mathfrak{g}$ (see §3 [GW2]). Let $SU_2(\tilde{\alpha})$ be the Lie subgroup of G_c having complexified Lie algebra $\mathfrak{su}_2(\tilde{\alpha})$. Let h be the nontrivial element in the center of $SU_2(\tilde{\alpha})$. Then the quaternionic real form G_0 of $G(\mathbb{C})$ is defined as the connected component of the identity element of the group

$$\{g \in G(\mathbb{C}) : \tau g = hgh^{-1}\}.$$

G_0 has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{su}_2(\tilde{\alpha}) \oplus \mathfrak{m}$ and $\mathfrak{p} = \mathbb{C}^2 \otimes V_M$. We denote the connected real Lie groups in $G(\mathbb{C})$ corresponding to the various Lie algebras by G_0, $K_0 = SU_2(\tilde{\alpha}) \times_{\mu_2} M$, $L_0 = U_1 \times_{\mu_2} M$. Let G denote the double cover of G_0 with maximal compact subgroup $K = SU_2(\tilde{\alpha}) \times M$. Define the subgroup $L := U_1 \times M$ in K. We will call $G \times H$ a *quaternionic* Lie group if G is a quaternionic simple Lie group and H is a compact Lie group.

We tabulate $M(\mathbb{C})$ and V_M below. Set $2d = \dim V_M$. If $\mathfrak{g}$ is of type D_4, F_4, E_6, E_7, E_8, then $d = 3s + 4$ where $s = 0, 1, 2, 4, 8$ respectively.

Table 1

	$\mathfrak{g}$	$M(\mathbb{C})$	V_M
(a)	A_{d+1}, $d \geq 2$	$U_1 \times SL_d$	$(U_1^{d+2} \otimes \mathbb{C}^d) \oplus (U_1^{-d-2} \otimes (\mathbb{C}^d)^*)$
(b)	$\mathfrak{so}(d,4)$, $d \geq 5$	$SL_2 \times SO_d$	$\mathbb{C}^2 \otimes \mathbb{C}^d$
(c)	C_{d+1}	Sp_{2d}	$\mathbb{C}^d$
(d)	D_4	SL_2^3	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
(e$_1$)	F_4	Sp_6	$\pi(\varpi_3)$
(e$_2$)	$E_6 \rtimes \mathbb{Z}/2\mathbb{Z}$	$SU_6 \rtimes \mathbb{Z}/2\mathbb{Z}$	$\pi(\varpi_3)$
(e$_4$)	E_7	$Spin(12)$	$\pi(\varpi_6)$
(e$_8$)	E_8	simply connected E_7	$\pi(\varpi_1)$
(f)	G_2	SL_2	$S^3(\mathbb{C}^2)$

2.3. It is well known that G/L has a complex structure and G/K has a quaternionic structure (cf. §3 [GW2]). Let $W[k] = e^{-k\tilde{\alpha}/2} \otimes W$ be an irreducible finite dimensional representation of $L = U_1 \times M$. Let $\mathcal{O}(W[k])$ denote the sheaf of holomorphic sections of the G-equivariant bundle on G/L induced from the representation $W[k]$.

Denote

$$\mathbf{H}(G, U) := \Gamma^1_{K/L}(\mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), U)_L).$$

as the Harish-Chandra module of G where Γ^1 is the first Zuckerman derived functor and U is a finite dimensional representation of L extended trivially to $\bar{\mathfrak{u}}$. If $k \geq 2$, then by the work of Schmid [S1], Wong [W1] [W2] and Gross and Wallach [GW1] [GW2], $H^1(G/L, \mathcal{O}(W[k]))$ is a complex Frechet space and it is the maximal globalization of $\mathbf{H}(G, W[k])$. It has infinitesimal character $\mu + \rho(G) - k\frac{\tilde{\alpha}}{2}$ and K-types ($K = SU_2(\tilde{\alpha}) \times M$)

$$\sum_{n=0}^{\infty} S^{k-2+n}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes (Sym^n(V_M) \otimes W). \tag{2}$$

It contains a unique irreducible G-submodule which is generated by the translates of the lowest K-types

$$S^{k-2}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes W. \tag{3}$$

Denote its Harish-Chandra module by $\sigma(G, W[k])$. Sometimes we will omit G and write $\sigma(G, W[k])$ and $\mathbf{H}(G, W[k])$ as $\sigma(W[k])$ and $\mathbf{H}(W[k])$ respectively. We will call $\mathbf{H}(W[k])$ and $\sigma(W[k])$ *quaternionic* representations.

The references for the proofs of the above results could be found in Theorem 3.3.1 of [L1] where we give a more thorough discussion. Also see §5 [GW2].

If $\mathbf{H}(W[k])$ is unitarizable, then it is irreducible for otherwise the orthogonal complement of $\sigma(W[k])$ would be a nontrivial submodule which does not contain the lowest K-type (3).

It is clear that if $W = \sum_i W_i$ is a decomposable representation, then

$$\mathbf{H}(W[k]) = \sum_i \mathbf{H}(W_i[k]) \quad \text{and} \quad \sigma(W[k]) = \sum_i \sigma(W_i[k]). \tag{4}$$

2.4. Suppose G' is a connected quaternionic real Lie subgroup of G containing $SU_2(\tilde{\alpha})$. We have correspondingly $K' = G' \cap K$, $M' = G' \cap M$, $L' = G' \cap L$ and the Lie algebras $\mathfrak{g}'$, $\mathfrak{m}'$, $\mathfrak{l}'$, $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$, $\overline{\mathfrak{q}}' = \mathfrak{l}' \oplus \overline{\mathfrak{u}}'$. Denote $\mathfrak{u}''$ and $\overline{\mathfrak{u}}''$ to be the sum of root spaces of G such that

$$\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{u}'' \quad \text{and } \overline{\mathfrak{u}} = \overline{\mathfrak{u}}' \oplus \overline{\mathfrak{u}}''.$$

We also have $V_{M'} \subset V_M$. Note that $V_0 := V_M/V_{M'} = \mathfrak{u}'' = \overline{\mathfrak{u}}''$ as representations of M'. In order to avoid confusion, from now on, $\overline{\mathfrak{u}}'$ and $\overline{\mathfrak{u}}''$ will strictly denote a representation of $L' = U_1 \times M'$ whereas $V_{M'}$ and V_0 will denote representations of M'.

2.5. Given a representation $W[k]$ of L, we extend this to a representation of $\mathfrak{q}$ by letting $\mathfrak{u}$ act trivially. We define the generalized Verma module

$$N(W[k]) = N(G, W[k]) = N(\mathfrak{g}, L, W[k]) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} W[k].$$

Note that this is a $(\mathfrak{g}, L)$-module. As a representation of M

$$N(\mathfrak{g}, L, W[k]) = \sum_{n=0}^{\infty} Sym^n(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} W \tag{5}$$

and the torus $U_1 \subset SU_2(\tilde{\alpha})$ acts on the n-th graded piece by $e^{-(k+n)\tilde{\alpha}/2}$.

$\mathcal{U}(\mathfrak{g})$ has a natural filtration

$$1 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots$$

where $\mathcal{U}_n$ is generated as a vector space by elements of the form $X_1 \cdots X_s$ where $X_i \in \mathfrak{g}$ and $s \leq n$.

Let V_n be the $\mathcal{U}(\mathfrak{g}')$-submodule of $N(\mathfrak{g}, L, W[k])$ defined by

$$V_n := (\mathcal{U}(\mathfrak{g}') \cdot \mathcal{U}_n) \otimes_{\mathcal{U}(\mathfrak{q})} W[k]. \tag{6}$$

If $n = 0$, we write $V_{(0)}$ so as to avoid confusion with V_0 defined above. V_n forms a filtration of $N(\mathfrak{g}, L, W[k])$. As a representation of L',

$$V_n = \sum_{r=0}^{\infty} \sum_{m=0}^{n} Sym^r(\overline{\mathfrak{u}}') \otimes_{\mathbb{C}} Sym^m(\overline{\mathfrak{u}}'') \otimes_{\mathbb{C}} W[k]. \tag{7}$$

It is clear that $\mathfrak{g} V_n = V_{n+1}$.

2.6. For the ease of notations, let

$$M_n = \mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{q}')} (Sym^n(\overline{\mathfrak{u}}'') \otimes_{\mathbb{C}} W[k])$$

where $\overline{\mathfrak{u}}''$ is considered as a representation of L' extended trivially to $\mathfrak{q}'$. With reference to (7) we define

$$\phi_n : V_n \longrightarrow M_n$$

to be the natural L'-module projection of V_n into

$$M_n = \sum_{r=0}^{\infty} Sym^r(\overline{\mathfrak{u}}') \otimes_{\mathbb{C}} (Sym^n(\overline{\mathfrak{u}}'') \otimes_{\mathbb{C}} W).$$

This is a well defined map of $(\mathfrak{g}', L')$-modules and

$$0 \longrightarrow V_{n-1} \longrightarrow V_n \xrightarrow{\phi_n} M_n \longrightarrow 0 \tag{8}$$

is an exact sequence of $(\mathfrak{g}', L')$-modules (see §1 [JV]).

2.7. Let $\Gamma^n = \Gamma^n_{K/L}$ be the n-th derived functor of the Zuckerman functor of taking K-finite vectors. A modified proof of Prop. 9.12 of [GW2] gives the following lemma.

Lemma 2.7.1. *Let Q be a $(\mathfrak{g}', L')$-subquotient of $N(\mathfrak{g}, L, W[k])$ for $k \geq 2$. Then $\Gamma^i_{K'/L'}Q = 0$ for $i \neq 1$.* □

By applying Lemma 2.7.1 to the exact sequence (8), we get an exact sequence

$$0 \to \Gamma^1(V_{n-1}) \to \Gamma^1(V_n) \to \Gamma^1(M_n) \to 0. \tag{9}$$

2.8. Given a complex vector space V, let $V^\wedge$ denote the space of conjugate linear complex functions on V. If V is a $(\mathfrak{g}, K)$-module, we refer to §6 [EPWW] for the definition of the $(\mathfrak{g}, K)$-module structure on $V^\wedge$.

Given an irreducible representation V of L, there is a Hermitian pairing

$$\langle \, , \, \rangle : \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), V)_L \times N(\mathfrak{g}, L, V^\wedge) \to \mathbb{C} \tag{10}$$

given by $\langle f, X \otimes v \rangle = v(f(X))$. Using this pairing one checks that each factor is the conjugate dual of the other. Since L is compact, $V \simeq V^\wedge$ as representations of L. The aim of introducing the conjugate dual is to state Thm 6.3 of [EPWW].

Theorem 2.8.1. *$A \mapsto (\Gamma^1 A)^\wedge$ and $A \mapsto \Gamma^1(A^\wedge)$ are natural equivalent functors from the category of $(\mathfrak{g}, L)$-modules to the category of $(\mathfrak{g}, K)$-modules.* □

2.9. The inclusion $V_{n-1} \subset N(W[k])$ induces a surjection

$$\mathbf{H}(G, W[k]) \to \Gamma^1(V_{n-1})^\wedge \to 0. \tag{11}$$

Denote the kernel of the above map by H'_n. These modules form a decreasing filtration of $\mathbf{H}(G, W[k])$ and each has K'-types ($K' = SU_2 \times M'$)

$$\sum_{p=0}^{\infty} \sum_{q=n}^{\infty} S^{k+p+q-2}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes (S^p(V_{M'}) \otimes S^q(V_0) \otimes W). \tag{12}$$

Moreover by (9)

$$H'_n / H'_{n+1} = \Gamma^1(V_n)^\wedge / \Gamma^1(V_{n-1})^\wedge = \mathbf{H}(G', S^n(\overline{\mathfrak{u}}'') \otimes W[k]). \tag{13}$$

H'_n are the Harish-Chandra modules of a decreasing filtration $\mathcal{H}_n$ of $H^1(G, \mathcal{O}(W[k]))$ constructed in §4 of [L1].

From now on, we will mainly work with Harish-Chandra modules of G and G'. The symbol $\mathrm{Res}^G_{G'}$ will denote the restriction of a Harish Chandra module of G to $(\mathfrak{g}', K')$. The next proposition follows from (13).

Proposition 2.9.1. *If $\mathbf{H}(G, W[k])$ is unitarizable, then*

$$\mathrm{Res}^G_{G'} \mathbf{H}(G, W[k]) = \sum_{n=0}^{\infty} \mathbf{H}(G', S^n(\overline{\mathfrak{u}}'') \otimes W[k])$$

and each summand on the right are unitarizable. □

3. Restrictions

3.1. We retain the notations of §2 where $G \supset G'$. The goal of this section is to give the motivation and the proof of Theorem 3.4.1.

3.2. First we review the work of [EPWW] and [GW2] in determining the unitarizability of $\sigma(G, W[k])$. There exists a quadratic form called the *Shapovalov* form on $N(G, W[k])$ (see [Sh]). Suppose the L-type $W_1[k+m]$ is the L-type with the smallest m among those of $N(G, W[k])^{\mathfrak{g}_2}$ which lies in the radical of the Shapovalov form. Then by (5)

$$W_1 \subset Sym^m V_M \otimes W \tag{14}$$

and we have an exact sequence

$$N(G, W_1[k+m]) \xrightarrow{\rho} N(G, W[k]) \to Q \to 0 \tag{15}$$

where Q denotes the quotient. Suppose that the image of ρ is the radical of the Shapovalov form and the quadratic form induced on Q is positive definite. By Lemma 2.7.1, the functor $\Gamma^1(-)^\wedge$ preserves the exactness of (15) and we have an exact sequence

$$0 \to \Gamma^1(Q)^\wedge \to \mathbf{H}(G, W[k]) \to \mathbf{H}(G, W_1[k+m]). \tag{16}$$

In addition, by Prop 6.6 of [EPWW] the positive definite quadratic form on Q induces a $(\mathfrak{g}, K)$-invariant positive definite quadratic form on $\Gamma^1(Q)^\wedge$ and thus $\Gamma^1(Q)^\wedge$ is decomposable. On the other hand $\mathbf{H}(G, W[k])$ has a unique irreducible submodule, namely $\sigma(G, W[k])$. Hence $\Gamma^1(Q)^\wedge = \sigma(G, W[k])$ and it is unitarizable.

3.3. In §2 we have a filtration of $(\mathfrak{g}', K')$-modules for each of the last two terms of (16). We are going to use them to determine the restriction of $\sigma(G, W[k])$ to $(\mathfrak{g}', K')$.

3.4. We need some notations in order to state Theorem 3.4.1. The inclusion in (14) gives rise to the following natural maps of M-modules

$$Sym^{n-m}(V_M) \otimes W_1 \to Sym^{n-m}(V_M) \otimes Sym^m(V_M) \otimes W \to Sym^n(V_M) \otimes W.$$

Let r'_n denote the composite of the above maps. The direct sum $V_M = V_{M'} \oplus V_0$ (cf. §2.4) induces a natural map of M'-modules

$$r''_n : Sym^n(V_M) \otimes W \to Sym^n(V_0) \otimes W.$$

We define $r_n = r''_n \circ r'_n$ for $n \geq m$. For $0 \leq n < m$, we set r_n to be the zero map into $Sym^n(V_0) \otimes W$. Let R_n denote the cokernel of r_n and denote $R_\bullet := \bigoplus_{n=0}^{\infty} R_n$. Note that R_n is a representation of M' and we write

$$R_n = \sum_j W_{n,j}$$

where $W_{n,j}$ are the irreducible subrepresentations of R_n.

Theorem 3.4.1. *Suppose $\sigma(G, W[k])$ is the unitarizable Harish-Chandra module obtained from the construction given in §3.2, then*

$$\begin{aligned} \mathrm{Res}^G_{G'}\sigma(G, W[k]) &= \sum_{n=0}^{\infty} \sigma(G', R_n[k+n]) && (17)\\ &= \sum_{n=0}^{\infty}\sum_j \sigma(G', W_{n,j}[k+n]). && (18) \end{aligned}$$

In particular the summands in (18) *are unitarizable.*

Recall §2.3 that $\sigma(G', W_{n,j}[k+n])$ is the unique irreducible submodule of

$$\Gamma^1_{K'/L'}(\mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{q}}')}(\mathcal{U}(\mathfrak{g}'), W_{n,j} \otimes e^{-(k+n)\tilde{\alpha}/2})_{L'}).$$

Proof. For the ease of notations, we denote

$$\begin{aligned} \mathbf{H}'_n &:= \mathbf{H}(G', S^n(\bar{\mathfrak{u}}'') \otimes W[k]) \\ \mathbf{H}''_n &:= \mathbf{H}(G', S^n(\bar{\mathfrak{u}}'') \otimes W_1[k+m]) \end{aligned}$$

Recall that (6) forms a filtration of $\mathcal{U}(\mathfrak{g}')$-modules for a generalized Verma module. For $N(G, W[k])$ and $N(G, W_1[k+m])$ we denote their filtrations by V'_n and V''_n respectively. We also denote their respective kernels defined in (11) by H'_n and H''_n. We define $V'_n = V''_n = \mathbf{H}''_n = 0$ and $H'_n = H''_n = H''_0$ for all $n < 0$.

In (15), ρ maps $1 \otimes W_1[k+m]$ to $S^m(\bar{\mathfrak{u}}) \otimes W[k]$ and thus it maps V''_n to V'_{n-m}. This in turn induces maps

$$t_n : H'_n \to H''_{n-m} \text{ and } s_n : \mathbf{H}'_n \to \mathbf{H}''_{n-m}.$$

Therefore we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & H'_{n+1} & \to & H'_n & \to & \mathbf{H}'_n & \to & 0 \\ & & \downarrow t_{n+1} & & \downarrow t_n & & \downarrow s_n & & \\ 0 & \to & H''_{n-m+1} & \to & H''_{n-m} & \to & \mathbf{H}''_{n-m} & \to & 0 \end{array}$$

Let K_n and σ_n be the kernels of t_n and s_n respectively. We apply the Snake Lemma to the commutative diagram to get an exact sequence

$$0 \to K_n/K_{n+1} \to \sigma_n \to H''_{n-m+1}/H'_{n+1}. \qquad (19)$$

Since $\sigma(G, W[k])$ is $SU_2(\tilde{\alpha})$-admissible and unitarizable, it is decomposable as a $(\mathfrak{g}', K')$-module and

$$\mathrm{Res}^G_{G'}\sigma(G, W[k]) = K_0 \oplus \bigoplus_{n=0}^{\infty} K_{n+1}/K_n.$$

Let σ be an irreducible submodule of σ_n. Since $\sigma_n \subset \mathbf{H}'_n$, σ is of the form

$$\sigma := \sigma(G', W'[k+n])$$

where W' is an irreducible M'-submodule. It is generated by its lowest K'-types $S^{k+n-2}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes W'$ which by (12) is not a K'-type of H''_{n-1}. Hence by the exact sequence (19), $\sigma \subset K_n/K_{n+1} \subset \sigma(G', W[k])$.

To find the irreducible representations in σ_n, it suffices to determine the restriction of s_n between the lowest K'-types of $\mathbf{H}'_n$ and $\mathbf{H}''_{n-m}$

$$s_n : S^{k+n-2}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes Sym^n(V_0) \otimes W \to S^{k+n-2}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes Sym^{n-m}(V_0) \otimes W_1.$$

Indeed the kernel of s_n will generate the irreducibles in σ_n. Therefore it suffices to know the kernel of the M'-homomorphism $r_n^\wedge$

$$r_n^\wedge : Sym^n(V_0) \otimes W \to Sym^{n-m}(V_0) \otimes W_1$$

by further restricting s_n. Recall that $r_n^\wedge$ is obtained from ρ in (15). Hence $r_n^\wedge$ is the conjugate dual of r_n and the kernel of $r_n^\wedge$ is the cokernel R_n of r_n as a representation of the compact group M'. This proves (17). Equation (18) follows from (17) and (4). □

3.5. It follows that if r_m is surjective, then r_n is surjective for all $n \geq m$ and $R_\bullet = \sum_{n=0}^{m-1} S^n V_0$.

4. Representations and Orbits

4.1. In [GW2] Gross and Wallach determine the unitarizability of $\sigma(G, \mathbb{C}[k])$ for all simple quaternionic Lie groups G using the method described in §3.2. In addition they show that the K-types can be obtained from the coordinate ring of certain associated orbits in $\mathbb{P}V_M$. In §4.2 we will review some facts about orbit associations. We refer the reader to [GW2] for details. Next we will deduce Corollary 4.2.1 from Theorem 3.4.1. Finally we illustrate our results with three examples.

4.2. $\mathbf{H}(G, \mathbb{C}[k])$ is irreducible and unitarizable if $k \geq k_0$ where k_0 is given in Table 2 below. If $k > \dim V_M$, then it belongs to the discrete series. If $k < k_0$, $\sigma(\mathbb{C}[k])$ is a proper submodule of $\mathbf{H}(\mathbb{C}[k])$. In order to describe those $\sigma(\mathbb{C}[k])$ which are unitarizable, we need the following definition.

Since V_M is a self dual representation of M, we identify its dual representation V_M^* with V_M. Let $\mathcal{O}$ be a $M(\mathbb{C})$ orbit in $\mathbb{P}(V_M)$ and $\overline{\mathcal{O}}$ be its Zariski closure. Let $I^\bullet(\overline{\mathcal{O}}) = \bigoplus_{n \geq m} I^n(\overline{\mathcal{O}})$, ($I^m \neq 0$) be the homogeneous ideal defining it and $A^\bullet(\overline{\mathcal{O}}) = \bigoplus A^n(\overline{\mathcal{O}})$ be its coordinate ring. Note that $I^n(\overline{\mathcal{O}})$ and $A^n(\overline{\mathcal{O}})$ are representations of M. We say that $\sigma(\mathbb{C}[k])$ is *associated* with $\mathcal{O}$ if it satisfies the following 2 conditions:

(i) It has K-types $\sum_{n=0}^{\infty} S^{k+n-2}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes A^n(\overline{\mathcal{O}})$.

(ii) It is unitarizable and it is obtained via the method given in §3.2. More specifically the inclusion $I^m(\overline{\mathcal{O}}) \subset S^m(V_M)$ induces a map as in (16)

$$0 \to \sigma(\mathbb{C}[k]) \to \mathbf{H}(G, \mathbb{C}[k]) \to \mathbf{H}(G, I^m(\overline{\mathcal{O}})[k+m]). \tag{20}$$

Note that [GW2] does not include (ii) in their definition of orbit association but it is a corollary of their proofs.

We tabulate the unitarizable $\sigma(\mathbb{C}[k])$ for $2 \leq k < k_0$ in Table 2 below. Each representation is associated with an orbit $\mathcal{O}$ in $\mathbb{P}V_0$ and it satisfies an equation of the form given in (20). The last 2 columns in Table 2 give the values of m and $I^m(\overline{\mathcal{O}})$ in (20). The associated orbits as well as the K-types are described in [GW2].

Table 2

	G_0	k_0		$\sigma(\mathbb{C}[k])$	m	$I^m(\overline{\mathcal{O}})$
(a)	$SU(d,2),\ d \geq 2$	$d+1$	(i)	$\sigma(\mathbb{C}[d]$	2	$\mathbb{C}$
(b)	$SO(d,4),\ d \geq 5$	d	(i)	$\sigma(\mathbb{C}[d-1])$	4	$\mathbb{C}$
			(ii)	$\sigma(\mathbb{C}[d-2])$	2	$S^2(\mathbb{C}^2) \otimes \mathbb{C}$
(c)	$Sp_{2d,2}$	2				
(d)	$SO(4,4)$	4	(i)	$\sigma(\mathbb{C}[3])$	4	$\mathbb{C}$
			(ii)	$\sigma(\mathbb{C}[2])$	2	$S^2 \oplus S^2 \oplus S^2$
(e$_s$)	$F_{4,4}$	$d = 3s+4$	(i)	$\sigma(\mathbb{C}[d-1])$	4	$\mathbb{C}$
	$E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$		(ii)	$\sigma(\mathbb{C}[2s+2])$	3	V_M
	$E_{7,4},\ E_{8,4}$		(iii)	$\sigma(\mathbb{C}[s+2])$	2	$\mathfrak{m}$
(f)	$G_{2,2}$	2				

For (e$_s$), $s = 1, 2, 4, 8$ if G_0 is of type F_4, E_6, E_7 and E_8 respectively. The representation $\sigma(\mathbb{C}[s+2])$ in (e$_s$)(iii) is the called the minimal representation. We will say more about it in §5.2.

Corollary 4.2.1. *Suppose $\sigma = \sigma(G, \mathbb{C}[k])$ is associated with the $M(\mathbb{C})$-orbit $\mathcal{O}$ in $\mathbb{P}V_M$. Let $\mathcal{O}' = \overline{\mathcal{O}} \cap \mathbb{P}V_0$ and denote its coordinate ring in $\mathbb{P}V_0$ by $A^\bullet(\mathcal{O}') = \bigoplus A^n(\mathcal{O}')$. We consider $A^n(\mathcal{O}')$ as a representation of M', then*

$$\mathrm{Res}^G_{G'}\sigma \supseteq \sum_{n=0}^{\infty} \sigma(G', A^n(\mathcal{O}')[k+n]). \tag{21}$$

Equality holds if and only if the restriction of $I^m(\overline{\mathcal{O}})$ to $\mathbb{P}V_0$ generates the homogeneous ideal of $\mathcal{O}'$.

Proof. By the definition of $R_\bullet = \bigoplus R_n$, $R_\bullet^{\mathrm{red}} := R_\bullet/\mathrm{Nil}(R_\bullet) = A^\bullet(\mathcal{O}')$. Hence (21) follows from Theorem 3.4.1. The last assertion follows from the fact that the sum of the images of r_n is the homogeneous ideal of $Sym^\bullet(V_0)$ generated by $r_m(I^m(\overline{\mathcal{O}}))$. □

Note that if G' is the maximal compact subgroup K of G, then the right hand side of (21) equals the K-types of σ.

4.3. The Hilbert polynomials of $R_\bullet$ and $A^\bullet(\mathcal{O}')$ have the same degree as they are both equal to the Krull dimension of $\mathcal{O}'$. Hence we are justified in calling the decomposition in (21) *generic*. In §4.6 we will see an example where $R_\bullet$ is non-reduced and the containment in (21) is proper.

Since $\overline{\mathcal{O}}$ is defined by $I^m(\overline{\mathcal{O}})$, $\mathcal{O}'$ is the projective variety cut out by $r_m(I^m(\overline{\mathcal{O}}))$. Hence $\mathcal{O}' = \mathbb{P}V_0$ if and only if $r_m = 0$ if and only if $r_n = 0$ for all n (cf. §3.5). On the other extreme, if $\mathcal{O}'$ is the empty set, then by Hilbert Nullstellensatz there exists n_0 such that r_n is surjective for all $n \geq n_0$. Thus the restriction of the associated representation decomposes into finitely many subrepresentations.

4.4. The projection map $V_M = V_{M'} + V_0 \to V_0$ induces a $M'(\mathbb{C})$-invariant rational map

$$\phi : \mathbb{P}(V_M) \cdots \to \mathbb{P}(V_0).$$

Let $\mathcal{O}'' = \overline{\phi(\mathcal{O})}$ and $\bigoplus_n A^n(\mathcal{O}'')$ be its coordinate ring. $\mathcal{O}''$ is the closure of a M'-orbit in $\mathbb{P}(V_0)$. Then the definition of r_n implies that $\sigma'' := \sigma(G', \mathbb{C}[k])$ has K-types

$$\sum_{n=0}^{\infty} S^{k+n}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes A^n(\mathcal{O}'').$$

Thus σ'' is the representation *associated* with $\mathcal{O}''$.

More generally, let W be an irreducible representation of $Sym^m(V_0)$ and let $\mathcal{W}$ be the corresponding M' equivariant vector bundle on $\mathbb{P}(V_0)$. Then the K'-types of $\sigma(W) := \sigma(G', W[m+k])$ is a quotient of

$$\sum_{n=0}^{\infty} S^{k+n+m}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes \Gamma(\mathcal{O}'', \mathcal{O}(n) \otimes \mathcal{W}|_{\mathcal{O}''}). \tag{22}$$

In particular $\sigma(W)$ has Gelfand-Kirillov dimension not bigger than $\dim\mathcal{O}'' + 2$. Unfortunately the K'-types of $\sigma(W)$ are seldom equal to those given in (22) except in the case when $\mathcal{O}'' = \mathbb{P}V_0$.

4.5. In the remaining part of this section we will give some examples on the restrictions of the representations of the classical groups given in Table 2.

4.6. Let $G = \widetilde{SO}(d,4) \supset G' = \widetilde{SO}(d-1,4)$ where the tildes above the groups denote their double covers. $V_0 = \mathbb{C}^2 \otimes \mathbb{C}$ as a representation of $M' = SU_2 \times SO(d-1)$.

Proposition 4.6.1. *Let $d \geq 4$ and $G = \widetilde{SO}(d,4) \supset G' = \widetilde{SO}(d-1,4)$. Then*
(a) $\mathrm{Res}^G_{G'}\sigma(G, \mathbb{C}[d-1]) = \sum_{n=0}^{\infty} \sigma(G', S^n(\mathbb{C}^2)[n+d-1])$.
(b) $\mathrm{Res}^G_{G'}\sigma(G, \mathbb{C}[d-2]) = \sigma(G', \mathbb{C}[d-2]) + \sigma(G', \mathbb{C}^2[d-1])$.

Proof. (a) From Table 2(b)(i) and (d)(i), we have $r_4 : \mathbb{C} \to S^4(\mathbb{C}^2)$ so $r_n = 0$. Thus $R_n = S^n(V_0)$.

(b) From Table 2(b)(ii) and Table 2(d)(ii) we have $r_2 : S^2(\mathbb{C}^2) \otimes \mathbb{C} \to S^2(\mathbb{C}^2) \otimes \mathbb{C}$. r_2 is either a surjection or the zero map. We claim that the associated orbit does not intersect $\mathbb{P}V_0$. The claim implies that r_2 is surjective and by §3.5, r_n is surjective for all $n \geq 2$. Hence $R_\bullet = R_0 + R_1 = \mathbb{C} + \mathbb{C}^2$ and it is non-reduced.

It remains to prove the claim. Let $e_1, \ldots, e_d$ be an orthonormal basis of $(\mathbb{C}^d, \langle,\rangle)$ and we identify $\mathbb{C}^2 \otimes \mathbb{C}^d = \mathbb{C}^d \oplus \mathbb{C}^d$. Suppose $d \geq 5$, then the closure of the associated orbit $\mathcal{O}$ is

$$\overline{\mathcal{O}} = \mathbb{P}\{(w_1, w_2) \in \mathbb{C}^d \oplus \mathbb{C}^d : \langle w_1, w_1\rangle = \langle w_1, w_2\rangle = \langle w_2, w_2\rangle = 0\}. \tag{23}$$

V_0 is spanned by $(e_1, 0)$ and $(0, e_1)$ and clearly $\mathbb{P}V_0$ does not intersect $\overline{\mathcal{O}}$.

Next suppose $d = 4$, then the closure of the associated orbit $\mathcal{O}$ is the image of Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^4).$$

$\mathcal{O}$ is a subvariety of (23) which does not intersect $\mathbb{P}V_0$. This proves the claim. □

4.7. Let $G = \widetilde{SU}(d,2) \supset G' = \widetilde{SU}(d-1,2) \times U_1$ and $V_0 = \mathbb{C} \oplus \mathbb{C}^*$. From Table 2(a)(i) the associated orbit of $\sigma(G, \mathbb{C}[d])$ is

$$\overline{\mathcal{O}} = \mathbb{P}\{(v, v^*) \in \mathbb{C}^d \oplus (\mathbb{C}^d)^* : \langle v, v^* \rangle = 0\}. \tag{24}$$

so $\mathcal{O} \cap \mathbb{P}V_0 = \mathbb{P}\{e_1, e_1^*\}$ (2 points) and $r_n : S^{n-2}V_0 \to S^n V_0$ is the map given by multiplication by $e_1 e_1^*$. Hence $R_n = U_1^n \oplus U_1^{-n}$. The one dimensional center U_1 of M' acts on the summand $U_1^{\pm n}$ of R_n by $U_1^{\pm n(d-1)}$. Therefore

$$\mathrm{Res}^G_{G'} \sigma(G, \mathbb{C}[d]) = \sum_{n=-\infty}^{\infty} \sigma(\widetilde{SU}(d-1,2), U_1^{n(d-1)}[d + |n|]) \otimes U_1^n.$$

4.8. Let $G = \widetilde{SU}(2d,2) \supset G' = \widetilde{Sp}_{2d,2}$ and $V_0 = \mathbb{C}^{2d}$. Then

$$\mathrm{Res}^G_{G'} \sigma(G, \mathbb{C}[2d]) = \sum_{n=0}^{\infty} \sigma(G', S^n(\mathbb{C}^{2d})[2d+n]).$$

Indeed (24) implies that $\overline{\mathcal{O}} \cap \mathbb{P}V_0 = \mathbb{P}V_0$. Therefore $r_2 = 0$ and $R_n = S^n(V_0)$.

5. Compact Dual Pairs

5.1. In this section we will investigate the dual pairs correspondences of the minimal representations of the four exceptional quaternionic groups of real rank 4.

5.2. Let G_0 be one of the four exceptional groups given in Table 1(e_s) indexed by $s = 1$, 2, 4, 8. Let G be its double cover with maximal compact subgroup $SU_2 \times M$. In each case there is a unique minimal closed $M(\mathbb{C})$-orbit Z in $\mathbb{P}V_M$. The unitarizable Harish-Chandra module $\sigma(\mathbb{C}[s+2])$ given in Table 2(e_s)(iii) is the unitary representation associated with Z. We will follow [GW1] and denote $\sigma(\mathbb{C}[s+2])$ by σ_Z. Its annihilator ideal is the Joseph ideal in $\mathcal{U}(\mathfrak{g})$ so σ_Z is called the *minimal* representation of G. It has K-types

$$\sum_{n=0}^{\infty} S^{s+n}_{\tilde{\alpha}}(\mathbb{C}^2) \otimes \pi_M(n\lambda)$$

where λ is the highest weight of V_M. Note that σ_Z descends to a Harish-Chandra module of G_0 for groups of type E.

5.3. We will consider compact dual pair correspondences. Let $G' = H_1 \times_C H_2$ be a dual pair in G and suppose H_1 is a quaternionic Lie subgroup containing $SU_2(\tilde{\alpha})$. Let $\mathfrak{h}_1$ be the Lie algebra of H_1 and $K_1 = SU_2(\tilde{\alpha}) \times M_1$ be its maximal compact subgroup. Since H_2 commutes with $SU_2(\tilde{\alpha})$, $H_2 \subset M$ and it is compact. Let S be an irreducible finite dimensional representation of H_2. We define the unitarizable $(\mathfrak{h}_1, K_1)$-module $\Theta(S)$ of H_1 by

$$\mathrm{Res}^G_{G'} \sigma_Z = \sum_S \Theta(S) \otimes S.$$

If there are exactly n copies of $\pi \otimes S$ in σ_Z, we say that π has *multiplicity* n in $\Theta(S)$. Θ is said to be 1-1 if all nontrivial $\Theta(S)$ are irreducible and the map $S \leftrightarrow \Theta(S)$ is a bijection.

5.4. We would like to apply Theorem 3.4.1 to $\mathrm{Res}^G_{G'}\sigma_Z$. Note that R_n is a representation of $M' = M_1 \times_C H_2$. Suppose $V_1 \otimes S$ is an irreducible representation of $M' = M_1 \times_C H_2$ in R_n, then by Theorem 3.4.1 $\mathrm{Res}^G_{G'}\sigma_Z$ contains the representation

$$\sigma(H_1 \times_C H_2, V_1 \otimes S[s+2+n]) = \sigma(H_1, V_1[s+2+n]) \otimes S.$$

This proves the next proposition.

Proposition 5.4.1. *$\Theta(S)$ is a sum of quaternionic unitarizable $(\mathfrak{h}_1, K_1)$-modules of H_1.* □

To determine $\Theta(S)$, it suffices to find R_n. By Table 2(e$_s$)(iii), $W_1 = I^2(Z) = \mathfrak{m} \subset Sym^2(V_M)$ and R_n is the cokernel of the map of $M' = M_1 \times_C H_2$ modules

$$r_n : \mathfrak{m} \otimes S^{n-2}(V_0) \to S^n(V_0). \tag{25}$$

In general it is difficult to determine R_n completely but it is relatively easy to find certain irreducible subrepresentations as well as their multiplicities in R_n. This gives the existence of a large number of correspondences which we state as the next proposition.

Proposition 5.4.2. *In Tables 1B-4B in the appendix, $\Theta_1(S)$ occurs with multiplicity one in $\Theta(S)$.*

Proof. Tables 1A - 4A in the appendix give the highest weight representations of V_0 and $\mathfrak{m}$ as representations of $M' = M_1 \times_C H_2$. This is done with the help of the tables in [KP].

We will use (25) and the data in Tables 1A - 4A to show that R_n contains the corresponding irreducible representations of M' and each occurs with multiplicity one. This is done on a case by case basis.

Table 3B(d) and Table 4B(c)(d) are trivial.

Suppose ω is one of the highest weight with maximal length in V_0. Furthermore assume that 2ω is not a weight of $\mathfrak{m}$ and it is not shorter than any weights of $\mathfrak{m}$. Then it is clear that $\pi(n\omega)$ is contained in R_n with multiplicity 1. This proves (b), (c), (d), (f) of Table 1B, (b), (d), (f), (g) of Table 2B, Table 3B(b) and Table 4B(b). A variation of this argument proves Table 2B(c) and Table 3B(c).

Consider the situation where $V_0 = \pi_{M'}(\varpi) \oplus \pi_{M'}(\varpi')$ and let X and Y be elements in the highest weight spaces of $\pi_{M'}(\varpi)$ and $\pi_{M'}(\varpi')$ respectively. Suppose ϖ (resp. ϖ') is not a weight of $\pi(\varpi)$ (resp. $\pi(\varpi')$) and suppose X^2, XY and Y^2 do not lie in $\mathfrak{m} \subset Sym^2(V_M)$. Then X^aY^b will not lie in the image of r_{a+b}. Hence X^aY^b spans the unique highest weight space of $\pi(a\varpi + b\varpi')$ in R_{a+b}. This proves the remaining cases. □

Corollary 5.4.3. *Θ-correspondences of the dual pairs*

$$F_{4,4} \times G_2 \quad \text{and} \quad E_{6,4} \times_{\mu_3} SU_3$$

in $E_{8,4}$ are not 1-1.

Proof. Table 1B(d)(e). □

5.5. The following correspondences given in Tables 1B - 4B have been shown to be the complete decomposition of σ_Z.

(i) Cases (b) of Tables 1B - 4B: This is due to [HPS].

(ii) Cases (a) of Tables 1B - 4B: We will prove it in Theorem 6.1.2 using a branching rule of [HPS].

(iii) Table 1B(f): See §6 [GW1].
(vi) Tables 2B(d), 2B(e) and 3B(d) : See [G].
(v) Table 4B(c), 4B(d): See [L2]

It is known that the correspondences given in the tables are not complete for the dual pairs $Spin(4,4) \times_{K_4} Spin(8)$ in $E_{8,4}$ (Table 1B(c)) and $Spin(4,4) \times SU_2^3$ in $E_{7,4}$ (Table 2B(c)) (see [L1]). The proofs of the above results are mainly done by considering the K-types. We can proceed as in §4.6 to §4.8 and provide alternative proofs by computing the intersections of the orbits. For example one can show that correspondences for $E_{6,4} \times_{\mu_3} SU_3$ in Table 1B is complete whereas that of $F_{4,4} \times G_2$ is not. However the calculations are quite tedious and we hope to give more details as well as the correspondences of the remaining dual pairs in our next paper. Finally we remark that [GS] and [MS] did some calculations on intersections of exceptional orbits in $\mathbb{P}V_M$.

6. The Dual Pair $SU(2,1) \times H_2$

6.1. Let $\widetilde{SU}(2,1)$ be the double cover of $SU(2,1)$ with maximal compact subgroup $SU_2(\tilde{\alpha}) \times U_1$. Let χ be the fundamental character of $M = U_1$. Hence $V_{U_1} = \chi^3 + \chi^{-3}$. The center μ_3 of $\widetilde{SU}(2,1)$ is contained in the torus U_1.

Lemma 6.1.1. *Let $a \in \mathbb{Z}$, then*

$$\mathbf{H}(\widetilde{SU}(2,1), \chi^a[|a| + n])$$

is unitarizable and irreducible if $n \geq 2$.

Proof. Consider $\widetilde{SU}(2,1) \subset \tilde{G}_{2,2}$. In this case $V_0 = \chi + \chi^{-1}$. Prop. 8.4 of [GW2] and Table 2(f) say that $\mathbf{H}(\tilde{G}_{2,2}, \mathbb{C}[n])$ is unitarizable for $n \geq 2$ and the lemma follows from Proposition 2.9.1. ☐

As before, let G_0 be one of the 4 exceptional groups indexed by s and G be its double cover. G contains the dual pair

$$\widetilde{SU}(2,1) \times_{\mu_3} H_2$$

where H_2 is given in (a) of Tables 1A - 4A in the appendix. From the tables, V_0 decomposes into

$$(\chi \otimes \pi_{H_2}(\varpi)) \oplus \left(\chi^{-1} \otimes \pi_{H_2}(\varpi')\right)$$

as representations of $U_1 \times H_2$. Note that $\pi(\varpi)$ and $\pi(\varpi')$ are dual representations of each other.

Theorem 6.1.2.

$$\mathrm{Res}^G_{\widetilde{SU}(2,1) \times H_2} \sigma_Z = \sum_{a,b \geq 0} \Theta(a,b) \otimes \pi_{H_2}(a\varpi + b\varpi')$$

where $\Theta(a,b)$ is the representation

$$H_{a,b} := \mathbf{H}(\widetilde{SU}(2,1), \chi^{a-b}[s + 2 + a + b]).$$

Proof. By Lemma 6.1.1, $H_{a,b}$ is unitarizable and irreducible. Tables 1B - 4B (a) read that $\Theta(a,b) \supset H_{a,b}$. Prop. 3.1 of [HPS] states that $\Theta(a,b)$ has K-types

$$\sum_{n=0}^{\infty}\sum_{p+q=n} S_{\tilde{\alpha}}^{s+n+a+b}(\mathbb{C}^2)\chi^{3p-3q+a-b} = \sum_{n=0}^{\infty} S_{\tilde{\alpha}}^{s+n+a+b}(\mathbb{C}^2) Sym^n(V_{U_1})\chi^{a-b}$$

which are equal to the K-types of $H_{a,b}$ (cf. (2)). Hence $\Theta(a,b) = H_{a,b}$ and this proves the theorem. □

7. The Holomorphic Representations

7.1. In this section we will study the continuations of the holomorphic discrete series [Wa] [RV]. In §7.3, we will briefly recall their constructions and we refer the reader to [Wa] for details. The situation is similar to the quaternionic case where one associates orbits to unitary representations. In §7.4 we derive results on the restrictions of such representations. These results are similar to Theorem 3.4.1 and Corollary 4.2.1. The proofs are almost identical but they are easier in this case because they do not involve the Zuckerman functors. Hence we have omitted them.

7.2. In this section $\mathfrak{g}_0$ will denote a real simple Lie algebra with maximal compact Lie subalgebra $\mathfrak{k}_0$. Let $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$ and $\mathfrak{k} = \mathfrak{k}_0(\mathbb{C})$ and we assume they satisfy the following properties:

(1) $\mathfrak{k}_0$ contains a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$
(2) $\mathfrak{k}_0$ is of the form $\mathbb{R}iH \oplus \mathfrak{m}_0$ and $\mathfrak{m}_0 = [\mathfrak{k}, \mathfrak{k}]$.
(3) $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{k} \oplus \mathfrak{u}_-$ where $\mathfrak{u}_\pm$ is the ± 1-eigenspace of $\mathrm{ad}(H)$ on $\mathfrak{g}$. Let $\mathfrak{q} = \mathfrak{u}_+ \oplus \mathfrak{k}$ and $\overline{\mathfrak{q}} = \mathfrak{u}_- \oplus \mathfrak{k}$ be the maximal parabolic algebras.

Let $G(\mathbb{C})$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$ and let G_0 be its connected real subgroup with Lie algebra $\mathfrak{g}_0$ and maximal compact subgroup $K_0 = U_1 \times M$. Let $\tilde{U}_1$ be the double cover of U_1 and let G be the double cover of G_0 with maximal compact subgroup $K = \tilde{U}_1 \times M$. Note that G/K is a bounded symmetric domain.

7.3. Let k be a non-positive integer and let $\tilde{U}_1^k \otimes W$ be an irreducible representation of $K = \tilde{U}_1 \times M$. Denote

$$H(G, \tilde{U}_1^k \otimes W) := \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}(\mathbb{C})), \tilde{U}_1^k \otimes W)_K. \tag{26}$$

It is conjugate dual to a generalized Verma module (cf. (10)). Let V_M denote the restriction of $\mathfrak{u}_+^*$ to M. Then (26) has K types ($K = \tilde{U}_1 \times M$)

$$\sum_{n=0}^{\infty} \tilde{U}_1^{k-2n} \otimes (Sym^n(V_M) \otimes W).$$

It contains a unique irreducible submodule $\sigma(G, \tilde{U}_1^k \otimes W)$ generated by the lowest K-type $\tilde{U}_1^k \otimes W$.

The Harish-Chandra module $\sigma(G, \tilde{U}_1^k \otimes \mathbb{C})$ is said to be *associated* with a $M(\mathbb{C})$ orbit $\mathcal{O}$ in $\mathbb{P}(V_M^*)$ if it has K-types ($K = \tilde{U}_1 \times M$)

$$\sum_{n=o}^{\infty} \tilde{U}_1^{-c-2n} \otimes A^n(\overline{\mathcal{O}})$$

where c is a nonnegative integer and $\bigoplus_n A^n(\overline{\mathcal{O}})$ is the coordinate ring of the Zariski closure of $\mathcal{O}$ in $\mathbb{P}(V_M^*)$.

Let r be the real split rank of $\mathfrak{g}$. The next proposition describes the $M(\mathbb{C})$-orbits in $\mathbb{P}(V_M^*)$.

Proposition 7.3.1. *$M(\mathbb{C})$ has $r-1$ orbits $\mathcal{O}_i$ where $i = 2, \ldots, r$ in $\mathbb{P}(V_M^*)$. They satisfy*

$$\{\} = \mathcal{O}_1 \subset \mathcal{O}_2 \subset \overline{\mathcal{O}}_3 \subset \ldots \subset \overline{\mathcal{O}}_r = \mathbb{P}(V_M^*).$$

$\mathcal{O}_2$ is the minimal closed orbit and its ideal is generated by quadratic polynomials.

We refer to [Wa] for the definition of c_i for $i = 1, \ldots, r$. We can now state a version of Thm 5.10 of [Wa].

Theorem 7.3.2. *(i) $H(G, \tilde{U}_1^k \otimes \mathbb{C})$ is unitarizable if $k < -c_r$.*
(ii) When $k = -c_i$ where $i = 1, \ldots, r$, the unique irreducible submodule $\sigma_i := \sigma(G, \tilde{U}_1^k \otimes \mathbb{C})$ of $H(G, \tilde{U}_1^k \otimes \mathbb{C})$ is unitarizable. σ_i is associated with the orbit $\mathcal{O}_i$ given in Proposition 7.3.1. In particular $c_1 = 0$ and $\sigma_1 = \mathbb{C}$ is the 1 dimensional trivial representation.

If $G_0 = Sp(2n, \mathbb{R})$ then the ideal of annihilation of σ_2 in the above theorem is the Joseph ideal and it is called the Weil representation.

7.4. Let $\mathfrak{g}_0'$ be a reductive Lie subalgebra of $\mathfrak{g}_0$ containing $\mathbb{R}iH$. Correspondingly we define $\mathfrak{g}' = \mathfrak{g}_0'(\mathbb{C})$, $\mathfrak{k}_0' = \mathbb{R}iH \oplus \mathfrak{m}_0'$, $\mathfrak{u}_\pm' \subset \mathfrak{u}_\pm$, $G' \subset G$ and $K' = \tilde{U}_1 \times M' \subset K = \tilde{U}_1 \times M$.

Suppose $\mathfrak{u}_+^* = (\mathfrak{u}_+')^* \oplus V_0$ as representations of M'. Using a similar argument as in §2.6 to §2.9 gives the next proposition (also see §1 [JV]).

Proposition 7.4.1. *Let k be a non-positive integer. Then $H(G, \tilde{U}_1^k \otimes W)$ has a filtration H_n' of $(\mathfrak{g}', K')$ submodule such that $\bigcap_n H_n' = 0$ and*

$$H_n'/H_{n+1}' \simeq \mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{q}'})}(\mathcal{U}(\mathfrak{g}'(\mathbb{C})), \tilde{U}_1^{k-2n} \otimes (Sym^n V_0 \otimes W))_{K'}. \quad \square$$

Suppose $I^m(\overline{\mathcal{O}}_i) \subset Sym^m(V_M)$ generates the ideal of $\overline{\mathcal{O}}_i$. Define R_n to be the cokernel of the following composition of natural maps of M'-modules

$$Sym^{n-m}(V_M) \otimes I^m(\overline{\mathcal{O}}_i) \to Sym^{n-m} V_0 \otimes Sym^m V_0 \to Sym^n V_0.$$

Note that R_n is a representation of M' and we write

$$R_n = \sum_j W_{n,j}.$$

A similar argument as in §3 and §4 gives the following results (cf. Thm 3.4.1 and Cor 4.2.1).

Theorem 7.4.2.

$$\begin{aligned}\mathrm{Res}^{\mathfrak{g},K}_{\mathfrak{g}',K'}\sigma_i &= \sum_{n=0}^{\infty}\sigma(G',\tilde{U}_1^{-c_i-2n}\otimes R_n)\\ &= \sum_{n=0}^{\infty}\sum_j \sigma(G',\tilde{U}_1^{-c_i-2n}\otimes W_{n,j}). \quad \square\end{aligned}$$

Corollary 7.4.3. *Let $\mathcal{O}' = \overline{\mathcal{O}}_i \cap \mathbb{P}V_0$ and denote its coordinate ring in $\mathbb{P}V_0$ by $A^\bullet(\mathcal{O}') = \bigoplus A^n(\mathcal{O}')$. We consider $A^n(\mathcal{O}')$ as a representation of M', then*

$$\mathrm{Res}^{\mathfrak{g},K}_{\mathfrak{g}',K'}\sigma_i \supseteq \sum_{n=0}^{\infty}\sigma(G',A^n(\mathcal{O}')[k+n]). \quad \square$$

Following a similar treatment as in §5, we can deduce results about compact dual pairs correspondences of the Weil representations. As we have mentioned in §1.4, such results are well known.

Appendix

All the exceptional Lie groups that appear in the tables are real forms of simply connected complex algebraic groups.

$H_1 \times_C H_2$ are dual pairs in G where C lies in the center of $H_1 \times H_2$. H_1 has maximal compact subgroup $SU_2 \times M_1$.

In Tables 1A, 2A, 3A and 4A, the last three columns give the highest weights of the irreducible components of $\mathfrak{m} = \mathrm{Lie}\, M(\mathbb{C})$, V_0 and $Sym^2(V_0)$ as representations of $M_1 \times_C H_2$. We use the method in [KP] but the numbering of [Bou] to denote the weights. For example, on line 3 of Table 1A,

$$\text{(-2.000001)} \;=\; U_1^{-2}\otimes\pi_{E_6}(\varpi_6).$$

In Tables 1B, 2B, 3B and 4B, S is a finite dimensional representation of H_1 and the column gives its highest weight. The column labeled $\Theta_1(S)$ denotes quaternionic representations of H_1. For example, on the last line of Table 1B, (a000000)[a+10] represents the $(\mathfrak{h}_1, K_1)$-module

$$\sigma(E_{7,4},\pi_{Spin(12)}(a\varpi_1)[a+10]).$$

Table 1A: $G = E_{8,4}$ and $M(\mathbb{C})$ is the simply connected $E_7(\mathbb{C})$.

	$H_1 \times_C H_2$	$M_1 \times_C H_2$	$\mathfrak{m}$	V_0	$Sym^2(V_0)$
(a)	$SU(2,1) \times_{\mu_3} E_6$	$U_1 \times_{\mu_3} E_6$	(0.010000) (-2.100000) (2.000001) (0.000000)	(1.100000) (-1.000001)	
(b)	$G_{2,2} \times F_4$	$SU_2 \times F_4$	(0.1000) (2.0001) (2.0000)	(1.0001)	
(c)	$Spin(4,4) \times_{K_4} Spin(8)$	$SU_2^3 \times_{K_4} Spin(8)$	(0.0.0.0100) (0.1.1.1000) (1.0.1.0010) (1.1.0.0001) (2.0.0.0000) (0.2.0.0000) (0.0.2.0000)	(1.0.0.1000) (0.1.0.0010) (0.0.1.0001)	
(d)	$F_{4,4} \times G_2$	$Sp_6 \times G_2$	(010.01) (000.10) (200.00)	(100.01)	(200.02) (200.00) (010.01) (010.10) (000.01) (000.10)
(e)	$E_{6,4} \times_{\mu_3} SU_3$	$SU_6 \times_{\mu_3} SU_3$	(10001.00) (01000.01) (00010.10) (00000.11)	(10000.10) (00001.01)	(20000.20) (00002.02) (01000.01) (00010.10) (10001.00) (10001.11) (00000.11) (00000.00)
(f)	$E_{7,4} \times_{\mu_2} SU_2$	$Spin(12) \times_{\mu_2} SU_2$	(010000.0) (000010.1) (000000.2)	(100000.1)	

Table 1B: $G = E_{8,4}$.

	$H_1 \times_C H_2$	S	$\Theta_1(S)$
(a)	$SU(2,1) \times_{\mu_3} E_6$	(a0000b)	(a-b)[a+b+10]
(b)	$G_{2,2} \times F_4$	$(000a)$	$(a)[a+10]$
(c)	$Spin(4,4) \times_{K_4} Spin(8)$	$(a0bc)$	$(a.b.c)[a+b+c+10]$
(d)	$F_{4,4} \times G_2$	$(0a)$ (01)	$(a00)[a+10]$ (000)[12]
(e)	$E_{6,4} \times_{\mu_3} SU_3$	(ab) (00)	$(a000b)[a+b+10]$ (00000)[12]
(f)	$E_{7,4} \times_{\mu_2} SU_2$	(a)	$(a00000)[a+10]$

Table 2A: $G = E_{7,4}$ and $M = Spin(12)$.

	$H_1 \times_C H_2$	$M_1 \times H_2$	$\mathfrak{m}$	V_0
(a)	$SU(2,1) \times_{\mu_3} SU_6$	$U_1 \times_{\mu_3} SU_6$	(2.01000) (-2.00010) (0.10001) (0.00000)	(1.00010) (-1.01000)
(b)	$G_{2,2} \times PSp_6$	$SU_2 \times PSp_6$	(2.010) (0.200) (2.000)	(1.010)
(c)	$Spin(4,4) \times_{K_4} SU_2^3$	$SU_2^3 \times_{K_4} SU_2^3$	(110.110) (101.101) (011.011) (200.000) (020.000) (002.000) (000.200) (000.020) (000.002)	(100.011) (010.101) (001.110)
(d)	$F_{4,4} \times SO_3$	$Sp_6 \times SO_3$	(010.2) (200.0) (000.2)	(100.2)
(e)	$E_{6,4} \times U_1$	$SU_6 \times U_1$	(10001.0) (01000.2) (00010.-2) (00000.0)	(10000.1) (00001.-1)
(f)	$Spin(8,4) \times SU_2$	$SU_2 \times Spin(8) \times SU_2$	(0.0100.1) (1.1000.1) (2.0000.0) (0.0000.2)	(0.0001.1)
(g)	$Sp_{4,2} \times G_2$	$Sp_4 \times G_2$	(20.00) (00.10) (00.01) (01.10)	(10.10)

Table 2B: $G = E_{7,4}$.

	$H_1 \times_C H_2$	S	$\Theta_1(S)$
(a)	$SU(2,1) \times_{\mu_3} SU_6$	$(0a0b0)$	$(a-b)[a+b+6]$
(b)	$G_{2,2} \times PSp_6$	$(0a0)$	$(a)[a+6]$
(c)	$Spin(4,4) \times_{K_4} SU_2^3$	$(b+c.c+a.a+b)$	$(abc)[a+b+c+6]$
(d)	$F_{4,4} \times SO_3$	$(2a)$	$(a00)[a+6]$
(e)	$E_{6,4} \times U_1$	(a) $(-b)$	$(a0000)[a+6]$ $(0000b)[b+6]$
(f)	$Spin(8,4) \times SU_2$	(a)	$(0.000a)[a+6]$
(g)	$Sp_{4,2} \times G_2$	$(a0)$	$(a0)[a+6]$

Table 3A: $G = E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$ and $M = SU_6 \rtimes \mathbb{Z}/2\mathbb{Z}$.

In (c), U_1^2 is defined as $\{(x,y,z) \in U_1^3 : xyz = 1\}$. The table gives the characters of U_1^2 which is sufficient to derive that of $\tilde{U}$ except when the character is trivial. In this situation, we write (.000+) and (.000-) to indicate that $\mathbb{Z}/2\mathbb{Z}$ acts trivially and nontrivially respectively.

	$H_1 \times H_2$	$M_1 \times H_2$	$\mathfrak{m}$	V_0
(a)	$SU(2,1)\times_{\mu_3}$ $(SU_3^2 \rtimes \mathbb{Z}/2\mathbb{Z})$	$U_1\times_{\mu_3}$ $(SU_2^3 \rtimes \mathbb{Z}/2\mathbb{Z})$	(-2.10.01) (2.01.10) { (0.11.00) (0.00.11) } (0.00.00)	(1.01.10) (-1.10.01)
(b)	$G_{2,2}\times$ $(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z})$	$SU_2\times$ $(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z})$	(2.11) (0.11) (2.00)	(1.11)
(c)	$Spin(4,4)\times_{K_4}$ $(U_1^2 \rtimes \mathbb{Z}/2\mathbb{Z})$	$SU_2^3\times_{K_4}$ $(U_1^2 \rtimes \mathbb{Z}/2\mathbb{Z})$	(110.1 -1 0) (101.1 0 -1) (011.0 1 -1) (200.000+) (020.000+) (002.000+) (000.000+) (000.000-)	(100.0 1 -1) (010.1 0 -1) (001.1 -1 0)
(d)	$F_{4,4} \times \mathbb{Z}/2\mathbb{Z}$	$Sp_6 \times \mathbb{Z}/2\mathbb{Z}$	(010.1) (200.0) (000.0)	(100.1)

Table 3B: $G = E_{6,4} \rtimes \mathbb{Z}/2\mathbb{Z}$.

	$H_1 \times_C H_2$	S	$\Theta_1(S)$
(a)	$SU(2,1) \times_{\mu_3} (SU_3^2 \rtimes \mathbb{Z}/2\mathbb{Z})$	$(ba.ab)$	$(a-b)[a+b+4]$
(b)	$G_{2,2} \times (PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z})$	(aa)	$(a)[a+4]$
(c)	$Spin(4,4) \times_{K_4} (U_1^2 \rtimes \mathbb{Z}/2\mathbb{Z})$	$(a+b.-a.-b)$ $(-b.a+b.-a)$ $(-a.-b.a+b)$	$(0ab)[a+b+4]$ $(b0a)[a+b+4]$ $(ab0)[a+b+4]$
(d)	$F_{4,4} \times \mathbb{Z}/2\mathbb{Z}$	(0) (1)	$(000)[4]$ $(100)[5]$

Table 4A: $G = \tilde{F}_{4,4}$ is the double cover of the split $F_{4,4}$. The maximal compact subgroup is $SU_2 \times M$ where $M(\mathbb{C}) = Sp_6$. The tilde $\tilde{}$ above the group indicates that it is a double cover.

For the dual pairs $\widetilde{Spin}(4,4) \times_{\mu_2^3} \mu_2^3$ and $\widetilde{Spin}(5,4) \times_{\mu_2^2} \mu_2^2$, we omit the characters of the finite center since it is clear how they act.

	$H_1 \times H_2$	$M_1 \times H_2$	$\mathfrak{m}$	V_0
(a)	$\widetilde{SU}(2,1) \times_{\mu_3} SU_3$	$U_1 \times_{\mu_3} SU_3$	(-2.20) (2.02) (0.11) (0.00)	(1.20) (-1.02)
(b)	$\tilde{G}_{2,2} \times_{\mu_2} O_3$	$SU_2 \times_{\mu_2} O_3$	(2.4) (0.2) (2.0)	(1.4)
(c)	$\widetilde{Spin}(4,4) \times_{\mu_2^3} \mu_2^3$	$SU_2^3 \times_{\mu_2^3} \mu_2^3$	(0.1.1) (1.0.1) (1.1.0) (2.0.0) (0.2.0) (0.0.2)	(1.0.0) (0.1.0) (0.0.1)
(d)	$\widetilde{Spin}(5,4) \times_{\mu_2^2} \mu_2^2$	$(SU_2 \times Spin(5)) \times_{\mu_2^3} \mu_2^3$	(2.00) (0.02) (1.01)	(0.01)

Table 4B: $G = \tilde{F}_{4,4}$.

	$H_1 \times_C H_2$	S	$\Theta_1(S)$
(a)	$\widetilde{SU}(2,1) \times_{\mu_3} SU_3$	$(2a, 2b)$	$(a-b)[a+b+3]$
(b)	$\tilde{G}_{2,2} \times_{\mu_2} O_3$	$(\{4a\})$	$(a)[a+3]$
(c)	$\widetilde{Spin}(4,4) \times_{\mu_2^3} \mu_2^3$	—	$(0.0.0)[3]$
		—	$(1.0.0)[4]$
		—	$(0.1.0)[4]$
		—	$(0.0.1)[4]$
(d)	$\widetilde{Spin}(5,4) \times_{\mu_2^2} \mu_2^2$	—	$(0.00)[3]$
		—	$(0.01)[4]$

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