

RATIONAL FORMS OF EXCEPTIONAL DUAL PAIRS

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ABSTRACT. Let \mathfrak{g} be a simple Lie algebra of type F_4 or E_n defined over a local or global field k of characteristic zero. We show that \mathfrak{g} can be obtained by the Tits construction from an octonion algebra \mathbb{O} and a cubic Jordan algebra \mathcal{J} . In particular, \mathfrak{g} contains a dual pair \mathfrak{h} defined over k which is the direct sum of the derivation algebras of \mathbb{O} and \mathcal{J} . We determine the conjugacy classes of k -forms of \mathfrak{h} in \mathfrak{g} .

1. INTRODUCTION

1.1. Let k be a field of characteristic 0. Let \mathbb{O} be an octonion k -algebra and let $\text{Der}(\mathbb{O})$ be the Lie algebra of derivatives. It is a simple Lie algebra of exceptional type G_2 . Let \mathcal{J} be a central simple cubic Jordan k -algebra and let $\text{Der}(\mathcal{J})$ be the Lie algebra of derivatives. Let \mathbb{O}_0 be the set of trace 0 elements in \mathbb{O} , and let \mathcal{J}_0 be the set of trace 0 elements in \mathcal{J} . Then $\mathfrak{h} = \text{Der}(\mathbb{O}) \oplus \text{Der}(\mathcal{J})$ acts on $\mathfrak{h}^\perp = \mathbb{O}_0 \otimes \mathcal{J}_0$. It is an absolutely irreducible representation of \mathfrak{h} . On the vector space

$$\mathfrak{g} = \mathfrak{g}(\mathbb{O}, \mathcal{J}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$$

Tits [T1] defines a Lie structure such that \mathfrak{h} is a subalgebra, and the bracket of $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^\perp$ is the action of \mathfrak{h} on \mathfrak{h}^\perp . The Lie algebra \mathfrak{g} is simple, and of type F_4 or E_n , depending on the Jordan algebra. The centralizer of $\text{Der}(\mathbb{O})$ in \mathfrak{g} is $\text{Der}(\mathcal{J})$, and vice versa. In other words, the two simple factors of \mathfrak{h} form a dual pair in \mathfrak{g} .

The algebra \mathfrak{h} is a maximal subalgebra in \mathfrak{g} . If k is an algebraically closed field, Dynkin [Dy] showed that any subalgebra of \mathfrak{g} isomorphic to \mathfrak{h} is conjugated to \mathfrak{h} by an automorphism of \mathfrak{g} . Let $A = \text{Aut}(\mathfrak{g})$ be the group of automorphisms of \mathfrak{g} . Let $X(\bar{k})$ denote the set of all $A(\bar{k})$ -conjugates of $\mathfrak{h} \otimes \bar{k}$. Then $X(\bar{k}) = A(\bar{k})/B(\bar{k})$ where B is the normalizer of \mathfrak{h} in A . It is an algebraic variety defined over k (see [Bo]). We have an exact sequence of pointed sets

$$1 \rightarrow B \rightarrow A \rightarrow X \rightarrow 1$$

where $g \in A(\bar{k})$ maps to $\text{Ad}_g(\mathfrak{h} \otimes \bar{k})$ in $X(\bar{k})$, and the corresponding long exact sequence of pointed sets

$$1 \longrightarrow B(k) \longrightarrow A(k) \longrightarrow X(k) \longrightarrow H^1(k, B) \xrightarrow{\gamma} H^1(k, A)$$

The group B acts on \mathfrak{h} by conjugation i.e. we have a canonical map from B to $\text{Aut}(\mathfrak{h})$. This map is an isomorphism. It follows that the sets $H^1(k, B)$ and $H^1(k, A)$ classify forms of \mathfrak{h}

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and \mathfrak{g} , respectively. It is not too difficult to see that the map γ is the Tits construction. Thus, in order to describe $B(k)$ -orbits on $X(k)$, i.e. rational conjugacy classes of dual pairs, it suffices to determine the fibers of the Tits construction. If k is a local field, the map γ can be completely described. It is always a surjection. In fact, it is a bijection unless $k \cong \mathbb{R}$ or \mathfrak{g} is of type F_4 . For a global field k we have the following.

Theorem 1.2. *Let k be a number field. Let S be the set of archimedean places, i.e. $k_\infty = k \otimes \mathbb{R} = \prod_{v \in S} k_v$. The map $\gamma : H^1(k, B) \rightarrow H^1(k, A)$ is surjective. If A is not of type F_4 then the fibers of γ are finite. More precisely, if the fiber of $[\mathfrak{g}] \in H^1(k, A)$ consists of n distinct classes $[\mathfrak{h}_1], \dots, [\mathfrak{h}_n] \in H^1(k, B)$, then the fiber of $[\mathfrak{g} \otimes \mathbb{R}] \in H^1(k_\infty, A) = \prod_{v \in S} H^1(k_v, A)$ consists of n distinct classes $[\mathfrak{h}_1 \otimes \mathbb{R}], \dots, [\mathfrak{h}_n \otimes \mathbb{R}] \in H^1(k_\infty, B) = \prod_{v \in S} H^1(k_v, B)$. For any finite place v of k , $[\mathfrak{h}_1 \otimes k_v] = \dots = [\mathfrak{h}_n \otimes k_v] \in H^1(k_v, B)$.*

The proof of this theorem is an application of the Hasse principle, keeping in mind that the groups B and A are not necessarily connected, which presents additional difficulties.

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2. PRELIMINARIES

2.1. Galois cohomology. Let \mathfrak{g} be a semi-simple Lie algebra over k . Let G and A be the groups of inner and outer automorphisms of \mathfrak{g} , respectively. The Galois cohomology $H^1(k, A)$ classifies k -forms of \mathfrak{g} . The forms of \mathfrak{g} whose cohomology classes are in the image of the map $j : H^1(k, G) \rightarrow H^1(k, A)$ are called inner forms. The map j is not injective in general, so the elements in $H^1(k, G)$ classify inner forms of \mathfrak{g} with some additional information. Assume that we have a short exact sequence

$$1 \longrightarrow G(\bar{k}) \longrightarrow A(\bar{k}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

and, hence, a long exact sequence

$$1 \longrightarrow G(k) \longrightarrow A(k) \longrightarrow H^0(k, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(k, G) \xrightarrow{j} H^1(k, A)$$

of pointed sets in Galois cohomology. The Galois group Γ_k must act trivially on $\mathbb{Z}/2\mathbb{Z}$ so $H^0(k, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Let $[\mathfrak{g}'] \in H^1(k, A)$ be in the image of j . By the standard twisting argument, the exact sequence shows that $j^{-1}([\mathfrak{g}'])$ consist of one or two points. It consist of one point if and only if $A'(k)/G'(k)$ is nontrivial. i.e. \mathfrak{g}' has an outer automorphism defined over k . The image of j contains a unique quasi split form \mathfrak{g}^{qs} . It is well known that \mathfrak{g}^{qs} has an outer automorphism defined over k . Hence $H^1(k, G)$ contains a unique point c_G such that $j(c_G) = [\mathfrak{g}^{qs}]$.

2.2. Lie algebras and groups of type G_2 . Let \mathbb{O} be an octonion k -algebra. The space of derivatives $\mathfrak{g}_2 = \text{Der}(\mathbb{O})$ is a simple Lie k -algebra of type G_2 , and $G = \text{Aut}(\mathfrak{g}_2) = \text{Aut}(\mathbb{O})$ is a connected, simply connected algebraic k -group of type G_2 . Hence $H^1(k, G)$ classifies any of the following the collections of objects (see page 190 in [Se]):

- (a) Octonion k -algebras.
- (b) Lie k -algebras of type G_2 .
- (c) Connected algebraic groups of type G_2 over k .

2.3. Jordan algebras. Let \mathcal{A} be a simple k -algebra. Let \mathcal{A}^+ be the Jordan algebra with the same space and multiplication $x \cdot y = \frac{1}{2}(xy + yx)$. An involution of \mathcal{A} is a linear map $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in \mathcal{A}$. Then $S(\sigma, \mathcal{A}) = \{x \in \mathcal{A} | \sigma(x) = x\}$ is a Jordan subalgebra of \mathcal{A}^+ . Let \mathcal{J} be a central simple cubic Jordan algebra. Then \mathcal{J} is one of the following five types (see Theorem 37.2(2) and Section 37.C in [BI]):

- (a) $\mathcal{J} = S(\sigma, \mathcal{A})$ where \mathcal{A} is a central simple k -algebra of degree 3, and σ is an orthogonal involution. Then $\dim(\mathcal{J}) = 6$, and the Lie algebra $\text{Der}(\mathcal{J})$ is a form of $\mathfrak{so}(3)$.
- (b) $\mathcal{J} = \mathcal{A}^+$ where \mathcal{A} is a central simple k -algebra of degree 3. Then $\dim(\mathcal{J}) = 9$, and the Lie algebra $\text{Der}(\mathcal{J})$ is an inner form of $\mathfrak{sl}(3)$.
- (c) $\mathcal{J} = S(\sigma, \mathcal{A})$ where \mathcal{A} is a central simple K -algebra of degree 3, K is a quadratic extension of k , and σ is a unitary involution. Then $\dim(\mathcal{J}) = 9$, and the Lie algebra $\text{Der}(\mathcal{J})$ is an inner form of $\mathfrak{su}(3)$.
- (d) $\mathcal{J} = S(\sigma, \mathcal{A})$ where \mathcal{A} is a central simple k -algebra of degree 6, and σ is a symplectic involution. Then $\dim(\mathcal{J}) = 15$, and the Lie algebra $\text{Der}(\mathcal{J})$ is a form of $\mathfrak{sp}(6)$.
- (e) \mathcal{J} is an exceptional Jordan algebra. Then $\dim(\mathcal{J}) = 27$, and the Lie algebra $\text{Der}(\mathcal{J})$ is a form of F_4 .

A convenient way to construct (some) cubic Jordan algebras is by means of composition algebras. Let \mathcal{C} be a composition k -algebra. Let $c \mapsto \bar{c}$, for $c \in \mathcal{C}$, be the conjugation in \mathcal{C} . Let $\varrho = \text{diag}(\varrho_1, \varrho_2, \varrho_3)$ be a 3 by 3 diagonal matrix with entries in k . We set

$$\mathcal{H}_3(\mathcal{C}, \varrho) = \{x \in M_3(\mathcal{C}) : \bar{x}^\top = \varrho x \varrho^{-1}\}.$$

This forms a Jordan algebra under the multiplication defined by $x \cdot y = \frac{1}{2}(xy + yx)$. See Section 37.C in [BI]. If ϱ is the identity matrix, then we denote the above Jordan algebra by $\mathcal{H}_3(\mathcal{C})$. The dimension of $\mathcal{H}_3(\mathcal{C}, \varrho)$ is $3 + 3 \dim(\mathcal{C})$. If $k = \mathbb{R}$ then we say that a composition algebra \mathcal{C} is definite if $c \mapsto c\bar{c}$ is a positive definite quadratic form. Otherwise, we say that \mathcal{C} is indefinite. Definite composition algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and Cayley octonions \mathbb{O} . The following is an exhaustive list of all real central simple cubic Jordan algebras (see pages 172-173 in [OV]):

- (a) $\mathcal{J}^c = \mathcal{H}_3(\mathcal{C}, \varrho)$ where \mathcal{C} is definite, and $\varrho = \text{diag}(1, 1, 1)$. The real Lie algebra $\text{Der}(\mathcal{J}^c)$ is compact.
- (b) $\mathcal{J}^{\text{II}} = \mathcal{H}_3(\mathcal{C}, \varrho)$ where \mathcal{C} is definite, and $\varrho = \text{diag}(1, -1, 1)$. The real Lie algebra $\text{Der}(\mathcal{J}^{\text{II}})$ has split rank one.

(c) $\mathcal{J}^s = \mathcal{H}_3(\mathcal{C}, \varrho)$ where \mathcal{C} is indefinite, and $\varrho = \text{diag}(1, 1, 1)$. The real Lie algebra $\text{Der}(\mathcal{J}^s)$ is split.

3. TITS CONSTRUCTION

3.1. Let \mathbb{O} be an octonion k -algebra and \mathcal{J} a central simple cubic Jordan k -algebra. Let $\text{Der}(\mathbb{O})$ and $\text{Der}(\mathcal{J})$ be the Lie algebras of derivatives. Let $\mathfrak{h} = \text{Der}(\mathbb{O}) \oplus \text{Der}(\mathcal{J})$. Let H be the group of inner automorphisms of \mathfrak{h} . Let \mathbb{O}_0 and \mathcal{J}_0 denote the respective subspaces of trace zero elements. Then $\mathfrak{h}^\perp = \mathbb{O}_0 \otimes \mathcal{J}_0$ is an irreducible \mathfrak{h} module. It is also an irreducible H -module. In [T1], Tits constructed an exceptional Lie algebra \mathfrak{g} over k on the vector space

$$\mathfrak{g} = \mathfrak{g}(\mathbb{O}, \mathcal{J}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$$

We refer to [T1] and [Ja] for the definition of the Lie brackets. If $\dim \mathcal{J} = 6, 9, 15$ or 27 , then \mathfrak{g} is a simple Lie algebra of type F_4, E_6, E_7 or E_8 respectively. Henceforth, we shall call $(\mathfrak{h}, \mathfrak{g})$ a Tits pair.

Let G be the group of inner automorphisms of \mathfrak{g} . As H acts on \mathfrak{h}^\perp , it is immediate that H is contained in G . Let G_{sc} and H_{sc} be simply connected covers of G and H , respectively. Let C_G and C_H be their centers. We have the following important commutative diagram with exact rows that will be used throughout the paper.

$$(1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & C_H & \longrightarrow & H_{sc} & \longrightarrow & H & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C_G & \longrightarrow & G_{sc} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

If G is of type E_6 (resp. E_7, E_8), then H_{sc} is a subgroup of G_{sc} and $C_H = C_G$ is a cyclic group of order 3 (resp. 2, 1). If G is of type F_4 , then C_G is trivial but C_H is a cyclic group of order 2.

Lemma 3.2. *Let $(\mathfrak{h}, \mathfrak{g})$ be a Tits pair defined over k . Let B be the normalizer of \mathfrak{h} in $A = \text{Aut}(\mathfrak{g})$.*

(i) *The centralizer of \mathfrak{h} in $A(k)$ is trivial.*

(ii) *The conjugation action of B on \mathfrak{h} gives an isomorphism of $B(k)$ and $\text{Aut}(\mathfrak{h})(k)$.*

Proof. Let $\varphi \in A(k)$ such that $\varphi(x) = x$ for all $x \in \mathfrak{h}$. Since \mathfrak{h}^\perp is an absolutely irreducible \mathfrak{h} -module, there exists a non-zero constant c such that $\varphi(x) = cx$ for all $x \in \mathfrak{h}^\perp$. From the explicit formula for the Lie bracket in [T1], it follows that there exist $x, y \in \mathfrak{h}^\perp$ such that $[x, y]$ is not contained in \mathfrak{h} i.e. $[x, y] = z + z^\perp \in \mathfrak{h} \oplus \mathfrak{h}^\perp$ and $z^\perp \neq 0$. Applying φ to this relation implies $c^2 = c$. Hence $c = 1$ and φ is the trivial automorphism. This proves (i).

Part (i) implies that the conjugation action of the normalizer B on \mathfrak{h} is faithful. Since $H(k) \subseteq B(k) \subseteq \text{Aut}(\mathfrak{h})(k)$, we are done, except in the E_6 -case, when $H(k) \neq \text{Aut}(\mathfrak{h})(k)$ i.e. when \mathfrak{h} has an outer automorphism defined over k . Recall that a form of $\mathfrak{sl}(3)$ corresponds to a central simple algebra \mathcal{A} of degree 3 with a unitary involution σ over an étale quadratic extension K of k . If the form of $\mathfrak{sl}(3)$ has an outer automorphism defined over k , then \mathcal{A}

is isomorphic to its opposite. Hence \mathcal{A} is isomorphic to the matrix algebra $M_3(K)$. Then $\mathcal{J} = \{x \in M_3(K) \mid \sigma(x) = x\}$, and $\text{Der}(\mathcal{J}) = \{x \in M_3(K) \mid \sigma(x) = -x\}$. For every $x \in M_3(K)$ let \bar{x} denote the action of the non-trivial automorphism of K over k . Then $\tau(x, y) = (x, \bar{y})$, for every $(x, y) \in \text{Der}(\mathbb{O}) \oplus \text{Der}(\mathcal{J})$, is an outer automorphism of \mathfrak{h} . It extends to an outer automorphism of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ by $\tau(x \otimes y) = x \otimes \bar{y}$, for every $x \otimes y \in \mathbb{O}_0 \otimes \mathcal{J}_0$. Part (ii) follows. \square

3.3. Let k be any field, and $\mathfrak{h} \subseteq \mathfrak{g}$ be two semi-simple Lie algebras defined over k . Let $A = \text{Aut}(\mathfrak{g})$ and B the normalizer of \mathfrak{h} in A . We have a natural map from B to $\text{Aut}(\mathfrak{h})$. Assume that this is a bijection, as it is in the case of a Tits pair $(\mathfrak{h}, \mathfrak{g})$. Let $\mathfrak{h}' \subseteq \mathfrak{g}'$ be a rational k -form of $\mathfrak{h} \subseteq \mathfrak{g}$, i.e. there exists an isomorphism $\varphi : \mathfrak{g}(\bar{k}) \rightarrow \mathfrak{g}'(\bar{k})$ such that $\varphi(\mathfrak{h}(\bar{k})) = \mathfrak{h}'(\bar{k})$.

Lemma 3.4. *The class of \mathfrak{h}' determines the class of \mathfrak{g}' . In particular, if \mathfrak{h} and \mathfrak{h}' are isomorphic over k , then \mathfrak{g} and \mathfrak{g}' are isomorphic over k too.*

Proof. The class $[\mathfrak{g}'] \in H^1(k, A)$ is given by the cocycle

$$\sigma \mapsto a_\sigma = \varphi^{-1} \circ \varphi^\sigma \in A(\bar{k})$$

where $\sigma \in \Gamma_k$. Since $\varphi^{-1} \circ \varphi^\sigma(\mathfrak{h}) = \mathfrak{h}$, it follows that $a_\sigma \in B(\bar{k})$. Hence $[\mathfrak{g}'] \in H^1(k, B)$. Under the isomorphism of B and $\text{Aut}(\mathfrak{h})$, the class $[\mathfrak{g}']$ corresponds to the cohomology class $[\mathfrak{h}']$. Hence the lemma. \square

Proposition 3.5. *Let $(\mathfrak{h}, \mathfrak{g})$ be a Tits pair. The map*

$$\gamma : H^1(k, B) \longrightarrow H^1(k, A)$$

is the Tits construction.

Proof. Indeed, if \mathfrak{h}' is a form of \mathfrak{h} , then $\gamma([\mathfrak{h}']) = [\mathfrak{g}']$ where \mathfrak{g}' is a form of \mathfrak{g} containing \mathfrak{h}' . We know, from Lemma 3.4, that there is at most one form of \mathfrak{g} containing \mathfrak{h}' . But one such form is obtained by the Tits construction. Hence \mathfrak{g}' is obtained from \mathfrak{h}' by the Tits construction. \square

Proposition 3.6. *Let $(\mathfrak{h}, \mathfrak{g})$ be a Tits pair. If \mathfrak{h} is split (respectively quasi-split) then \mathfrak{g} is split (respectively quasi-split).*

Proof. By Lemma 3.4, it suffices to construct, by any means, a rational k -form $\mathfrak{h}' \subseteq \mathfrak{g}'$ of $\mathfrak{h} \subseteq \mathfrak{g}$ where both \mathfrak{h}' and \mathfrak{g}' are (quasi) split. Let \mathfrak{g}' be a split Lie algebra over k . It is well known that there is a split dual pair \mathfrak{h}' in \mathfrak{g}' see, for example, [Sa] for a construction. We shall extend it to the quasi-split case. We may assume that \mathfrak{g}' has type E_6 . Lift the nontrivial automorphism of the Dynkin diagram to an outer automorphism τ of \mathfrak{g}' . Extend the Dynkin diagram by adding the lowest root, and remove the branch point. This is a diagram of type $A_2 \times A_2 \times A_2$. Hence \mathfrak{g}' contains a subalgebra isomorphic to $\mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k)$. One can choose an identification such that

(a) $\tau(x, y, z) = (x, z, y)$ for all $(x, y, z) \in \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k)$ and,

(b) under the adjoint action of $\mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k)$, the Lie algebra \mathfrak{g}' decomposes as

$$\mathfrak{g}' = \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \oplus (V \otimes V \otimes V) \oplus (V^* \otimes V^* \otimes V^*)$$

where V is the standard representation of $\mathfrak{sl}_3(k)$.

Lift the nontrivial automorphism of the Dynkin diagram of type A_2 to an outer automorphism ρ of $\mathfrak{sl}_3(k)$. Let $\varphi : \mathfrak{sl}_3(k) \rightarrow \mathfrak{g}'$ be a homomorphism defined by

$$\varphi(x) = (0, x, \rho(x)) \in \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \oplus \mathfrak{sl}_3(k) \subseteq \mathfrak{g}'$$

for every $x \in \mathfrak{sl}_3(k)$. The centralizer of $\varphi(\mathfrak{sl}_3(k))$ in \mathfrak{g}' is

$$\mathfrak{g}_2 \cong \mathfrak{sl}_3(k) \oplus V \oplus V^*$$

a split Lie algebra of type G_2 . Thus \mathfrak{g}' contains a split dual pair $\mathfrak{h}' \cong \mathfrak{g}_2 \oplus \mathfrak{sl}_3(k)$. Moreover, the automorphism τ of \mathfrak{g}' restricts to an automorphism of \mathfrak{h}' , trivial on \mathfrak{g}_2 , and ρ on $\mathfrak{sl}_3(k)$. In particular, if K is a quadratic extension of k , we can twist simultaneously \mathfrak{h}' and \mathfrak{g}' to construct a quasi split form of \mathfrak{h}' contained in a quasi split form of \mathfrak{g}' . \square

4. TITS ALGEBRAS

In this section we will review representations of algebraic groups over k from [T2].

4.1. Let G_{sc} be a semi-simple, simply connected algebraic group defined over k . Let T be a maximal torus of G_{sc} defined over k . Let Λ denote the lattice of weights of $T(\bar{k})$. Let Λ_r denote the root sublattice. We choose a positive root system Φ^+ in Λ . Let Λ_+ denote the subset of dominant weights. We set $C^* = \Lambda/\Lambda_r$ which is the abelian group dual to the center $C(\bar{k})$ of $G_{sc}(\bar{k})$.

The Galois group Γ_k acts on the Dynkin diagram as follows. If $\sigma \in \Gamma_k$ then the conjugate of a weight λ is defined by ${}^\sigma\lambda = \sigma \circ \lambda \circ \sigma^{-1}$. For every $\sigma \in \Gamma_k$ there exists a unique element w in the Weyl group such that $w({}^\sigma\Phi^+) = \Phi^+$. In particular, $\lambda \mapsto w({}^\sigma\lambda)$ defines a permutation of simple roots. For example, let K be a quadratic extension of k and consider a hermitian form

$$h(z_1, \dots, z_n) = a_1|z_1|^2 + \dots + a_n|z_n|^2$$

on K^n where $a_1, \dots, a_n \in k$. Let G_{sc} be the group of special unitary transformations, and T the torus consisting of diagonal matrices. Then $T(k) \cong (K^1)^n$ where K^1 is the group of norm one elements in K^\times . Let $\sigma \in \Gamma_k \setminus \Gamma_K$. Then $\sigma(z) = z^{-1}$ for every $z \in K^1$. Hence ${}^\sigma\lambda = -\lambda$ for all weights. Since -1 is not in the Weyl group of the root system of type A_{n-1} , σ acts non-trivially on the Dynkin diagram.

4.2. Let K be the extension of k such that the Galois group Γ_K is the stabilizer of all vertices of the Dynkin diagram. For every $\lambda \in \Lambda_+$ there exists a unique irreducible \bar{k} -representation V_λ of $G_{sc}(\bar{k})$ with the highest weight λ . Tits shows that V_λ can be defined over K in the following sense. There exists a central simple algebra \mathcal{A}_λ over K of dimension $(\dim V_\lambda)^2$ and a homomorphism

$$\rho_\lambda : G_{sc}(K) \rightarrow \mathrm{GL}_1(\mathcal{A}_\lambda)$$

such that V_λ is obtained from ρ_λ by extension of scalars. Define $\beta_G(\lambda) \in \text{Br}(K)$ to be the class of \mathcal{A}_λ in the Brauer group of K . In [T2], Tits shows that β_G induces a group homomorphism

$$\beta_G : C^* \rightarrow \text{Br}(K).$$

For example, if G_{sc} is a form of SL_n , corresponding to a degree n central simple algebra \mathcal{A} over a quadratic extension K , with a unitary involution, then $\beta_G(\lambda)$ is the class of \mathcal{A} , where λ is the highest weight of an n -dimensional representation of $G_{sc}(\bar{k})$.

4.3. We now give a cohomological interpretation of the map β_G . Let $G = G_{sc}/C$ be the adjoint quotient of G_{sc} . We have an exact sequence of algebraic groups

$$1 \rightarrow C \rightarrow G_{sc} \rightarrow G \rightarrow 1.$$

which gives a coboundary map $\delta : H^1(k, G) \rightarrow H^2(k, C)$. Let $A = \text{Aut}(\mathfrak{g})$. The inclusion of G in A gives rise to a map of pointed sets $j : H^1(k, G) \rightarrow H^1(k, A)$. As in Section 2.1, let $c_G \in H^1(k, G)$ be the unique point such that $j(c_G)$ is a class of a quasi split Lie algebra. Let $t_G = -\delta(c_G) \in H^2(k, C)$. Any weight λ gives a homomorphism $C \rightarrow \bar{k}^\times$. Hence we have a sequence of maps

$$H^2(k, C) \longrightarrow H^2(K, C) \longrightarrow H^2(K, \bar{k}^\times) = \text{Br}(K)$$

where the first map is the restriction. Let $h_\lambda : H^2(k, C) \longrightarrow \text{Br}(K)$ be the composite. By Proposition 31.6 in [BI]

$$(2) \quad \beta_G(\lambda) = h_\lambda(t_G).$$

4.4. Let $(\mathfrak{h}, \mathfrak{g})$ be a Tits pair. Assume that the type of \mathfrak{g} is not F_4 . In this case $C_H = C_G = C$, hence the commutative diagram (1) gives a commutative diagram

$$\begin{array}{ccc} H^1(k, H) & \xrightarrow{\delta_H} & H^2(k, C) \\ \alpha \downarrow & & \parallel \\ H^1(k, G) & \xrightarrow{\delta_G} & H^2(k, C) \end{array}$$

Let \mathfrak{h}^{qs} be the unique quasi split inner form of \mathfrak{h} . This gives a unique class $c_H \in H^1(k, H)$. By Proposition 3.6, $\gamma(\mathfrak{h}^{qs})$ is a class of a quasi split \mathfrak{g}^{qs} . Hence $\alpha(c_H) = c_G$, and the formula (2) implies the following proposition.

Proposition 4.5. *Let $(\mathfrak{h}, \mathfrak{g})$ be a Tits pair. Assume that the type of \mathfrak{g} is not F_4 . Then*

$$\beta_H = \beta_G. \quad \square$$

5. FORMS OF TYPE E_6

5.1. Assume that G_{sc} has type E_6 . Let K be the smallest extensions of k such that the Galois group Γ_K acts trivially on the Dynkin diagram of G_{sc} . The center C of G_{sc} is Γ_K -isomorphic to μ_3 . Hence the image of β_G is contained in the 3-torsion of the Brauer group:

$$\beta_G : C^* \rightarrow \text{Br}_3(K).$$

Let ϖ and ϖ' be the fundamental weights of the two 27 dimensional representations of $G_{sc}(\bar{k})$. Since $\varpi + \varpi'$ is in the root lattice, $\beta_G(\varpi)$ and $\beta_G(\varpi')$ are classes of two opposite division algebras. If K is a quadratic extension of k , then $\sigma \in \Gamma_k \setminus \Gamma_K$ switches ϖ and ϖ' . This implies that $\beta_G(\varpi)$, with K -structure twisted by σ , is isomorphic to its opposite $\beta_G(\varpi')$ i.e. $\beta_G(\varpi)$ admits an involution of the second kind with respect to K , see also [T2].

Proposition 5.2. *Assume that $(\mathfrak{h}, \mathfrak{g})$ is a Tits pair over k such that \mathfrak{g} has type E_6 .*

- (i) \mathfrak{g} has an outer automorphism defined over k if and only if $\beta_G = 1$.
- (ii) \mathfrak{g} has an outer automorphism defined over k if and only if \mathfrak{h} has.
- (iii) If $k = \mathbb{R}$ then both \mathfrak{g} and \mathfrak{h} have an outer automorphism defined over \mathbb{R} .

Proof. If $G_{sc}(k)$ has an outer automorphism then $\beta_G(\varpi) = \beta_G(\varpi')$. Hence $\beta_G = 1$, since $\beta_G(\varpi)$ and $\beta_G(\varpi')$ are inverses of each other, and of order dividing 3. If $\beta_G = 1$, then $\beta_H = 1$ by Proposition 4.5. Since $\beta_H = 1$, the factor of \mathfrak{h} of type A_2 is defined by means of a unitary involution on the matrix algebra $M_3(K)$. But then \mathfrak{g} has an outer automorphism defined over k , as constructed in the proof of Lemma 3.2. This proves (i). Part (ii) now also follows, as existence of the outer automorphism for both algebras is equivalent to the vanishing of $\beta_G = \beta_H$. Part (iii) follows from the fact that $\text{Br}_3(\mathbb{R}) = \text{Br}_3(\mathbb{C}) = 1$, hence β_G is trivial. \square

5.3. Assume that $(\mathfrak{h}, \mathfrak{g})$ be a Tits pair such that \mathfrak{g} is of type E_6 . We have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H(\bar{k}) & \longrightarrow & B(\bar{k}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & G(\bar{k}) & \longrightarrow & A(\bar{k}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \end{array}$$

where the two horizontal rows are exact sequences. This in turn gives a commutative diagram

$$(3) \quad \begin{array}{ccccccccccc} 1 & \longrightarrow & H(k) & \longrightarrow & B(k) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & H^1(k, H) & \xrightarrow{j_H} & H^1(k, B) & \longrightarrow \\ & & \downarrow & & \downarrow & & \parallel & & \alpha \downarrow & & \gamma \downarrow & \\ 1 & \longrightarrow & G(k) & \longrightarrow & A(k) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & H^1(k, G) & \xrightarrow{j_G} & H^1(k, A) & \longrightarrow \end{array}$$

Lemma 5.4. *Let $x \in H^1(k, G)$ and $y = j_G(x) \in H^1(k, A)$. Then j_H is a bijection from $\alpha^{-1}(x)$ to $\gamma^{-1}(y)$.*

Proof. Let $z \in \gamma^{-1}(y)$. A diagram chase, using $H^1(k, \mathbb{Z}/2\mathbb{Z})$ (not pictured) shows that $z = j_H(u)$. By a twisting argument, we can assume that $y = [\mathfrak{g}]$ and $z = [\mathfrak{h}]$. Then the diagram (3) and Proposition 5.2, part (ii) imply that α gives a bijection between $j_H^{-1}(z)$ and $j_G^{-1}(y)$. \square

Lemma 5.5. *The map γ is surjective (respectively injective) for all pairs $(\mathfrak{h}, \mathfrak{g})$ if and only if the map α is surjective (respectively injective) for all pairs $(\mathfrak{h}, \mathfrak{g})$.*

Proof. Surjectivity is obvious. Injectivity follows from Lemma 5.4. \square

6. REAL LIE ALGEBRAS

In this section, all groups and Lie algebras are defined over \mathbb{R} unless otherwise stated.

6.1. Real forms of type G_2 . The compact real Lie algebra $\mathfrak{g}_2^c = \text{Der}(\mathbb{O}^c)$ and the split real Lie algebra $\mathfrak{g}_2^s = \text{Der}(\mathbb{O}^s)$ are the only real Lie algebras of type G_2 .

6.2. Real Jordan algebras. Recall, from Section 2.3, the three Jordan algebras \mathcal{J}^s , \mathcal{J}^c and \mathcal{J}^{II} constructed from a composition algebra of dimension d . Then $\text{Der}(\mathcal{J}^s)$ is split, $\text{Der}(\mathcal{J}^c)$ is compact, and $\text{Der}(\mathcal{J}^{\text{II}})$ has split rank one, as specified by Table 1.

d	1	2	4	8
$\text{Der}(\mathcal{J}^{\text{II}})$	$\mathfrak{so}_{1,2}(\mathbb{R})$	$\mathfrak{su}_{1,2}(\mathbb{C})$	$\mathfrak{su}_{1,2}(\mathbb{H})$	$\mathfrak{f}_{4,1}$

TABLE 1. The derivation algebras of real Jordan algebras.

In the literature, $\mathfrak{f}_{4,1}$ is also denoted by $\mathfrak{f}_{4(-20)}$.

6.3. Tits construction over real numbers. Let $\mathfrak{g} = \mathfrak{g}(\mathbb{O}, \mathcal{J})$ be a real exceptional Lie algebra obtained by the Tits construction in Section 3. We shall determine this real form by computing the signature of the Killing form. This is the same as computing the trace of the corresponding Cartan involution. Let $\mathfrak{h} = \text{Der}(\mathbb{O}) \oplus \text{Der}(\mathcal{J}) \subseteq \mathfrak{g}$. Let θ_H be a Cartan involution of \mathfrak{h} corresponding to a maximal compact subalgebra of \mathfrak{h} . Since the maximal compact subalgebra of \mathfrak{h} is contained in a maximal compact subalgebra of \mathfrak{g} , the Cartan involution θ_H extends to a Cartan involution θ_G of \mathfrak{g} corresponding to the maximal compact subalgebra of \mathfrak{g} . We claim that θ_G is given by the natural action of $\theta_H \in \text{Aut}(\mathfrak{h}) \subseteq \text{Aut}(\mathfrak{g})$ on \mathfrak{g} . Indeed, $\theta_H \theta_G^{-1} \in \text{Aut}(\mathfrak{g})$ centralizes \mathfrak{h} . By Lemma 3.2, the centralizer of \mathfrak{h} in $\text{Aut}(\mathfrak{g})$ is trivial and the claim follows. Using the explicit action of θ_H on $\mathbb{O}^0 \otimes \mathcal{J}^0$, it is not difficult to compute the trace of θ_G on \mathfrak{g} . We tabulate the results in Table 2 ([Ja]).

In Table 2, d in $\mathcal{J}(d)$ is the dimension of the composition algebra used to construct \mathcal{J} . The upper-scripts s and c denote the split and compact Lie algebras, respectively. The algebra $\mathfrak{e}_{6(-14)}$ (respectively $\mathfrak{e}_{6(-26)}$) is the unique Lie algebra of type E_6 with real rank 2 whose Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ satisfies $\dim \mathfrak{p} - \dim \mathfrak{k} = -14$ (respectively -26). By inspection, we see that Table 2 contains all real exceptional Lie algebras of type F_4 and E_n .

\mathbb{O}	$\mathcal{J}^c(1)$	$\mathcal{J}^s(1)$	$\mathcal{J}^c(2)$	$\mathcal{J}^\Pi(2)$	$\mathcal{J}^s(2)$	$\mathcal{J}^c(4)$	$\mathcal{J}^\Pi(4)$	$\mathcal{J}^s(4)$	$\mathcal{J}^c(8)$	$\mathcal{J}^\Pi(8)$	$\mathcal{J}^s(8)$
\mathbb{O}^c	\mathfrak{f}_4^c	$\mathfrak{f}_{4,1}$	\mathfrak{e}_6^c	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-26)}$	\mathfrak{e}_7^c	$\mathfrak{e}_{7,4}$	$\mathfrak{e}_{7,3}$	\mathfrak{e}_8^c	\mathfrak{e}_8^s	$\mathfrak{e}_{8,4}$
\mathbb{O}^s	\mathfrak{f}_4^s	\mathfrak{f}_4^s	$\mathfrak{e}_{6,4}$	$\mathfrak{e}_{6,4}$	\mathfrak{e}_6^s	$\mathfrak{e}_{7,4}$	$\mathfrak{e}_{7,4}$	\mathfrak{e}_7^s	$\mathfrak{e}_{8,4}$	$\mathfrak{e}_{8,4}$	\mathfrak{e}_8^s

TABLE 2. The real Lie algebras obtained using Tits construction.

Thus we have obtained the following (see [Kv] for a more thorough discussion of dual pairs in \mathfrak{g} of type E_n).

Proposition 6.4. *If k is an archimedean field, then $\gamma : H^1(k, B) \rightarrow H^1(k, A)$ is a surjection. If $k \cong \mathbb{C}$ then γ is a bijection. If $k \cong \mathbb{R}$ then γ is given by Table 2.*

6.5. Real forms of type E_6 . Recall that H and G are the connected components of $B = \text{Aut}(\mathfrak{h})$ and $A = \text{Aut}(\mathfrak{g})$, respectively. We have a commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{R}, H) & \xrightarrow{j_H} & H^1(\mathbb{R}, B) \\ \alpha \downarrow & & \downarrow \gamma \\ H^1(\mathbb{R}, G) & \xrightarrow{j_G} & H^1(\mathbb{R}, A) \end{array}$$

By Proposition 5.2(iii) the maps j_H and j_G are injections. Thus, the elements of $H^1(\mathbb{R}, H)$ and $H^1(\mathbb{R}, G)$ can be identified with the inner forms of \mathfrak{h} and \mathfrak{g} respectively. Hence the map α is also given by Table 2. In particular, it is surjective.

7. RATIONAL FORMS OVER p -ADIC FIELDS

7.1. In this section k is a p -adic field. There is a unique (split) octonion algebra over k , denoted by \mathbb{O} . The algebra $\text{Der}(\mathbb{O})$ is a split simple Lie algebra of type G_2 . Let \mathcal{J} be a cubic Jordan algebra over k and $\mathfrak{g} = \mathfrak{g}(\mathbb{O}, \mathcal{J})$ be the corresponding exceptional Lie algebra obtained by the Tits construction. If \mathcal{J} is split, then $\text{Der}(\mathcal{J})$ is also split. Other cases are tabulated in Table 3.

\mathcal{J}	$\mathcal{H}_3(k, \rho)$	D^+	$\mathcal{H}_3(K)$	$\mathcal{H}_3(Q)$
$\text{Der}(\mathcal{J})$	$\mathfrak{so}_3(k)$	$\mathfrak{sl}_1(D)$	$\mathfrak{su}_3(K)$	$\mathfrak{su}_3(Q)$

TABLE 3. The derivation algebras of Jordan algebras defined over a p -adic field k .

In Table 3 D is a cubic division algebra over k and $\mathfrak{sl}_1(D)$ is the Lie algebra consisting of the trace zero elements in D ; K is a quadratic extension of k , and Q is the unique quaternion algebra over k . Over the field k there are two cubic division algebras. If D is one then the other is its opposite D° . The algebras $\mathfrak{sl}_1(D)$ and $\mathfrak{sl}_1(D^\circ)$ are isomorphic by the map $x \mapsto -x$ for all $x \in \mathfrak{sl}_1(D)$.

7.2. By a result of Kneser, see [Kn], $H^1(k, H_{sc})$ and $H^1(k, G_{sc})$ are trivial. Hence the commutative diagram (1) gives a commutative diagram

$$\begin{array}{ccc} H^1(k, H) & \xrightarrow{\delta_H} & H^2(k, C_H) \\ \alpha \downarrow & & \downarrow \\ H^1(k, G) & \xrightarrow{\delta_G} & H^2(k, C_G) \end{array}$$

where horizontal maps are bijections. The following proposition now follows from Lemma 5.4.

Proposition 7.3. *If k is a p -adic field, then $\gamma : H^1(k, B) \rightarrow H^1(k, A)$ is a surjection. It is a bijection if \mathfrak{g} is not of type F_4 .*

The group $H^2(k, C_H)$ can be computed by identifying C_H with a concrete group as in Table 4. In the table, $\mu_3[K]$ is the group of cube roots of 1, where the usual action of Γ_k is twisted by $\sigma(\zeta) = \zeta^{-1}$ for every $\sigma \in \Gamma_k \setminus \Gamma_K$. The data given in the last row follow from results in local class theory, such as $H^2(k, \mu_n) = Br_n(k) = \mathbb{Z}/n\mathbb{Z}$. The group $H^2(k, \mu_n[K])$, for n odd, parameterizes division algebras over K of degree dividing n , that admit a unitary involution, see (30.15) in [BI]. On the other hand, a division algebra of degree dividing n over K admits a unitary involution if and only if it is in the kernel of the corestriction map from $Br_n(K)$ to $Br_n(k)$. However this map is a bijection for the p -adic field k and n odd, hence $H^2(k, \mu_n[K])$ is trivial.

G	F_4	E_6	E_6^K	E_7	E_8
C_H	μ_2	μ_3	$\mu_3[K]$	μ_2	1
$H^2(k, C_H)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	1

TABLE 4. The centers and their cohomologies.

Completeness of data in Table 3 follows from Table 4. We have a unique dual pair, except when \mathfrak{g} has type F_4 . If \mathfrak{g} is of type F_4 , then it is split and it contains two conjugacy classes of dual pairs, with split and non-split $\mathfrak{so}_3(k)$.

8. RATIONAL FORMS OVER GLOBAL FIELDS

8.1. We have a commutative diagram

$$\begin{array}{ccc} H^1(k, H) & \xrightarrow{i_H} & \prod_v H^1(k_v, H) \xrightarrow{\prod_v \delta_{H_v}} \prod_v H^2(k_v, C_H) \\ \alpha \downarrow & & \prod_v \alpha_v \downarrow \qquad \qquad \qquad \downarrow \\ H^1(k, G) & \xrightarrow{i_G} & \prod_v H^1(k_v, G) \xrightarrow{\prod_v \delta_{G_v}} \prod_v H^2(k_v, C_G) \end{array}$$

where \prod_v denotes a restricted direct product of pointed sets over all places v of k , i.e. if (x_v) is an element of the product, then x_v is the distinguished point for almost all v . The

maps i_H and i_G are injections ([PR] Theorem 6.22) while $\prod_v \alpha_v$ is a surjection. The image of i_H is equal to the inverse image of $H^2(k, C_H) \subseteq \prod_v H^2(k_v, C_H)$ ([Kz], 2.6 Proposition). The same also applies to the image of i_G . Now a simple diagram chase shows that α is a surjection for every pair $(\mathfrak{h}, \mathfrak{g})$. It follows, from Lemma 5.5, that

$$\gamma : H^1(k, B) \longrightarrow H^1(k, A)$$

is surjective. The question of fibers of γ can be translated to the question of fibers for α by Lemma 5.4. If \mathfrak{g} is not of type F_4 then, by Proposition 7.3, α_v is a bijection for every finite place, hence the fibers of α can be identified with the fibers of $\alpha_\infty : H^1(k_\infty, H) \rightarrow H^1(k_\infty, G)$. These are the same as the fibers of $\gamma_\infty : H^1(k_\infty, B) \rightarrow H^1(k_\infty, A)$ by Section 6.5. Theorem 1.2 is proved.

8.2. We shall now show how to construct given \mathfrak{g} using the Tits construction. Let \mathbb{O} be an octonion k -algebra. By the Hasse principle for G_2 the octonion algebra \mathbb{O} is specified by isomorphism classes of its completions \mathbb{O}_v for all real places v .

- (a) F_4 Let $\mathcal{A} = M_3(k)$. Let $\mathcal{J} = S(\sigma, \mathcal{A})$ where σ is an orthogonal involution of \mathcal{A} . Pick \mathbb{O} and the involution σ such that $\mathfrak{g}_v \cong \mathfrak{g}(\mathbb{O}_v, \mathcal{J}_v)$ for all real places v . Then $\mathfrak{g} \cong \mathfrak{g}(\mathbb{O}, \mathcal{J})$.
- (b) E_6 (inner) Let ϖ be the highest weight of an irreducible representation of dimension 27. Let \mathcal{A} be a central simple algebra of degree 3 such that $[\mathcal{A}] = \beta_G(\varpi) \in \text{Br}_3(k)$. Let $\mathcal{J} = \mathcal{A}^+$. Pick \mathbb{O} such that $\mathfrak{g}_v \cong \mathfrak{g}(\mathbb{O}_v, \mathcal{J}_v)$ for all real places v . Then $\mathfrak{g} \cong \mathfrak{g}(\mathbb{O}, \mathcal{J})$.
- (c) E_6 (outer) Let ϖ be the highest weight of an irreducible representation of dimension 27. Let \mathcal{A} be a central simple algebra of degree 3 over K such that $[\mathcal{A}] = \beta_G(\varpi) \in \text{Br}_3(K)$. Let $\mathcal{J} = S(\sigma, \mathcal{A})$ where σ is a unitary involution of \mathcal{A} . Pick \mathbb{O} and the involution σ such that $\mathfrak{g}_v \cong \mathfrak{g}(\mathbb{O}_v, \mathcal{J}_v)$ for all real places v . Then $\mathfrak{g} \cong \mathfrak{g}(\mathbb{O}, \mathcal{J})$.
- (d) E_7 Let ϖ be the highest weight of the irreducible representation of dimension 56. Let \mathcal{A} be a central simple algebra of degree 6 such that $[\mathcal{A}] = \beta_G(\varpi) \in \text{Br}_2(k)$. Let $\mathcal{J} = S(\sigma, \mathcal{A})$ where σ is a symplectic involution of \mathcal{A} . Pick \mathbb{O} and the involution σ such that $\mathfrak{g}_v \cong \mathfrak{g}(\mathbb{O}_v, \mathcal{J}_v)$ for all real places v . Then $\mathfrak{g} \cong \mathfrak{g}(\mathbb{O}, \mathcal{J})$.
- (e) E_8 By the Hasse principle for F_4 , an exceptional Jordan k -algebra \mathcal{J} is specified by isomorphism classes of its completions \mathcal{J}_v for all real places v . Pick \mathbb{O} and the algebra \mathcal{J} such that $\mathfrak{g}_v \cong \mathfrak{g}(\mathbb{O}_v, \mathcal{J}_v)$ for all real places v . Then $\mathfrak{g} \cong \mathfrak{g}(\mathbb{O}, \mathcal{J})$.

We refer the reader to the book by Scharlau [Sch], where existence of involutions with prescribed local desiderata is discussed.

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