INVARIANTS AND K-SPECTRUMS OF LOCAL THETA LIFTS

HUNG YEAN LOKE AND JIAJUN MA

Abstract. Let (G, G') be a type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$. We assume that (G, G') is in the stable range where G is the smaller member. Let K and K' be maximal compact subgroups of G and G' respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ be the complexified Cartan decompositions of the Lie algebras of G and G' respectively. Let $\widetilde{K}$ and $\widetilde{K}'$ be the inverse images of K and K' in the metaplectic double cover $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$ of $\mathrm{Sp}(W_{\mathbb{R}})$. Let ρ be a genuine irreducible $(\mathfrak{g}, \widetilde{K})$-module. Our first main result is that if ρ is unitarizable, then except for one special case, the full local theta lift $\rho' = \Theta(\rho)$ is equal to the local theta lift $\theta(\rho)$. Thus excluding the special case, the full theta lift ρ' is an irreducible and unitarizable $(\mathfrak{g}', \widetilde{K}')$-module. Our second main result is that the associated variety and the associated cycle of ρ' are the theta lifts of the associated variety and the associated cycle of the contragredient representation ρ^* respectively. Finally we obtain some interesting $(\mathfrak{g}, \widetilde{K})$-modules whose $\widetilde{K}$-spectrums are isomorphic to the spaces of global sections of some vector bundles on some nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$.

1. Introduction

1.1. Let $W_{\mathbb{R}}$ be a finite dimensional symplectic real vector space. Throughout this paper (G, G') will denote a type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$. Such dual pairs are listed in Table 1 in Section 2.1.

We follow the notation in [15] closely. Let $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$ be the metaplectic double cover of $\mathrm{Sp}(W_{\mathbb{R}})$. For any subgroup E of $\mathrm{Sp}(W_{\mathbb{R}})$, let $\widetilde{E}$ denote its inverse image in $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$. We choose a maximal compact subgroup U of $\mathrm{Sp}(W_{\mathbb{R}})$ such that $K := G \cap \mathrm{U}$ and $K' := G' \cap \mathrm{U}$ are maximal compact subgroups of G and G' respectively. Hence $\widetilde{K}$ and $\widetilde{K}'$ are maximal compact subgroups of $\widetilde{G}$ and $\widetilde{G}'$ respectively. The choice of U determines a unique complex structure on $W_{\mathbb{R}}$ with the following property: there is a positive definite Hermitian form $\langle\, , \,\rangle$ on the resulting complex vector space W so that the imaginary part of $\langle\, , \,\rangle$ coincides with the symplectic form on $W_{\mathbb{R}}$, and U coincides with the unitary group attached to $(W, \langle\, , \,\rangle)$. We choose the oscillator representation of $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$ whose Fock model $\mathscr{Y}$ is realized as the space $\mathbb{C}[W]$ of complex polynomials on W with the $\widetilde{\mathrm{U}}$ action as described in Appendix A.1. Let ς denote the minimal $\widetilde{\mathrm{U}}$-type of $\mathscr{Y}$. It is a one dimensional representation of $\widetilde{\mathrm{U}}$ acting on the space of constant functions in $\mathbb{C}[W]$. Let $\varsigma|_{\widetilde{E}}$ denote the restriction of ς to $\widetilde{E}$ for any subgroup E of U.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the complexified Cartan decomposition of the Lie algebra of $\widetilde{G}$ corresponding to the maximal compact subgroup $\widetilde{K}$. Likewise we define $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ for $\widetilde{G}'$. Let ρ be an irreducible admissible genuine $(\mathfrak{g}, \widetilde{K})$-module. By (2.5) in [15],

$$\mathscr{Y}/(\cap_{\psi \in \mathrm{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathscr{Y}, \rho)} \ker \psi) \simeq \rho \otimes \Theta(\rho)$$

where $\Theta(\rho)$ is a $(\mathfrak{g}', \widetilde{K}')$-module called the *full (local) theta lift* of ρ. Theorem 2.1 in [15] states that if $\Theta(\rho) \neq 0$, then $\Theta(\rho)$ is a $(\mathfrak{g}', \widetilde{K}')$-module of finite length with an infinitesimal character and it has a unique irreducible quotient $\theta(\rho)$ called the *(local) theta lift* of ρ.

2010 *Mathematics Subject Classification.* 22E46, 22E47.

Key words and phrases. local theta lifts, nilpotent orbits, associated varieties, associated cycles, moment maps.

Moreover if $\theta(\rho_1)$ and $\theta(\rho_2)$ are nonzero, then they are isomorphic if and only if ρ_1 and ρ_2 are isomorphic.

It is a result of Protsak and Przebinda [33] that in the stable range, $\theta(\rho)$ is nonzero. This partially generalizes a previous result of Li [21] which states that if ρ is irreducible and unitarizable, then $\theta(\rho)$ is nonzero and unitarizable.

In order to state our first result, we exclude following special case.

(†) The dual pair $(G, G') = (\mathrm{Sp}(n, \mathbb{R}), \mathrm{O}(2n, 2n))$ and ρ is the one dimensional genuine representation of $\widetilde{\mathrm{Sp}}(n, \mathbb{R})$.

Theorem A. *Suppose (G, G') is in the stable range where G is the smaller member. Let ρ be an irreducible unitarizable genuine $(\mathfrak{g}, \widetilde{K})$-module. We exclude the case (†) above. Then*

$$\Theta(\rho) = \theta(\rho)$$

as $(\mathfrak{g}', \widetilde{K}')$-modules. In other words, $\Theta(\rho)$ is already irreducible and unitarizable.

The proof is given in Section 2.2.

In Case (†), $\Theta(\rho)$ is reducible by Lee's appendix in [23].

The above theorem is useful because invariants attached to $\Theta(\rho)$ are usually easier to describe than that of $\theta(\rho)$. For example, we could deduce a formula for the $\widetilde{K}'$-types of $\Theta(\rho)$ in Proposition 2.1.

1.2. Before stating other results, we briefly review the definitions of some invariants of Harish-Chandra modules. See Section 2 in Vogan [36] for details.

Let (g, K) denote the Harish-Chandra pair of a real reductive group G. Let $\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$ denote the complexified Cartan decomposition the Lie algebra of G corresponding to the maximal compact subgroup K. Let (ϱ, V_ϱ) be a (g, K)-module of finite length and let $0 \subset F_0 \subset \cdots \subset F_j \subset F_{j+1} \subset \cdots$ be a good filtration of ϱ, i.e. $\dim F_j$ is finite, $\cup_{j \in \mathbb{N}} F_j = V_\varrho$ and $\mathcal{U}_p(\mathsf{g}) F_q = F_{p+q}$ for all q sufficiently large and for all $p > 0$. Then $\mathrm{Gr}\,\varrho = \bigoplus F_j / F_{j-1}$ is a finitely generated $(\mathcal{S}(\mathsf{p}), \mathsf{K})$-module where $\mathcal{S}(\mathsf{p})$ is the symmetric algebra on p.

Let $\mathscr{A}$ be the associated $\mathsf{K}_\mathbb{C}$-equivariant coherent sheaf of $\mathrm{Gr}\,\varrho$ on $\mathsf{p}^* = \mathrm{Spec}\,(\mathcal{S}(\mathsf{p}))$. The *associated variety* of ϱ is defined to be $\mathrm{AV}(\varrho) := \mathrm{Supp}(\mathscr{A})$ in p^*. Its dimension is called the *Gelfand-Kirillov dimension* of ϱ. It is a well known fact that $\mathrm{AV}(\varrho)$ is a closed subset of the null cone of p^*.

Let $\mathrm{AV}(\varrho) = \bigcup_{j=1}^r \overline{\mathcal{O}_j}$ where $\mathcal{O}_j$ are the distinct open $\mathsf{K}_\mathbb{C}$-orbits in $\mathrm{AV}(\varrho)$. By Lemma 2.11 in [36] (c.f. Proposition 4.3), there is a finite $(\mathcal{S}(\mathsf{p}), \mathsf{K}_\mathbb{C})$-invariant filtration $0 \subset \mathscr{A}_0 \subset \cdots \subset \mathscr{A}_l \subset \cdots \subset \mathscr{A}_n = \mathscr{A}$ of $\mathscr{A}$ such that $\mathscr{A}_l / \mathscr{A}_{l-1}$ is generically reduced on each $\overline{\mathcal{O}_j}$. For a closed point $x_j \in \mathcal{O}_j$, let $i_{x_j} \colon \{x_j\} \hookrightarrow \mathsf{p}^*$ be the natural inclusion map and let K_{x_j} be the stabilizer of x_j in $\mathsf{K}_\mathbb{C}$. Now

$$\chi_{x_j} := \bigoplus\nolimits_l (i_{x_j})^* (\mathscr{A}_l / \mathscr{A}_{l-1})$$

is a nonzero finite dimensional rational representation of K_{x_j}. We call χ_{x_j} an *isotropy representation* of ϱ at x_j. Its image $[\chi_{x_j}]$ in the Grothendieck group of finite dimensional rational K_x-modules is called the *isotropy character* of ϱ at x_j. The isotropy representation is dependent on the filtration but the isotropy character is independent of the filtration.

We call $\{ (\mathcal{O}_j, x_j, \chi_{x_j}) : j = 1, \ldots, r \}$ the set of *orbit data attached to the filtration* $\{ \mathscr{A}_j \}$. On the other hand, $\{ (\mathcal{O}_j, x_j, [\chi_{x_j}]) : j = 1, \ldots, r \}$ is independent of the filtration and we call it the set of *orbit data attached to* ϱ. Two orbit data are equivalent if they are conjugate to each other by the $\mathsf{K}_\mathbb{C}$-action. We define the *multiplicity of ϱ along $\mathcal{O}_j$* to be $m(\mathcal{O}_j, \varrho) = \dim_\mathbb{C} \chi_{x_j}$ and the *associated cycle* of ϱ to be $\mathrm{AC}(\varrho) = \sum_{j=1}^r m(\mathcal{O}_j, \varrho)[\overline{\mathcal{O}_j}]$.

In summary, the associated variety, the associated cycle and isotropy character(s) are invariants of ϱ, i.e. they are independent of the choices of filtrations.

Suppose G is a member group of a type I reductive dual pair in $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$. Then by [1], [26], [22] and [35], the above invariants of ϱ and of its contragredient ϱ^* are related by an automorphism C of G. We call C a *dualizing automorphism*. We will review these in Appendix B.

1.3. Now we describe a result about the associated variety of $\Theta(\rho)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ as in Section 1.1. In Appendix A.2 (also see [5]), we recall the definitions of the two moment maps

$$\mathfrak{p}^* \xleftarrow{\phi} W \xrightarrow{\phi'} \mathfrak{p}'^*. \tag{1}$$

The maps ϕ and ϕ' are given explicitly in Table 2. For a $K_{\mathbb{C}}$-invariant closed subset S of $\mathfrak{p}^*$, we define *the theta lift of S* to be $\theta(S) = \phi'(\phi^{-1}(S))$, which is a $K'_{\mathbb{C}}$-invariant closed subset of $\mathfrak{p}'^*$. Let $N(\mathfrak{p}^*) := \{\, x \in \mathfrak{p}^* \mid 0 \in \overline{K_{\mathbb{C}} \cdot x} \,\}$ be the nilpotent cone in $\mathfrak{p}^*$. Let $\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ be the set of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$. We define $N(\mathfrak{p}'^*)$ and $\mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ in the same way. It is well known that $\theta(S) \subseteq N(\mathfrak{p}'^*)$ if $S \subseteq N(\mathfrak{p}^*)$.

Since $\Theta(\rho)$ has finite length, the associated variety $\mathrm{AV}(\Theta(\rho))$ of $\Theta(\rho)$ is a closed subvariety of $N(\mathfrak{p}'^*)$.

Theorem B. *For any real reductive dual pair (G, G') (not necessary in the stable range) and any irreducible admissible genuine $(\mathfrak{g}, \widetilde{K})$-module, there is an upper bound of the associated variety of $\Theta(\rho)$ given by $\theta(\mathrm{AV}(\rho^*))$. In other words, we have*

$$\mathrm{AV}(\Theta(\rho)) \subseteq \theta(\mathrm{AV}(\rho^*))).$$

The proof is given in Section 3.4.

The above theorem is a correction to Proposition 3.12 in Nishiyama-Zhu [30].

1.4. We now assume (G, G') is in the stable range where G is the smaller member. Given $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$, it is a result of [31], [6] and [29] that there is a unique $\mathcal{O}' \in \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ such that $\overline{\mathcal{O}'} = \theta(\overline{\mathcal{O}})$. We call $\mathcal{O}'$ the *theta lift of $\mathcal{O}$* and we write $\mathcal{O}' = \theta(\mathcal{O})$. Moreover,

$$\begin{aligned} \theta\colon \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*) &\to \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*) \\ \mathcal{O} &\mapsto \mathcal{O}' \end{aligned}$$

is an injective map preserving the closure relations, i.e. $\theta(\mathcal{O}_2) \subset \overline{\theta(\mathcal{O}_1)}$ if $\mathcal{O}_2 \subset \overline{\mathcal{O}_1}$.

Definition. We define the following notion of theta lifts of objects in the stable range.

(1) Let $c = \sum_j m_j[\overline{\mathcal{O}_j}]$ be a formal sum of closures of nilpotent orbits. We define the *theta lift of the cycle c* to be $\theta(c) := \sum_j m_j[\overline{\theta(\mathcal{O}_j)}]$.

(2) Let $(\mathcal{O}, x, \chi_x)$ be an orbit datum where $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$, $x \in \mathcal{O}$ and χ_x is a finite-dimensional rational $\widetilde{K}_x$-module where $\widetilde{K}_x$ is the stabilizer of x in $\widetilde{K}_{\mathbb{C}}$. Let $\mathcal{O}' = \theta(\mathcal{O})$. Fixing points $x \in \mathcal{O}$, $w \in W$, $x' \in \mathcal{O}'$ such that $\phi(w) = x$ and $\phi'(w) = x'$, we will define a group homomorphism $\alpha\colon K'_{x'} \to K_x$ in Proposition 4.1. We define the *theta lift of the orbit datum* $(\mathcal{O}, x, \chi_x)$ to be $(\mathcal{O}', x', \chi_{x'})$ where

$$\chi_{x'} := \varsigma|_{\widetilde{K}'_{x'}} \otimes (\varsigma|_{\widetilde{K}_x} \otimes \chi_x) \circ \alpha.$$

We write $\theta(\mathcal{O}, x, \chi_x) = (\mathcal{O}', x', \chi_{x'})$ which is well defined up to $\widetilde{K}_{\mathbb{C}}$-conjugation. Similarly we define the theta lift of $(\mathcal{O}, x, [\chi_x])$ to be $\theta(\mathcal{O}, x, [\chi_x]) := (\mathcal{O}', x', [\chi_{x'}])$.

Theorem C. *Suppose (G, G') is in the stable range where G is the smaller member. Let ρ be a genuine irreducible $(\mathfrak{g}, \widetilde{K})$-module. Suppose $\{\, (\mathcal{O}_j, x_j, [\chi_{x_j}]) : j = 1, \ldots, r \,\}$ is the set of orbit data attached to ρ^*. Then $\{\, \theta(\mathcal{O}_j, x_j, [\chi_{x_j}]) : j = 1, \ldots, r \,\}$ is the set of orbit data attached to $\Theta(\rho)$.*

The next theorem is a corollary of Theorem C.

Theorem D. *Suppose (G, G') is in the stable range where G is the smaller member. Then*

$$\mathrm{AV}(\Theta(\rho)) = \theta(\mathrm{AV}(\rho^*)) \text{ and } \mathrm{AC}(\Theta(\rho)) = \theta(\mathrm{AC}(\rho^*)).$$

In particular if ρ is unitarizable and excluding (†), then $\mathrm{AV}(\theta(\rho)) = \theta(\mathrm{AV}(\rho^*))$ and $\mathrm{AC}(\theta(\rho)) = \theta(\mathrm{AC}(\rho^*))$ by Theorem A.

The proofs of Theorems C and D are given in Section 4.5. In these two theorems, we do not require that ρ^* is unitarizable. We will show in the proof of Lemma 4.4 that the dimension of every $\theta(\overline{\mathcal{O}_j})$ is equal to $\dim \mathrm{AV}(\Theta(\rho))$, i.e. the Gelfand-Kirillov dimension of $\Theta(\rho)$. However there are examples where $\Theta(\rho)$ is reducible and $\theta(\rho)$ has smaller Gelfand-Kirillov dimension than that of $\Theta(\rho)$. In particular $\mathrm{AV}(\theta(\rho))$ does not contain any $\theta(\mathcal{O}_j)$.

Theorem D overlaps with the previous work of [28] and [38] where G is a compact group. It also extends the work [30] where ρ is a unitarizable lowest weight module.

We would like to relate a recent result of [10] where Gomez and Zhu show that the dimensions of the generalized Whittaker functionals of the Casselman-Wallach globalizations of ρ and $\Theta(\rho)$ are the same. It is a famous result of [27] that in the p-adic case, the dimension of a space of generalized Whittaker functionals of an algebraic irreducible representation is equal to the corresponding multiplicity in its wavefront cycle. Theorem D together with [10] could be interpreted as an evidence for the corresponding phenomenon for real classical groups.

1.5. Let (g, K) and G as in Section 1.2. For a (g, K)-module ϱ of finite length, we define $\mathrm{V}_{\mathbb{C}}(\varrho)$ to be the complex variety cut out by the ideal $\mathrm{Gr}\,(\mathrm{Ann}_{\mathcal{U}(\mathsf{g})}\varrho)$ in g^*, where $\mathrm{Gr}\,(\mathrm{Ann}_{\mathcal{U}(\mathsf{g})}\varrho)$ is the graded ideal of $\mathrm{Ann}_{\mathcal{U}(\mathsf{g})}\rho$ in $\mathrm{Gr}\,\mathcal{U}(\mathsf{g}) = \mathbb{C}[\mathsf{g}^*]$. Alternatively $\mathrm{V}_{\mathbb{C}}(\varrho)$ is the associated variety of the $(\mathsf{g} \oplus \mathsf{g}, \mathrm{Ad}\mathsf{G})$-module $\mathcal{U}(\mathsf{g})/\mathrm{Ann}_{\mathcal{U}(\mathsf{g})}\varrho$. It is an $(\mathrm{Ad}^*\mathsf{G})_{\mathbb{C}}$-invariant complex variety in $\mathfrak{g}^*$ whose dimension is equal to $2\dim \mathrm{AV}(\varrho)$. By Proposition B.1, $\mathrm{V}_{\mathbb{C}}(\varrho^*) = \mathrm{V}_{\mathbb{C}}(\varrho)$.

We recall that (G, G') is a type I irreducible dual pair in the stable range where G is the smaller member. The actions of G and G' on the symplectic manifold $W_{\mathbb{R}}$ give two moment maps (see [5])

$$\mathfrak{g}^* \xleftarrow{\phi_G} W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\phi_{G'}} \mathfrak{g}'^*. \tag{2}$$

For an $\mathrm{Ad}^*G_{\mathbb{C}}$-invariant complex subvariety S of $\mathfrak{g}^*$, we define $\theta_{\mathbb{C}}(S) = \phi_{G'}(\phi_G^{-1}(S))$. This is an $\mathrm{Ad}^*G'_{\mathbb{C}}$-invariant complex subvariety of $\mathfrak{g}'^*$. We state a corollary of Theorem D.

Corollary E. *Suppose (G, G') is in the stable range where G is the smaller member. Let ρ be a genuine irreducible $(\mathfrak{g}, \widetilde{K})$-module. Then*

$$\mathrm{V}_{\mathbb{C}}(\Theta(\rho)) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho)).$$

The proof is given in Section 4.6.

The above corollary overlaps with Theorem 0.9 in [34] where Przebinda proves the identity $\mathrm{V}_{\mathbb{C}}(\theta(\rho)) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho))$ for dual pairs and unitarizable ρ satisfying some technical conditions.

1.6. In Section 5, we consider representations whose $\widetilde{K}$-spectrums are the same as the global sections of $\widetilde{K}_{\mathbb{C}}$-equivariant algebraic vector bundles on nilpotent orbits. We will show that theta lifts in the stable range preserve such property.

First we set up some notation. Let K be a compact group. Let $\mathcal{O}$ be a $\mathsf{K}_{\mathbb{C}}$-homogeneous space and let $x \in \mathcal{O}$. Let $\pi : \mathsf{K}_{\mathbb{C}} \to \mathcal{O}$ be the natural quotient map given by $\pi(k) =$

$(\mathrm{Ad}^*k)x$. Let K_x be the stabilizer of x in $\mathsf{K}_{\mathbb{C}}$. For a rational K_x-module (χ_x, V_x), we define the $\mathsf{K}_{\mathbb{C}}$-equivariant pre-sheaf $\mathscr{L}$ on $\mathcal{O}$ by $\mathscr{L}(U) = (\mathbb{C}[\pi^{-1}(U)] \otimes_{\mathbb{C}} V_x)^{\mathsf{K}_x}$ for all open subsets U of $\mathcal{O}$. By [4], $\mathscr{L}$ is a $\mathsf{K}_{\mathbb{C}}$-equivariant quasi-coherent sheaf with fiber χ_x at x. Moreover, by Theorem 2.7 in [4], $\chi_x \leftrightarrow \mathscr{L}$ gives an equivalence of categories between the category of rational representations of K_x and the category of $\mathsf{K}_{\mathbb{C}}$-equivariant quasi-coherent sheaves on $\mathcal{O} \simeq \mathsf{K}_{\mathbb{C}}/\mathsf{K}_x$. We define the $(\mathbb{C}[\mathcal{O}], \mathsf{K}_{\mathbb{C}})$-module

$$\mathrm{Ind}_{\mathsf{K}_x}^{\mathsf{K}_{\mathbb{C}}} \chi_x = (\mathbb{C}[\mathsf{K}_{\mathbb{C}}] \otimes V_x)^{\mathsf{K}_x} = H^0(\mathcal{O}, \mathscr{L}).$$

If $(\mathcal{O}, x, \chi_x)$ appears in the orbit data attached to a filtration of a finite length $(\mathfrak{g}, \widetilde{K})$-module, then we have

$$\mathscr{L} \text{ is generated by its space of global sections } \mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}} \chi_x. \tag{3}$$

For the rest of this section we will assume that data $(\mathcal{O}, x, \chi_x)$ satisfy (3).

We exclude following special cases:

$$(\dagger\dagger) \qquad \begin{aligned} &(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q)) \text{ where } p = 2n \text{ or } q = 2n; \\ &(G, G') = (\mathrm{Sp}(2n, \mathbb{C}), \mathrm{O}(4n, \mathbb{C})). \end{aligned}$$

Theorem F. *Suppose (G, G') is in the stable range where G is the smaller member. We exclude the special case $(\dagger\dagger)$ above. Let ρ be an irreducible admissible genuine $(\mathfrak{g}, \widetilde{K})$-module. Let $(\mathcal{O}, x, \chi_x)$ be an orbit datum satisfying (3) such that, as $\widetilde{K}$-modules*

$$\rho^* \;\simeq\; \mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}} \chi_x.$$

Let $(\mathcal{O}', x', \chi_{x'})$ be the theta lifting of $(\mathcal{O}, x, \chi_x)$. Then, as $\widetilde{K}'$-modules,

$$\Theta(\rho) \;\simeq\; \mathrm{Ind}_{\widetilde{K}'_{x'}}^{\widetilde{K}'_{\mathbb{C}}} \chi_{x'}.$$

The proof is given in Section 5.2.

1.7. We relate our results with a conjecture of Vogan on geometric quantizations and unipotent representations.

Definition (Definition 7.13 in [36])**.** Let $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ and $x \in \mathcal{O}$. The stabilizer K_x acts on the cotangent space $\mathrm{T}_x^*\mathcal{O} = (\mathfrak{k}/\mathfrak{k}_x)^*$. We define the character γ_x of K_x by

$$\gamma_x(k) = \det(\mathrm{Ad}(k)|_{(\mathfrak{k}/\mathfrak{k}_x)^*}) \qquad \forall k \in K_x.$$

A rational representation χ_x of the double cover $\widetilde{K}_x$ is called *admissible* if

$$\chi_x(\exp(X)) = \gamma_x(\exp(X/2)) \cdot \mathrm{Id} \qquad \forall X \in \mathfrak{k}_x. \tag{4}$$

An orbit datum $(\mathcal{O}, x, [\chi_x])$ is called an *admissible orbit datum* if χ_x is admissible. An orbit $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ is called *admissible* if it is part of an admissible datum. A representation χ_x of $\widetilde{K}_x$ satisfying (4) is uniquely determined by its character $[\chi_x]$.

A $(\mathfrak{g}, \widetilde{K})$-module ρ is said to have $\widetilde{K}$-spectrum determined by an admissible orbit datum $(\mathcal{O}, x, [\chi_x])$ if

$$\rho|_{\widetilde{K}} \;\simeq\; \mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}} \chi_x \tag{5}$$

as a $\widetilde{K}$-module. Such a representation ρ could be considered as a quantization of the orbit $\mathcal{O}$. In Conjecture 12.1 of [36], Vogan conjectured that, for every admissible orbit datum $(\mathcal{O}, x, [\chi_x])$ satisfying certain technical conditions and $\partial\mathcal{O}$ has codimension at least 2 in $\overline{\mathcal{O}}$, one can attach a unipotent representation ρ to this orbit datum and ρ satisfies (5).

In Section 6, we will show that the notion of admissibility is compatible with theta lifts in the stable range.

Proposition G. *Suppose (G, G') is in the stable range where G is the smaller member. Let $(\mathcal{O}, x, [\chi_x])$ be an admissible orbit datum for $\widetilde{G}$. Then its theta lift $\theta(\mathcal{O}, x, [\chi_x])$ is an admissible orbit datum for $\widetilde{G}'$.*

The above is a direct consequence of Proposition 6.1.

Suppose (G, G') is in the stable range where G is the smaller member and excluding the special case (††). Let ρ be an irreducible unitarizable $(\mathfrak{g}, \widetilde{K})$-module whose $\widetilde{K}$-spectrum is given by some admissible orbit datum $(\mathcal{O}, x, [\chi_x])$. It follows from Appendix B.1 that ρ^* is an irreducible unitarizable $(\mathfrak{g}, \widetilde{K})$-module whose $\widetilde{K}$-spectrum is given by the admissible orbit datum

$$C(\mathcal{O}, x, [\chi_x]) := (C(\mathcal{O}), \mathrm{Ad}^* C(x), [\chi_x \circ C])$$

where C is a dualizing automorphism on $\widetilde{G}$. By Theorem A, Theorem F and Proposition G, $\theta(\rho)$ is an irreducible unitarizable $(\mathfrak{g}', \widetilde{K}')$-module whose $\widetilde{K}'$-spectrum is given by the admissible orbit datum $\theta(C(\mathcal{O}, x, [\chi_x]))$.

1.8. Finally we construct a series of candidates for unipotent representations. Let

$$G_0, G_1, G_2, \cdots, G_n, \cdots$$

be a sequence of real classical groups satisfying the following properties:

(i) each pair (G_n, G_{n+1}) is an irreducible type I reductive dual pair with G_n being the smaller member excluding the special case (††).
(ii) The corresponding double covers $\widetilde{G}_n$ of G_n for the dual pairs (G_{n-1}, G_n) and (G_n, G_{n+1}) are isomorphic. We fix an isomorphism between these two double covers of G_n.
(iii) The covering group $\widetilde{G}_0$ has an irreducible genuine one dimensional unitary representation ρ_0 such that $\rho_0|_{\mathfrak{g}_0}$ is trivial.

It is clear that ρ_0 is attached to the admissible datum $(\{0\}, 0, \rho_0|_{(\widetilde{K}_0)_{\mathbb{C}}})$.

Let C_n be a dualizing automorphism on $\widetilde{G}_n$. Starting from ρ_0, we define inductively $\rho_{n+1} = \theta(\rho_n)$ and $(\mathcal{O}_{n+1}, x_{n+1}, \chi_{n+1}) = \theta(C_n(\mathcal{O}_n, x_n, \chi_n))$. The following theorem follows from Section 1.7.

Theorem H. *The $(\mathfrak{g}_n, \widetilde{K}_n)$-module ρ_n is an irreducible and unitarizable representation attached to the admissible orbit datum $(\mathcal{O}_n, x_n, \chi_n)$. Moreover, as $\widetilde{K}_n$-module,*

$$\rho_n \simeq \mathrm{Ind}_{\widetilde{K}_{x_n}}^{(\widetilde{K}_n)_{\mathbb{C}}} \chi_n.$$

The above theorem generalizes a result of Yang [39] [40] where he proves the above theorem for ρ_1. A related result on Dixmier algebras is given in [3].

Acknowledgment. We would like to thank Chen-bo Zhu, CheeWhye Chin, Roger Howe, Jian-Shu Li, Kyo Nishiyama and Peter Trapa for enlightening discussions. We also thank the referees and C. Chin for careful reading and pointing out a number of inaccuracies in the first draft. The first author is supported by a MOE-NUS grant MOE2010-T2-2-113. The second author would like to thank the hospitality of Aoyama Gakuin University and Hong Kong University of Science and Technology where part of this paper is written.

	G	G'	Stable range
Case $\mathbb{R}$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	$2n \leq p,q$
	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$p+q \leq n$
Case $\mathbb{C}$	$\mathrm{U}(n_1,n_2)$	$\mathrm{U}(p,q)$	$n_1+n_2 \leq p,q$
Case $\mathbb{H}$	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p,q)$	$n \leq p,q$
	$\mathrm{Sp}(p,q)$	$\mathrm{O}^*(2n)$	$2(p+q) \leq n$
Complex groups	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{O}(p,\mathbb{C})$	$4n \leq p$
	$\mathrm{O}(p,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	$p \leq n$

TABLE 1. Stable range for irreducible Type I dual pairs

2. THETA LIFTS OF UNITARY REPRESENTATIONS IN THE STABLE RANGE

2.1. Let (G,G') be a type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$. We list them in Table 1 below. We say it is in the stable range with G being the smaller member if it satisfies the conditions in the last column of the table.

We follow the notation in [15]. By Fact 1 in [15], K' is a member of a reductive dual pair (K',M) in $\mathrm{Sp}(W_{\mathbb{R}})$. We form the following see-saw pair in $\mathrm{Sp}(W_{\mathbb{R}})$:

$$\begin{tikzcd} G' \arrow[d, dash] \arrow[dr, dash] & M \arrow[d, dash] \arrow[dl, dash] \\ K' & G. \end{tikzcd} \tag{6}$$

The complex Lie algebra of M has Cartan decomposition $\mathfrak{m} = \mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(0,2)}$ where $\mathfrak{m}^{(1,1)}$ is the complexified Lie algebra of a maximal compact subgroup $M^{(1,1)}$ of M.

Let $\widetilde{\mathcal{H}} = \{\, v \in \mathscr{Y} \mid Xv = 0, \forall X \in \mathfrak{m}^{(0,2)} \,\}$ be the space of $\widetilde{K}'$-harmonics in $\mathscr{Y}$. As an $\widetilde{M}^{(1,1)} \times \widetilde{K}'$-module,

$$\widetilde{\mathcal{H}} = \bigoplus\nolimits_{\sigma' \in \widehat{\widetilde{K}'}} \sigma \otimes \sigma' \tag{7}$$

where each σ is either zero or an irreducible genuine $\widetilde{M^{(1,1)}}$-module uniquely determined by σ'.

Proposition 2.1. *Suppose (G,G') is in the stable range where G is the smaller member. Let ρ be an irreducible genuine $(\mathfrak{g},\widetilde{K})$-module. Then as $\widetilde{K}'$-modules*

$$\Theta(\rho)|_{\widetilde{K}'} = \bigoplus_{\sigma' \in \widehat{\widetilde{K}'}} m_{\sigma'}\sigma' \simeq (\widetilde{\mathcal{H}} \otimes \rho^*|_{\widetilde{K}})^K$$

where $m_{\sigma'}$ is the multiplicity of σ' in $\Theta(\rho)$. We have $m_{\sigma'} = \dim \mathrm{Hom}_{\widetilde{K}}(\sigma,\rho)$.

If ρ is the Harish-Chandra module of a discrete series representation of $\widetilde{G}$, the above proposition is Corollary 5.3 in [14].

Proof. Let $L(\sigma)$ denote the (full) theta lift of σ', which is a unitarizable lowest weight module of $\widetilde{M}$. The fact that the pair (G,G') is in the stable range implies that

$$L(\sigma) = \mathcal{U}(\mathfrak{m}) \otimes_{\mathcal{U}(\mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(0,2)})} \sigma \simeq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \sigma \tag{8}$$

as a $\mathfrak{g}$-module. The first equality follows from Jantzen irreducibility criterion (see Section 6 in [9]). In this case $L(\sigma)$ is a Harish-Chandra module of a (limit of) holomorphic discrete series representations (see Sections II.8.2 and III.8.1 in [18]). The second equality follows from $\mathfrak{k} = \mathfrak{g} \cap (\mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(0,2)})$ and $\mathfrak{m} = (\mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(0,2)}) + \mathfrak{g}$ (see (3.4) in [15]). Applying this to the see-saw pair (6), we get

$$m_{\sigma'} = \dim \operatorname{Hom}_{\widetilde{K}'}(\sigma', \Theta(\rho)) = \dim \operatorname{Hom}_{\mathfrak{g},\widetilde{K}}(L(\sigma), \rho) = \dim \operatorname{Hom}_{\widetilde{K}}(\sigma, \rho). \tag{9}$$

This proves the proposition. □

2.2. Let (ρ, V_ρ) be an irreducible unitarizable Harish-Chandra module of $\widetilde{G}$. For the rest of this section we will prove Theorem A.

First we recall Li's construction of $\theta(\rho)$ [21]. We denote an element in the inverse image of $g \in M$ by $\tilde{g} \in \widetilde{M}$. The actual choice of $\tilde{g}$ will not affect the calculation. Define

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \int_G \langle \rho^*(\tilde{g}) v_1, v_2 \rangle_{\rho^*} \langle \tilde{g} \cdot w_1, w_2 \rangle_{\mathscr{Y}} \, dg$$

for all $v_1 \otimes w_1, v_2 \otimes w_2 \in V_{\rho^*} \otimes \mathscr{Y}$. All pairings are done in the completion of the Harish-Chandra modules. We set

$$\operatorname{Rad}(\langle , \rangle) = \{ \Phi \in V_{\rho^*} \otimes \mathscr{Y} \mid \langle \Phi, \Phi' \rangle = 0, \forall \Phi' \in V_{\rho^*} \otimes \mathscr{Y} \}.$$

Let

$$H = (V_{\rho^*} \otimes \mathscr{Y}) / \operatorname{Rad}(\langle , \rangle).$$

We claim that $H \simeq \theta(\rho)$ as irreducible unitarizable Harish-Chandra modules of G'. Indeed Li [21] uses smooth vectors in the definition of $\langle , \rangle$ and likewise defines $H^\infty = ((V_{\rho^*})^\infty \otimes \mathscr{Y}^\infty)/\operatorname{Rad}(\langle , \rangle^\infty)$. Theorem 6.1 in [21] shows that $\theta(\rho)$ is the Harish-Chandra module of H^∞. Since H is $\widetilde{K}'$-finite and dense in H^∞, it is equal to the Harish-Chandra module $\theta(\rho)$ of H^∞. This proves our claim.

We refer to (σ, V_σ) in (7) and $L(\sigma)$ in (8). Then $L(\sigma)$ is an irreducible unitarizable Harish-Chandra module of $\widetilde{M}$ and $\mathscr{Y} = \bigoplus_{\sigma'} L(\sigma) \otimes \sigma'$.

We set

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\rho^*}^{\sigma'} = \int_G \langle \rho^*(\tilde{g}) v_1, v_2 \rangle_{\rho^*} \langle \tilde{g} \cdot w_1, w_2 \rangle_{L(\sigma)} \, dg \quad \forall v_i \otimes w_i \in V_{\rho^*} \otimes L(\sigma)$$

and define

$$H(\sigma') = (V_{\rho^*} \otimes L(\sigma)) / \operatorname{Rad}(\langle , \rangle_{\rho^*}^{\sigma'}).$$

Now $\dim H(\sigma')$ is the multiplicity of σ' in H and we have

$$H = \bigoplus_{\sigma'} H(\sigma') \otimes \sigma'.$$

We consider following embeddings:

(10)

$$H(\sigma') = (V_{\rho^*} \otimes L(\sigma))/\operatorname{Rad}(\langle , \rangle_{\rho^*}^{\sigma'}) \xhookrightarrow{\iota} \operatorname{Hom}_G(V_{\rho^*}^\infty \otimes L(\sigma)^\infty, \mathbb{C}) \xhookrightarrow{\text{rest.}} \operatorname{Hom}_{\mathfrak{g},K}(V_{\rho^*} \otimes L(\sigma), \mathbb{C})$$

where $\iota(\Phi)$ is given by

$$\Phi \mapsto (\Phi' \mapsto \langle \Phi', \Phi \rangle_{\rho^*}^{\sigma'}) \quad \forall \Phi' \in V_{\rho^*}^\infty \otimes L(\sigma)^\infty. \tag{11}$$

The last term on the right hand side of (10) is

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{g},K}(V_{\rho^*} \otimes L(\sigma), \mathbb{C}) &= \operatorname{Hom}_{\mathfrak{g},\widetilde{K}}(L(\sigma), \operatorname{Hom}_{\mathbb{C}}(V_{\rho^*}, \mathbb{C})) \\
&= \operatorname{Hom}_{\mathfrak{g},\widetilde{K}}(L(\sigma), V_\rho) \\
&\quad (L(\sigma) \text{ is } \widetilde{K}\text{-finite, so its image is in } \operatorname{Hom}_{\mathbb{C}}(V_{\rho^*}, \mathbb{C})_{\widetilde{K}-\text{finite}} = V_\rho) \\
&= \operatorname{Hom}_{\widetilde{K}}(V_\sigma, V_\rho) \qquad (\text{by } (8)) \\
&= \operatorname{Hom}_K(V_{\rho^*} \otimes V_\sigma, \mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes V_\sigma)^K, \mathbb{C}).
\end{aligned}$$

The isomorphism between the first term and the last term in above equalities is given by restriction. Combining these with (10) gives an inclusion map

$$H(\sigma') \hookrightarrow \operatorname{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes V_\sigma)^K, \mathbb{C}) \tag{12}$$

given by (11) but for $\Phi' \in (V_{\rho^*} \otimes V_\sigma)^K$. By (9)

$$\dim H(\sigma') \leq \dim \operatorname{Hom}_{\widetilde{K}}(V_\sigma, V_\rho) = \dim \operatorname{Hom}_{\widetilde{K}'}(\sigma', \Theta(\rho))$$

and it is finite.

Lemma 2.2. *Let* $\mathbf{d} \in (V_{\rho^*} \otimes V_\sigma)^K$ *be a nonzero vector. Then the pairing between* $V_{\rho^*} \otimes L(\sigma)$ *and* $\mathbf{d}$ *in* (11) *is non-vanishing.*

The above lemma implies that (12) is an isomorphism so that $\Theta(\rho)$ and H have the same $\widetilde{K}$-multiplicities. This will prove Theorem A.

In order to prove Lemma 2.2, we first exhibit a globalization of the Harish-Chandra module $L(\sigma)$. Our references are [18] and [17]. We refer to M in (6). Let

$$\operatorname{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}, V_\sigma) = \left\{ f : \widetilde{M} \to V_\sigma \;\middle|\; \begin{array}{l} f \text{ is analytic,} \\ f(\tilde{g}\tilde{k}) = \sigma(\tilde{k}^{-1}) f(\tilde{g}) \;\; \forall \tilde{g} \in \widetilde{M}, \tilde{k} \in \widetilde{M}^{(1,1)}, \\ r(X) f = 0 \;\; \forall X \in \mathfrak{m}^{(0,2)}. \end{array} \right\}.$$

Here $r(X)$ denote the right derivation action. Let $\{ v_i \}$ be an orthonormal basis of $V_\sigma \subset L(\sigma)$. Then

$$\xi : v \mapsto \left(\tilde{g} \mapsto \sum_i \left\langle \tilde{g}^{-1} \cdot v, v_i \right\rangle_{L(\sigma)} v_i \right)$$

defines an injective $(\mathfrak{m}, \widetilde{M}^{(1,1)})$-module homomorphism $\xi \colon L(\sigma) \to \operatorname{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}; V_\sigma)$.

For any $g \in M$, there are unique elements $z(g) \in \mathfrak{m}^{(0,2)}$, $k(g) \in M_{\mathbb{C}}^{(1,1)}$ and $z'(g) \in \mathfrak{m}^{(2,0)}$ such that $g = \exp(z(g)) k(g) \exp(z'(g))$. The map $g \mapsto k(g)$ lifts to a map $\tilde{k} : \widetilde{M} \to \widetilde{M}_{\mathbb{C}}^{(1,1)}$ (see page 8 in [18]). Let Ω denote the image of the map z and let $\zeta : \widetilde{M} \to M \xrightarrow{z} \Omega$ denote the composite map. Then

$$\Omega = \{ z(g) \in \mathfrak{m}^{(0,2)} : g \in M \} \simeq M / M^{(1,1)} \simeq \widetilde{M} / \widetilde{M}^{(1,1)} \tag{13}$$

is a bounded symmetric domain in $\mathfrak{m}^{(0,2)}$ and

$$M \subset \exp(\Omega) \cdot M_{\mathbb{C}}^{(1,1)} \cdot \exp(\mathfrak{m}^{(2,0)}).$$

Let $\operatorname{Hol}(\Omega, V_\sigma)$ denote the space of holomorphic functions on Ω with values in V_σ. We define $\mathrm{P} : \operatorname{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}, V_\sigma) \to \operatorname{Hol}(\Omega, V_\sigma)$ in the following way: For $f \in \operatorname{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}, V_\sigma)$, we set $\mathrm{P} f \in \operatorname{Hol}(\Omega, V_\sigma)$ by $\mathrm{P} f(\tilde{g} \widetilde{M}^{(1,1)}) = \sigma(\tilde{k}(\tilde{g})) f(\tilde{g})$. Then P is a bijection using (13).

Let $\overline{\xi} = \mathrm{P} \circ \xi : L(\sigma) \to \operatorname{Hol}(\Omega, V_\sigma)$. Let $\mathbb{C}[\mathfrak{m}^{(0,2)}]$ denote the space of polynomials on $\mathfrak{m}^{(0,2)}$. Then $\overline{\xi}(L(\sigma))$ is the linear span of

$$\left\{ p \times \overline{\xi}(v) \;\middle|\; p \in \mathbb{C}[\mathfrak{m}^{(0,2)}], v \in V_\sigma \right\}$$

because $L(\sigma)$ is a full generalized Verma module.

We write $V_\sigma = \bigoplus_{l \in L} D_l$ and $V_{\rho^*} = \bigoplus_{j \in J} D_j$ as direct sums of irreducible $\widetilde{K}$-modules. Then

$$(V_{\rho^*} \otimes V_\sigma)^K = \bigoplus_{j \in J} \bigoplus_{l \in L} (D_j \otimes D_l)^K = \bigoplus_{l \in L} \bigoplus_{D_j \simeq D_l^*} (D_j \otimes D_l)^K.$$

Let $\{d_{l\lambda} : \lambda = 1, \dots, \dim D_l\}$ be an orthonormal basis of D_l and let $\{d^*_{j\lambda}\}$ be a basis of $D_j \simeq D_l^*$ which is dual to $\{d_{l\lambda}\}$. Then a vector $\mathbf{d} \in (V_{\rho^*} \otimes V_\sigma)^K$ in Lemma 2.2 is of the form

$$\mathbf{d} = \sum_{j\in J, l\in L} c_{jl} \left(\sum_\lambda d^*_{j\lambda} \otimes d_{l\lambda} \right)$$

where $c_{jl} \in \mathbb{C}$. Here $c_{jl} = 0$ unless $D_j \simeq D_l^*$. We suppose $c_{j_0 l_0} \neq 0$ for some $j_0 \in J$ and $l_0 \in L$.

Let

$$\mathcal{C}(\widetilde{G}, \widetilde{K}; V_\sigma) = \left\{ f \in \mathcal{C}(\widetilde{G}, V_\sigma) \;\middle|\; f(\tilde{g}\tilde{k}) = \sigma(\tilde{k}^{-1}) f(\tilde{g}) \; \forall k \in \widetilde{K} \right\}$$

be the space of continuous sections. We define a G-module homomorphism $\xi_{\mathbf{d}} \colon V^\infty_{\rho^*} \to \mathcal{C}(\widetilde{G}, \widetilde{K}; V_\sigma)$ by

$$\xi_{\mathbf{d}} : v \mapsto \left(\tilde{g} \mapsto \sum_{j\in J, l\in L} c_{jl} \sum_\lambda \overline{\left\langle \rho^*(\tilde{g}^{-1}) v, d^*_{j\lambda} \right\rangle_{\rho^*}} d_{l\lambda} \right) \quad (\forall \tilde{g} \in \widetilde{G}).$$

Let $\Omega_0 \simeq \widetilde{G}/\widetilde{K}$ denote the image $\zeta(\widetilde{G})$ in Ω. We have $\mathrm{P}_0 \colon \mathcal{C}(\widetilde{G}, \widetilde{K}; V_\sigma) \to \mathcal{C}(\Omega_0; V_\sigma)$ defined by the same formula as P. We denote $\overline{\xi}_{\mathbf{d}} = \mathrm{P}_0 \circ \xi_{\mathbf{d}}$. Let dx be the G-invariant measure on $\Omega_0 \simeq G/K$ compatible with the Haar measure on G i.e. $dg = dx\, dk$. We recall that $c_{j_0 l_0} \neq 0$. Let $d_{l_0\lambda_0}$ be a unit vector in the orthonormal basis of D_{l_0}. Let w in $L(\sigma)$ such that $\overline{\xi}(w) = p \times \overline{\xi}(d_{l_0\lambda_0})$. Let $v = d^*_{j_0\lambda_0}$ be the corresponding unit vector in the orthonormal basis of $D_{j_0} \subseteq V_{\rho^*}$. Then $v \otimes w \in V_{\rho^*} \otimes L(\sigma)$ and we have

$$\begin{aligned}
\langle v \otimes w, \mathbf{d} \rangle^\sigma_{\rho^*} &= \sum_{j\in J, l\in L} c_{jl} \int_G \sum_\lambda \left\langle \rho^*(\tilde{g}^{-1}) v, d^*_{j\lambda} \right\rangle_{\rho^*} \left\langle \tilde{g}^{-1} \cdot w, d_{l\lambda} \right\rangle_{L(\sigma)} dg \\
&= \int_G \langle \xi(w)(\tilde{g}), \xi_{\mathbf{d}}(v)(\tilde{g}) \rangle_{V_\sigma} \, dg \\
&= \int_G \left\langle \sigma(\widetilde{k(g)}) \xi(w)(\tilde{g}), \sigma(\widetilde{k(g)}) \xi_{\mathbf{d}}(v)(\tilde{g}) \right\rangle_{V_\sigma} \, dg \\
&= \int_{G/K} \left\langle \overline{\xi}(w)(gK), \overline{\xi}_{\mathbf{d}}(v)(gK) \right\rangle_{V_\sigma} \, dgK \\
&= \int_{\Omega_0} p(x) \left\langle \overline{\xi}(d_{l_0\lambda_0})(x), \overline{\xi}_{\mathbf{d}}(d^*_{j_0\lambda_0})(x) \right\rangle_{V_\sigma} \, dx \\
&= \int_{\Omega_0} p(x) f(x) dx \qquad (14)
\end{aligned}$$

where $f(x) = \left\langle \overline{\xi}(d_{l_0\lambda_0})(x), \overline{\xi}_{\mathbf{d}}(d^*_{j_0\lambda_0})(x) \right\rangle_{V_\sigma}$. The function $f(x)$ is a nonzero continuous function because $f(0) = \sum_{j\in J, l\in L} c_{jl} \sum_\lambda \left\langle d^*_{j_0\lambda_0}, d^*_{j\lambda} \right\rangle_{\rho^*} \langle d_{l_0\lambda_0}, d_{l\lambda} \rangle_{L(\sigma)} = c_{j_0 l_0} \neq 0$. We extend $f(x)$ to the boundary of Ω_0 by 0.

By Li [21], the integration (14) is absolutely convergent for every $p \in \mathbb{C}[\mathfrak{m}^{(0,2)}]$. This is the place where we exclude the Case (†) in Section 1.1.

It remains to show that (14) is nonzero for some $p(x) \in \mathbb{C}[\mathfrak{m}^{(0,2)}]$. By [14], the restriction of $\mathbb{C}[\mathfrak{m}^{(0,2)}]$ to the compact subset $\overline{\Omega_0}$ forms a dense subset in $\mathcal{C}(\overline{\Omega_0})$ under sup-norm by the Stone-Weierstrass Theorem. Note that any open subset of $\overline{\Omega_0}$ has non-zero measure. Hence $\int_{\overline{\Omega_0}} p(x) f(x)\, dx$ is non-zero for some $p(x)$ by an approximation of identity argument. This completes the proof of Lemma 2.2 and Theorem A.

3. Natural filtrations and corresponding $(\mathcal{S}(\mathfrak{p}), K)$-modules

3.1. Let (G, G') be an irreducible type I dual pair as in Table 1. We do not assume that it is in the stable range except in Lemma 3.2. Let ρ be an irreducible genuine $(\mathfrak{g}, \widetilde{K})$-module. Let ρ^* denote its dual (contragredient) $(\mathfrak{g}, \widetilde{K})$-module and let $\rho' = \Theta(\rho)$ denote its full theta lift. For any module ϱ, we denote its underlying space by V_ϱ.

3.2. The Fock model $\mathscr{Y}$ is realized as complex polynomials on W, so $\mathscr{Y} = \bigcup_b \mathscr{Y}_b$ is filtered by degrees. See Appendix A.1. Let (τ, V_τ) be a lowest degree $\widetilde{K}$-type of (ρ, V_ρ) with degree j_0. Let $V_\tau \otimes V_{\tau'}$ be the image of joint harmonics in $V_\rho \otimes V_{\rho'}$. By [15], $V_{\rho'} = \mathcal{U}(\mathfrak{g}')V_{\tau'}$. Thus we define a good filtration on $V_{\rho'} = \bigcup_j V'_j$ by setting $V'_j = \mathcal{U}_j(\mathfrak{g}')V_{\tau'}$.

We view $V_{\rho^*} = \mathrm{Hom}_{\mathbb{C}}(V_\rho, \mathbb{C})_{\widetilde{K}-\text{finite}}$. Let $V_{\tau^*} \subset V_{\rho^*}$ be an irreducible $\widetilde{K}$-submodule with type τ^* which pairs perfectly with V_τ. By Theorem 13 (5) in [13], the lowest degree $\widetilde{K}$-type has multiplicity one in ρ. Hence V_τ and V_{τ^*} are well defined.

Likewise we define filtrations on V_ρ and V_{ρ^*} by $\{\, V_j := \mathcal{U}_j(\mathfrak{g})V_\tau \,\}_{j\in\mathbb{N}}$ and $\{\, V^*_j := \mathcal{U}_j(\mathfrak{g})V_{\tau^*} \,\}_{j\in\mathbb{N}}$ respectively. We will clarify the relationships between them in Appendix B.2.

Let $\mathbf{E} = V_{\rho^*} \otimes \mathscr{Y}$. We set $\mathbf{E}_{\mathfrak{g},K} = \mathbf{E}/\mathrm{Span}\{\, Xv, kv - v \mid v \in \mathbf{E}, X \in \mathfrak{g}, k \in K \,\}$ and $\mathbf{E}_{\mathfrak{p}} = \mathbf{E}/\mathrm{Span}\{\, X'v \mid v \in \mathbf{E}, X' \in \mathfrak{p} \,\}$. By Proposition 2.3 in [24],

$$\Theta(\rho) \simeq \mathbf{E}_{\mathfrak{g},K} = (\mathbf{E}_{\mathfrak{p}})^K .$$

Let $\eta\colon \mathbf{E} \twoheadrightarrow (\mathbf{E}_{\mathfrak{p}})^K \simeq V_{\rho'}$ be the natural quotient map. We define a filtration on $\mathbf{E}$ by

$$\mathbf{E}_j = \sum_{2a+b=j} V^*_a \otimes \mathscr{Y}_b.$$

Lemma 3.1 (Section 2 [24]). *We have $\eta(\mathbf{E}_{j_0+2j}) = \eta(\mathbf{E}_{2j_0+2j+1}) = V'_j$.* □

Let $\mathrm{pr}_{\mathfrak{p}}$ and pr_K be the projection to $\mathfrak{p}$-coinvariants and K-invariants respectively. The last lemma says that $\mathbf{E}_j$ is compatible with the filtration $\{V'_j\}_{j\in\mathbb{N}}$ on $V_{\rho'}$. Taking the graded module, η induces a map

$$\mathrm{Gr}\, V_{\rho^*} \otimes \mathrm{Gr}\, \mathscr{Y} \xrightarrow{\epsilon\otimes 1} \mathrm{Gr}\, V_{\rho^*} \otimes_{\mathcal{S}(\mathfrak{p})} \mathrm{Gr}\, \mathscr{Y} \longrightarrow \mathrm{Gr}\,(\mathrm{pr}_{\mathfrak{p}}(\mathbf{E})) \xrightarrow{\mathrm{Gr}\,\mathrm{pr}_K} \mathrm{Gr}\, V_{\rho'}. \tag{15}$$

Here $\epsilon : \mathrm{Gr}\, V_{\rho^*} \to \mathrm{Gr}\, V_{\rho^*}$ is the $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-module isomorphism such that $\epsilon(x) = (-1)^a x$ for all $x \in \mathrm{Gr}^a V_{\rho^*}$.

3.3. We recall that U is a maximal compact subgroup of $\mathrm{Sp}(W_{\mathbb{R}})$. Let $\mathfrak{sp}^{(1,1)}$ be the complexified Lie algebra of U. Let $\mathfrak{sp}(W_{\mathbb{R}}) \otimes \mathbb{C} = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(0,2)}$ denote the complexified Cartan decomposition (see Section A.2). Let $\mathfrak{s} = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)}$. We recall that ς is the minimal one dimensional $\widetilde{\mathrm{U}}$-type of the Fock model $\mathscr{Y}$. We extend ς to an $(\mathcal{S}(\mathfrak{s}), \widetilde{\mathrm{U}})$-module where $\mathfrak{s}$ acts trivially. We will continue to denote this one dimensional module by ς. In this way, $\mathrm{Gr}\, \mathscr{Y} = \bigoplus(\mathscr{Y}_{a+1}/\mathscr{Y}_a) \simeq \varsigma \otimes \mathbb{C}[W]$. Here U acts on $\mathbb{C}[W]$ by $(k \cdot f)(w) = f(k^{-1}w)$ for $k \in \mathrm{U}$, $f \in \mathbb{C}[W]$ and $w \in W$ (c.f. Section A.1). The algebra $\mathcal{S}(\mathfrak{sp}^{(0,2)})$ acts trivially on $\mathbb{C}[W]$ while $\mathcal{S}(\mathfrak{sp}^{(2,0)})$ acts by multiplication by degree two homogeneous polynomials. Since (G, G') is a reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$, we denote the restriction of ς as a $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-module by $\varsigma|_{\widetilde{K}}$. Similarly we get a one dimensional $(\mathcal{S}(\mathfrak{p}'), \widetilde{K}')$-module $\varsigma|_{\widetilde{K}'}$.

Let $\mathbf{A} = \varsigma|_{\widetilde{K}} \otimes \mathrm{Gr}\, V_{\rho^*}$ and $\mathbf{B} = \varsigma|_{\widetilde{K}'}^{-1} \otimes \mathrm{Gr}\, V_{\rho'}$. Since ρ is a genuine Harish-Chandra module of $\widetilde{G}$, $\mathbf{A}$ is an $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module. Similarly $\mathbf{B}$ is an $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module.

We note that that $K_{\mathbb{C}}$ acts on $\mathbf{A}\otimes\mathbb{C}[W]$ reductively and preserves the degrees. Then (15) gives the following $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module morphisms

$$\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W] \twoheadrightarrow \big(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W]\big)^{K_{\mathbb{C}}} \xrightarrow{\eta_0}\!\!\twoheadrightarrow \varsigma|_{\widetilde{K}'}^{-1}\otimes\big(\mathrm{Gr}\,(\mathrm{pr}_{\mathfrak{p}}(\mathbf{E}))\big)^{K_{\mathbb{C}}} \simeq \mathbf{B}. \tag{16}$$

The merit of introducing ς is that the $\widetilde{K}\cdot\widetilde{K}'$ action on $\mathrm{Gr}\,\mathscr{Y}$ descends to a geometric $K_{\mathbb{C}}\cdot K'_{\mathbb{C}}$ action on $\mathbb{C}[W]$.

Since $\mathbb{C}[W]$ is an $(\mathcal{S}(\mathfrak{s}), \mathrm{U}_{\mathbb{C}})$-module, it is also an $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})\times(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module.

Lemma 3.2. *Suppose (G, G') is in the stable range where G is the smaller member. Then η_0 in* (16) *is an isomorphism, i.e.*

$$\mathbf{B}\simeq(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W])^{K_{\mathbb{C}}}$$

as $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-modules. Here $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$ acts trivially on $\mathbf{A}$.

Proof. We recall that $\widetilde{\mathcal{H}}$ denotes the space of harmonics in (7). Let $\mathcal{H}=\varsigma^{-1}\widetilde{\mathcal{H}}$. Under the stable range assumption $\mathbb{C}[W]=\mathcal{S}(\mathfrak{p})\otimes\mathcal{H}$ as an $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})\times K'_{\mathbb{C}}$-module. Since $\mathbf{A}=\varsigma|_{\widetilde{K}}\otimes V_{\rho^*}$, as $K'_{\mathbb{C}}$-modules,

$$(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W])^{K_{\mathbb{C}}}=(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}(\mathcal{S}(\mathfrak{p})\otimes\mathcal{H}))^{K_{\mathbb{C}}}=(\mathbf{A}|_{K_{\mathbb{C}}}\otimes\mathcal{H})^{K_{\mathbb{C}}}\simeq\mathbf{B}|_{K'_{\mathbb{C}}}$$

by Proposition 2.1. The map η_0 is a surjection and $\mathbf{B}$ is an admissible $K'_{\mathbb{C}}$-module. The lemma follows from the equality of K'-types. □

3.4. *Proof of Theorem B.* Since the filtration on ρ^* (resp. ρ') is good, the graded module $\mathbf{A}$ (resp. $\mathbf{B}$) is a finitely generated $\mathcal{S}(\mathfrak{p})$-module (resp. $\mathcal{S}(\mathfrak{p}')$-module). Let $\mathscr{A}$ be the associated coherent sheaf of $\mathbf{A}$ on $\mathfrak{p}^*$. Using the moment maps

$$\mathfrak{p}^*\xleftarrow{\ \phi\ }W\xrightarrow{\ \phi'\ }\mathfrak{p}'^*$$

we see that the associated quasi-coherent sheaf of $\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W]$ on $\mathfrak{p}'^*$ is $\phi'_*\phi^*\mathscr{A}$. Let $\mathscr{B}$ be the associated quasi-coherent sheaf of $\mathbf{B}$ on $\mathfrak{p}'^*$.

By definition, $\mathrm{AV}(\rho^*)=\mathrm{Supp}(\mathscr{A})$ and $\mathrm{AV}(\Theta(\rho))=\mathrm{Supp}(\mathscr{B})$. By (16), $\mathscr{B}$ is a subquotient of the quasi-coherent sheaf $\phi'_*\phi^*\mathscr{A}$ so

$$\mathrm{Supp}(\mathscr{B})\subseteq\mathrm{Supp}(\phi'_*\phi^*\mathscr{A})\subseteq\overline{\phi'(\mathrm{Supp}(\phi^*\mathscr{A}))}\subseteq\overline{\phi'(\phi^{-1}(\mathrm{Supp}(\mathscr{A})))}=\theta(\mathrm{Supp}(\mathscr{A})).$$

This proves the theorem. □

The above proof also applies to type II reductive dual pairs.

4. Associated cycles

4.1. Throughout this section, we suppose (G, G') is in the stable range where G is the smaller member. Let ρ be an irreducible genuine $(\mathfrak{g}, \widetilde{K})$-module. The objective of this section is to prove Theorems C and D.

Proposition 4.1. *Suppose (G, G') is in the stable range where G is the smaller member. Let $\mathcal{O}\in\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ and $\mathcal{O}'=\theta(\mathcal{O})$. We fix a $w\in W$ such that $x=\phi(w)\in\mathcal{O}$ and $x'=\phi'(w)\in\mathcal{O}'$. Let $K_x=\mathrm{Stab}_{K_{\mathbb{C}}}(x)$ and $K'_{x'}=\mathrm{Stab}_{K'_{\mathbb{C}}}(x')$.*

- *(i) For every $k'\in K'_{x'}$, there exists a unique $k\in K_x$ such that $(k')^{-1}\cdot w=k\cdot w$. We denote k by $\alpha(k')$.*
- *(ii) The function $\alpha: K'_{x'}\twoheadrightarrow K_x$ defined by $k'\mapsto\alpha(k')$ in (i) is a surjective group homomorphism. In particular,*

$$\mathrm{Stab}_{K_{\mathbb{C}}\times K'_{\mathbb{C}}}(w)=K_x\times_\alpha K'_{x'}:=\{\,(\alpha(k'),k')\mid k'\in K'_{x'}\,\}. \tag{17}$$

We will prove this proposition in Section A.4 after we study some properties of the moment maps. The group homomorphism α depends on the choice of w. Indeed if we replace w by $(k_0, k_0')\cdot w$, then the corresponding group homomorphism becomes $\tilde{\alpha}\colon K'_{k_0'\cdot x'} \to K_{k_0\cdot x}$ which is given by $k' \mapsto k_0\alpha({k_0'}^{-1}k'k_0')k_0^{-1}$.

Pre-composition with α defines a map α^* from the set of K_x-modules (resp. virtual characters of K_x) to the set of $K'_{x'}$-modules (resp. virtual characters of $K'_{x'}$).

4.2. The next result is a key lemma which could be viewed as an enhancement of Section 1.3 in [30]. One may skip its proof in the first reading.

Lemma 4.2. *Let A be a $(\mathbb{C}[\overline{\mathcal{O}}], K_{\mathbb{C}})$-module. Define an $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module by*

$$B = (\mathbb{C}[W] \otimes_{\mathcal{S}(\mathfrak{p})} A)^{K_{\mathbb{C}}}.$$

Then B is a $(\mathbb{C}[\overline{\mathcal{O}'}], K'_{\mathbb{C}})$-module.

Let $\mathscr{A}$ and $\mathscr{B}$ be the quasi-coherent sheaves on $\overline{\mathcal{O}}$ and $\overline{\mathcal{O}'}$ associated to A and B respectively[1]*. Then we have the following isomorphism of $K'_{x'}$-modules:*

$$i_{x'}^*\mathscr{B} \simeq \alpha^*(i_x^*\mathscr{A}).$$

In particular, $\dim i_{x'}^*\mathscr{B} = \dim i_x^*\mathscr{A}$ *if A is finitely generated.*

Proof. See [25]. Let $Z = \phi^{-1}(\overline{\mathcal{O}})$ be the set theoretical inverse image of $\overline{\mathcal{O}}$. We consider following diagram

$$\begin{tikzcd}
\{x\} \arrow[dd, hook, "i_x"'] & \{w\} \arrow[l, "\simeq"'] \arrow[d, hook, "i_w"] \arrow[dr, "\simeq"] & \\
 & Z_{x'} \arrow[r] \arrow[d, hook, "i_{Z_{x'}}"] & \{x'\} \arrow[d, hook, "i_{x'}"] \\
\overline{\mathcal{O}} \arrow[d, hook, "i_{\overline{\mathcal{O}}}"] & Z \arrow[l, two heads, "\phi|_Z"'] \arrow[r, "\phi'|_Z"] \arrow[d, hook, "i_Z"] & \overline{\mathcal{O}'} \arrow[d, hook, "i_{\overline{\mathcal{O}'}}"] \\
\mathfrak{p}^* & W \arrow[l, "\phi"] \arrow[r, "\phi'"'] & \mathfrak{p}'^*
\end{tikzcd}$$

By Lemma A.6, the scheme theoretical inverse image $W \times_{\mathfrak{p}^*} \overline{\mathcal{O}}$ is reduced, i.e. $\mathbb{C}[Z] = \mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}]$. Then

$$B = (\mathbb{C}[W] \otimes_{\mathcal{S}(\mathfrak{p})} A)^{K_{\mathbb{C}}} = (\mathbb{C}[W] \otimes_{\mathcal{S}(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_{\mathbb{C}}} = (\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_{\mathbb{C}}}$$

as an $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module. By Lemma A.7, $\mathbb{C}[\overline{\mathcal{O}'}] = \mathbb{C}[Z]^{K_{\mathbb{C}}}$ so B is a $\mathbb{C}[\overline{\mathcal{O}'}]$-module.

We recall that x' is a point in $\mathcal{O}'$. Let $Z_{x'} = Z \times_{\overline{\mathcal{O}'}} \{x'\}$ be the scheme theoretical fiber. Since $\phi'|_Z\colon Z \to \overline{\mathcal{O}'}$ is dominant and we are in characteristic zero, $Z_{x'}$ is reduced. Let $m(x')$ be the maximal ideal in $\mathcal{S}(\mathfrak{p}')$ corresponding to the point x'.

Since taking $K_{\mathbb{C}}$-invariant is an exact functor and $\phi'^*(\mathcal{S}(\mathfrak{p}'))$ is $K_{\mathbb{C}}$-invariant, we have

$$\begin{aligned}
i_{x'}^*\mathscr{B} &= (\mathcal{S}(\mathfrak{p}')/m(x')) \otimes_{\mathcal{S}(\mathfrak{p}')} (\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_{\mathbb{C}}} \\
&= (\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_{\mathbb{C}}}/(m(x')\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_{\mathbb{C}}} \\
&= \Big((\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)/(m(x')\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)\Big)^{K_{\mathbb{C}}} \\
&= \Big((\mathbb{C}[Z]/m(x')\mathbb{C}[Z]) \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A\Big)^{K_{\mathbb{C}}}
\end{aligned}$$

[1]We will abuse notation and continue to denote their extensions by zero to $\mathfrak{p}^*$ and $\mathfrak{p}'^*$ by $\mathscr{A}$ and $\mathscr{B}$ respectively.

$$= (\mathbb{C}[Z_{x'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_{\mathbb{C}}}. \tag{18}$$

Let $\mathscr{Z} := i^*_{Z_{x'}}(\phi|_Z)^*\mathscr{A}$. Then $\mathscr{Z}(Z_{x'}) = \mathbb{C}[Z_{x'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A$. By Lemma A.2 (ii), $Z_{x'}$ is a $K_{\mathbb{C}} \times K_{x'}$-orbit generated by w. Let $S_w = \mathrm{Stab}_{K_{\mathbb{C}} \times K'_{x'}}(w)$. By (17), $S_w = K_x \times_\alpha K'_{x'} = \{(\alpha(k'), k') \in K_x \times K'_{x'}\}$. Then by Theorem 2.7 in [4],

$$\mathscr{Z}(Z_{x'}) = \mathrm{Ind}_{K_x \times_\alpha K'_{x'}}^{K_{\mathbb{C}} \times K'_{x'}} \chi$$

where χ is the fiber of $\mathscr{Z}$ at w. By the above commutative diagram, we have S_w-module isomorphisms

$$\chi = i^*_w \mathscr{Z} = i^*_w i^*_{Z_{x'}}(\phi|_Z)^*\mathscr{A} \simeq i^*_x \mathscr{A}$$

where $(\alpha(k'), k') \in S_w = K_x \times_\alpha K'_{x'}$ acts on $i^*_x\mathscr{A}$ via the natural action of $\alpha(k')$ on $i^*_x\mathscr{A}$.[2]

Putting the above into (18) gives

$$i^*_{x'}\mathscr{B} = (\mathscr{Z}(Z_{x'}))^{K_{\mathbb{C}}} = (\mathrm{Ind}_{K_x \times_\alpha K'_{x'}}^{K_{\mathbb{C}} \times K'_{x'}} \chi)^{K_{\mathbb{C}}} \simeq \chi \circ \alpha \tag{19}$$

as $K'_{x'}$-modules. Indeed if $f \in (\mathrm{Ind}_{K_x \times_\alpha K'_{x'}}^{K_{\mathbb{C}} \times K'_{x'}} \chi)^{K_{\mathbb{C}}}$, then $f : K_{\mathbb{C}} \times K'_{x'} \to V_\chi$ satisfies $f(k, k') = \chi(\alpha(k'))f(\alpha(k')^{-1}k, 1) = \chi(\alpha(k'))f(1, 1)$. Hence f is uniquely determined by $f(1, 1) \in V_\chi$. This proves the isomorphism on the right in (19). It also completes the proof of the lemma. $\square$

4.3. Let $\rho' = \Theta(\rho)$, $\mathbf{A} = \varsigma|_{\widetilde{K}} \otimes \mathrm{Gr}\, V_{\rho^*}$ and $\mathbf{B} = \varsigma|^{-1}_{\widetilde{K}'} \otimes \mathrm{Gr}\,\Theta(V_\rho)$ as before. For a subset Z of $\mathfrak{p}^*$, we let $I(Z)$ denote the ideal of $\mathcal{S}(\mathfrak{p})$ vanishing on Z.

Proposition 4.3. *There is a finite filtration $0 = \mathbf{A}_0 \subset \cdots \subset \mathbf{A}_l \subset \mathbf{A}_{l+1} \subset \cdots \subset \mathbf{A}_n = \mathbf{A}$ of $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-modules with the following property: For each l, there is a $K_{\mathbb{C}}$-orbit $\mathcal{O}_l$ such that the annihilator ideal of $\mathbf{A}_l/\mathbf{A}_{l-1}$ in $\mathcal{S}(\mathfrak{p})$ is the ideal $I(\overline{\mathcal{O}_l})$.*

In particular $\mathbf{A}_l/\mathbf{A}_{l-1}$ is a $\mathbb{C}[\overline{\mathcal{O}_l}]$-module and $\bigcup_{l=1}^n \overline{\mathcal{O}_l} = \mathrm{AV}(\rho^*)$.

Remark. We warn that the orbit $\mathcal{O}_l$ may not be connected since $K_{\mathbb{C}}$ may not be connected. Furthermore $\mathcal{O}_l$ may not be an open orbit in $\mathrm{AV}(\rho^*)$.

Proof. The proof essentially follows that of Lemma 2.11 in [36].

Let K_0 be the connected component of $K_{\mathbb{C}}$. The set of associated primes of $\mathbf{A}$ is finite. The connected group K_0 acts trivially on this finite set of associated primes.

Let $a \in \mathbf{A}$ such that its annihilator ideal $\mathscr{D} = \mathrm{Ann}_{\mathcal{S}(\mathfrak{p})}(a)$ is a minimal associated prime of $\mathbf{A}$. Let $\mathbf{A}_1 = \mathcal{S}(\mathfrak{p})K_{\mathbb{C}}a$ be the $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-submodule in $\mathbf{A}$ generated by a. Let $\mathrm{V}(\mathscr{D})$ be the subset of $\mathfrak{p}^*$ cut out by $\mathscr{D}$. Since $\mathrm{V}(\mathscr{D})$ is irreducible and K_0-invariant, it is the closure of single K_0-orbit $\mathcal{O}_0$. Let $\mathcal{O}_1 = K_{\mathbb{C}}\mathcal{O}_0$.

We claim that $\mathrm{Ann}_{\mathcal{S}(\mathfrak{p})}(\mathbf{A}_1) = I(\overline{\mathcal{O}_1})$. Indeed,

$$\begin{aligned} \mathrm{Ann}_{\mathcal{S}(\mathfrak{p})}(\mathbf{A}_1) &= \bigcap_{k \in K_{\mathbb{C}}} \mathrm{Ann}_{\mathcal{S}(\mathfrak{p})}(k \cdot a) = \bigcap_{k \in K_{\mathbb{C}}} k \cdot \mathscr{D} = \bigcap_{[k] \in K_{\mathbb{C}}/K_0} [k] \cdot \mathscr{D} \\ &= I(\bigcup_{[k] \in K_{\mathbb{C}}/K_0} [k] \cdot \mathrm{V}(\mathscr{D})) = I(\overline{\mathcal{O}_1}). \end{aligned}$$

[2]Let $m(w)$ be the maximal ideal of $\mathbb{C}[Z]$ corresponding to w and let $m(x)$ be the maximal ideal of $\mathbb{C}[\overline{\mathcal{O}}]$ corresponding to x. Then the map $\phi : w \mapsto x$ gives a $\mathbb{C}[\overline{\mathcal{O}}]$-algebra isomorphism: $L : \mathbb{C}[\overline{\mathcal{O}}]/m(x) \xrightarrow{\simeq} \mathbb{C}[Z]/m(w) = \mathbb{C}$. The group S_w acts on the right hand side while the group K_x acts on the left hand side. These two actions are compatible in the sense that for $(\alpha(k'), k') \in S_w \subset K_x \times K'_{x'}$, we have $L \circ \alpha(k') = (\alpha(k'), k') \circ L$.

Similarly $K'_{\mathbb{C}}$ acts on $\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A$ via translation on $\mathbb{C}[Z]$ while $K_{\mathbb{C}}$ acts via the tensor product of its action on A and the translation action on $\mathbb{C}[Z]$. Then $\chi = (\mathbb{C}[Z]/m(w)) \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A \simeq (\mathbb{C}[\overline{\mathcal{O}}]/m(x)) \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A = i^*_x\mathscr{A}$. Let $(\alpha(k'), k') \in S_w$. Then it acts on the right hand side via its natural action of $\alpha(k')$ on $i^*_x\mathscr{A}$.

The above second last equality holds because $\bigcap_{[k]\in K_{\mathbb{C}}/K_0}[k]\cdot\mathscr{D}$ is a finite intersection of prime ideals. The last equality holds by the definition of $\mathcal{O}_1$. This proves our claim.

Now, we could construct $\mathbf{A}_l$ and $\mathcal{O}_l$ inductively by applying the above construction to the $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module $\mathbf{A}/\mathbf{A}_{l-1}$. This procedure will eventually stop because $\mathbf{A}$ is a finitely generated module over the Noetherian ring $\mathcal{S}(\mathfrak{p})$. □

Let $\mathbf{A}_l$ be as in Proposition 4.3 and let $\mathbf{A}^l = \mathbf{A}_l/\mathbf{A}_{l-1}$. It is a finitely generated $\mathbb{C}[\overline{\mathcal{O}_l}]$-module and we let $\mathscr{A}^l$ be its associated coherent sheaf on $\overline{\mathcal{O}_l}$.

By Lemma 3.2, $\mathbf{B} = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}\mathbf{A})^{K_{\mathbb{C}}}$. Let $\mathbf{B}_l = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}\mathbf{A}_l)^{K_{\mathbb{C}}}$. Since ϕ is flat by Theorem A.4 and taking $K_{\mathbb{C}}$-invariants is exact, we may identify $\mathbf{B}_l$ with a submodule of $\mathbf{B}$. Hence $\mathbf{B}_l$ is an $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-equivariant filtration of $\mathbf{B}$. We set $\mathbf{B}^l = \mathbf{B}_l/\mathbf{B}_{l-1}$. Then

$$\mathbf{B}^l = \mathbf{B}_l/\mathbf{B}_{l-1} = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}(\mathbf{A}_l/\mathbf{A}_{l-1}))^{K_{\mathbb{C}}} = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}\mathbf{A}^l)^{K_{\mathbb{C}}}.$$

4.4. We define a partial ordering on the $K_{\mathbb{C}}$-orbits by containments in the Zariski closures. Let $\{\mathcal{O}_{l_1},\dots,\mathcal{O}_{l_r}\}$ be the set of (distinct) maximal nilpotent $K_{\mathbb{C}}$-orbits appearing in Proposition 4.3. For each $\mathcal{O}_{l_j}$ in this set, we fix a closed point $x_j\in\mathcal{O}_{l_j}$ and define the K_{x_j}-module

$$\chi(x_j,\operatorname{Gr}\mathscr{A}) = \bigoplus_{\mathcal{O}_{l_j}=\mathcal{O}_l} i^*_{x_j}\mathscr{A}^l. \tag{20}$$

Let $m_j = \dim_{\mathbb{C}}\chi(x_j,\operatorname{Gr}\mathscr{A})$. The integer m_j is independent of the choice of $x_j\in\mathcal{O}_{l_j}$. Moreover $m_j\neq 0$. Indeed all K_{x_j}-modules on the right hand side of (20) are non-zero because $\operatorname{Supp}(\mathscr{A}^l) = \mathrm{V}(\operatorname{Ann}_{\mathcal{S}(\mathfrak{p})}\mathbf{A}^l) = \overline{\mathcal{O}_l} = \overline{\mathcal{O}_{l_j}}$.

Recall that $\mathbf{A} = \varsigma|_{\widetilde{K}}\otimes\operatorname{Gr}V_{\rho^*}$. Let

$$\chi_{x_j} = \varsigma|_{\widetilde{K}}^{-1}\otimes\chi(x_j,\operatorname{Gr}\mathscr{A}).$$

Then $\{\,(\mathcal{O}_{l_j},x_j,\chi_{x_j})\,\}$ is the set of orbit data attached to the filtrations given by Proposition 4.3.

Now the associated cycle of ρ^* is

$$\operatorname{AC}(\rho^*) = \operatorname{AC}(\mathbf{A}) = \sum_{j=1}^r m_j[\overline{\mathcal{O}_{l_j}}].$$

and the associated variety is $\operatorname{AV}(\rho^*) = \bigcup_{j=1}^r\overline{\mathcal{O}_{l_j}}$.

4.5. *Proof of Theorems C and D.* First we observe following lemma.

Lemma 4.4. *Let $\{\,\mathcal{O}_{l_j} : j=1,\dots,r\,\}$ be the set of all distinct (open) maximal $K_{\mathbb{C}}$-orbits in* $\operatorname{AV}(\rho^*)$. *Then $\{\,\theta(\mathcal{O}_{l_j}) : j=1,\dots,r\,\}$ forms the set of all distinct (open) maximal $K'_{\mathbb{C}}$-orbits in $\theta(\operatorname{AV}(\rho^*))$.*

Proof. By Theorem A.1(i) the map $\theta\colon\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)\to\mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ is injective so all the $\theta(\mathcal{O}_{l_j})$'s are distinct. We also have $\theta(\operatorname{AV}(\rho^*)) = \phi'(\phi^{-1}(\bigcup_{j=1}^r\overline{\mathcal{O}_{l_j}})) = \bigcup_{j=1}^r\theta(\overline{\mathcal{O}_{l_j}}) = \bigcup_{j=1}^r\overline{\theta(\mathcal{O}_{l_j})}$. It suffices to show that $\dim\theta(\mathcal{O}_{l_j}) = \dim\theta(\operatorname{AV}(\rho^*))$.

By Theorem 8.4 in [36], every $K_{\mathbb{C}}$-orbit $\mathcal{O}_{l_j}$ generates the same $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\mathbb{C}}$ in $\mathfrak{g}^*$. Indeed $\overline{\mathcal{O}_{\mathbb{C}}}$ is the variety cut out by $\operatorname{Gr}(\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}(\rho^*))$.

Nilpotent $K_{\mathbb{C}}$-orbits for classical groups are parametrized by signed Young diagrams. In particular the underlying Young diagrams of different $\mathcal{O}_{l_j}$'s are the same and they have the same dimension equals to $\frac{1}{2}\dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}}$. By [31], the signed Young diagram of the orbit $\theta(\mathcal{O}_l)$ is obtained by adding a column to the signed Young diagram of $\mathcal{O}_{l_j}$. Hence every $K'_{\mathbb{C}}$-orbit $\theta(\mathcal{O}_{l_j})$ generates the same $G'_{\mathbb{C}}$-orbit $\mathcal{O}'_{\mathbb{C}}$ in $\mathfrak{g}'^*$ and its dimension is

$\frac{1}{2}\dim_{\mathbb{C}}\mathcal{O}'_{\mathbb{C}}$. This proves that $\dim\theta(\mathcal{O}_{l_j}) = \dim\theta(\mathrm{AV}(\rho^*))$ and completes the proof of the lemma. In fact $\overline{\mathcal{O}'_{\mathbb{C}}} = \theta_{\mathbb{C}}(\overline{\mathcal{O}_{\mathbb{C}}})$ where $\theta_{\mathbb{C}}$ was defined after (2) (see [5], [6] or [20]). ☐

Let $\mathcal{O}'_l = \theta(\mathcal{O}_l)$. By Theorem B, $\mathrm{AV}(\rho') \subseteq \bigcup_{j=1}^r \theta(\overline{\mathcal{O}_{l_j}}) = \bigcup_{j=1}^r \overline{\mathcal{O}'_{l_j}}$.

Let $\mathscr{B}_l$ and $\mathscr{B}^l$ be the associated coherent sheaves of $\mathbf{B}_l$ and $\mathbf{B}^l$ respectively. Now we apply Lemma 4.2 to $\mathbf{B}^l$ and we have

$$\chi(x'_j, \mathrm{Gr}\,\mathscr{B}) := \bigoplus_{\mathcal{O}'_l=\mathcal{O}'_{l_j}} i^*_{x'_j}\mathscr{B}^l \simeq \bigoplus_{\mathcal{O}_l=\mathcal{O}_{l_j}} \alpha^*_j(i^*_{x_j}\mathscr{A}^l)$$

where x'_j and α_j are x' and α respectively in Lemma 4.2.

Since $\mathbf{A} = \varsigma|_{\widetilde{K}} \otimes \mathrm{Gr}\,V_{\rho^*}$ and $\mathbf{B} = \varsigma|_{\widetilde{K}'}^{-1} \otimes \mathrm{Gr}\,\Theta(V_\rho)$, the isotropy representation of $\Theta(\rho)$ at x'_j with respect to the filtration $\mathscr{B}_l$ is

$$\chi_{x'_j} = \varsigma|_{\widetilde{K}'} \otimes \chi(x'_j, \mathrm{Gr}\,\mathscr{B}) = \varsigma|_{\widetilde{K}'} \otimes \chi(x_j, \mathrm{Gr}\,\mathscr{A}) \circ \alpha_j = \varsigma|_{\widetilde{K}'} \otimes (\varsigma|_{\widetilde{K}} \otimes \chi_{x_j}) \circ \alpha_j.$$

In particular, $\chi_{x'_j} \neq 0$ since $\chi_{x_j} \neq 0$. Therefore $\{(\mathcal{O}'_{l_j}, x'_j, \chi_{x'_j}) : j = 1, \ldots, r\}$ forms the set of orbit data attached to the filtration $\mathscr{B}_l$. This proves Theorem C.

Now

$$\mathrm{AC}(\Theta(\rho)) = \sum_{j=1}^r (\dim\chi_{x'_j})[\overline{\mathcal{O}'_{l_j}}] = \sum_{j=1}^r m_j[\overline{\theta(\mathcal{O}_{l_j})}] = \theta(\mathrm{AC}(\rho^*)).$$

This proves Theorem D. ☐

The proof also shows that the theta lift of a Harish-Chandra module in stable range is nonzero since $\chi_{x'_j} \neq 0$.

4.6. *Proof of Corollary E.* We recall $\rho' = \Theta(\rho)$. From the proof of Lemma 4.4,

$$\overline{G'_{\mathbb{C}}\mathrm{AV}(\rho')} = \overline{\mathcal{O}'_{\mathbb{C}}} = \theta_{\mathbb{C}}(\overline{\mathcal{O}_{\mathbb{C}}}) = \theta_{\mathbb{C}}(\overline{G_{\mathbb{C}}\mathrm{AV}(\rho^*)}) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho^*)) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho)).$$

The last equality follows from Proposition B.1. Although ρ' may not be irreducible, we claim that $\mathrm{V}_{\mathbb{C}}(\rho') = \overline{G'_{\mathbb{C}}\mathrm{AV}(\rho')}$ and this would prove the corollary. First $\mathrm{V}_{\mathbb{C}}$ is an additive map, i.e. $\mathrm{V}_{\mathbb{C}}(B) = \mathrm{V}_{\mathbb{C}}(A) \cup \mathrm{V}_{\mathbb{C}}(C)$ for every exact sequence $0 \to A \to B \to C \to 0$. This is well known to the experts (for example see Lemma 1.5 in [2]), which follows by taking the graded version of

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(A)\mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(C) \subseteq \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(B) \subseteq \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(A) \cap \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(C).$$

Next let $\rho'_1, \ldots, \rho'_s$ be all the irreducible subquotients of the $(\mathfrak{g}', \widetilde{K}')$-module ρ' of finite length. Using Theorem 8.4 in [36] again,

$$\mathrm{V}_{\mathbb{C}}(\mathrm{Ann}\,\rho') = \bigcup_{k=1}^s \mathrm{V}_{\mathbb{C}}(\mathrm{Ann}\,\rho'_k) = \bigcup_{k=1}^s \overline{G'_{\mathbb{C}}\mathrm{AV}(\rho'_k)} = \overline{G'_{\mathbb{C}}\bigcup_{k=1}^s \mathrm{AV}(\rho'_k)} = \overline{G'_{\mathbb{C}}\mathrm{AV}(\rho')}.$$

This proves our claim and Corollary E. ☐

5. The K-spectrum equation

In this section, we suppose (G, G') is in the stable range with G the smaller member excluding (††). We will also retain the notation in the previous section. The objective of this section is to prove Proposition 5.1 which implies Theorem F.

5.1. Let $x \in \mathcal{O}$ and let χ_x be a finite dimensional rational representation of K_x as in Section 1.6. We recall Theorem 2.7 in [4] that there is an equivalence of categories between the category of rational representations of K_x and the category of certain $K_{\mathbb{C}}$-equivariant sheaves on $\mathcal{O} \simeq K/K_x$. Let $\mathscr{L}$ be the $K_{\mathbb{C}}$-equivariant sheaf on $\mathcal{O}$ corresponding to χ_x. We assume that $\mathscr{L}$ is generated by its global sections (c.f. (3)). Let $i_{\mathcal{O}} : \mathcal{O} \to \mathfrak{p}^*$ denote the inclusion map and let $\mathscr{A} = (i_{\mathcal{O}})_* \mathscr{L}$. We also set

$$A := \mathscr{A}(\mathfrak{p}^*) = \mathscr{L}(\mathcal{O}) = \operatorname{Ind}_{K_x}^{K_{\mathbb{C}}} \chi_x \tag{21}$$

as an $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module. Clearly A is a $(\mathbb{C}[\overline{\mathcal{O}}], K_{\mathbb{C}})$-module.

Let $\mathcal{O}' = \theta(\mathcal{O})$. We fix a $w \in W$ such that $x = \phi(w) \in \mathcal{O}$ and $x' = \phi'(w) \in \mathcal{O}'$ in (1). Let $\alpha\colon K'_{x'} \to K_x$ be the map defined in Proposition 4.1 in Appendix A. Let $\chi_{x'} = \chi_x \circ \alpha$ be the representation of $K'_{x'}$. Let $\mathscr{L}'$ be the $K'_{\mathbb{C}}$-equivariant sheaf on $\mathcal{O}'$ corresponding to $\chi'_{x'}$. We define the $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module

$$B = (\mathbb{C}[W] \otimes_{\mathcal{S}(\mathfrak{p})} A)^{K_{\mathbb{C}}}.$$

By Lemma 4.2, B is a $(\mathbb{C}[\overline{\mathcal{O}'}], K'_{\mathbb{C}})$-module.

Proposition 5.1. *Suppose (G, G') is in the stable range with G the smaller member excluding (††). Let $i_{\mathcal{O}'} : \mathcal{O}' \to \overline{\mathcal{O}'}$ denote the inclusion map. Then the sheaf $(i_{\mathcal{O}'})_* \mathscr{L}'$ is the coherent sheaf associated to the $(\mathbb{C}[\overline{\mathcal{O}'}], K'_{\mathbb{C}})$-module B.*

Proof. Let $Y := W \times_{\mathfrak{p}} \mathcal{O}$. By Lemma A.6, Y is a reduced scheme. We consider following diagram where $Z^\circ = (\phi')^{-1}(\mathcal{O}') \cap Y$.

$$\begin{tikzcd}
 & Z^\circ \arrow[r, "\phi'"] \arrow[d, hook] & \mathcal{O}' \arrow[d, hook, "i_{\mathcal{O}'}"] \\
\mathcal{O} \arrow[d, hook, "i_{\mathcal{O}}"] & Y \arrow[l, "\phi|_Y"'] \arrow[r, "\phi'|_Y"] \arrow[d, hook, "i_Y"] & \overline{\mathcal{O}'} \arrow[d, hook] \\
\mathfrak{p}^* & W \arrow[l, "\phi"'] \arrow[r, "\phi'"] & \mathfrak{p}'^*
\end{tikzcd}$$

Since ϕ is flat, Proposition 9.3 in Chapter 3 in [12] gives

$$\phi^*(i_{\mathcal{O}})_* \mathscr{L} = (i_Y)_*(\phi|_Y)^* \mathscr{L}$$

as sheaves on W. Let $\mathscr{Q} = (\phi|_Y)^* \mathscr{L}$ and let $\mathscr{B}$ be the quasi-coherent sheaf on $\overline{\mathcal{O}'}$ associated to B. Note that $((\phi'|_Y)_* \mathscr{Q}(\overline{\mathcal{O}'}))^{K_{\mathbb{C}}} = (\phi'_* i_{Y*} \mathscr{Q}(\mathfrak{p}'^*))^{K_{\mathbb{C}}} = B = \mathscr{B}(\overline{\mathcal{O}'})$. By the exactness of taking $K_{\mathbb{C}}$-invariants and the fact that $\mathbb{C}[\overline{\mathcal{O}'}]$ is $K_{\mathbb{C}}$-invariant, we have $\mathscr{B} = ((\phi'|_Y)_* \mathscr{Q})^{K_{\mathbb{C}}}$, i.e. $\mathscr{B}(U) = (\mathscr{Q}((\phi'|_Y)^{-1}(U)))^{K_{\mathbb{C}}}$ for every open set $U \subset \overline{\mathcal{O}'}$.

Lemma 5.2. *We have $\mathscr{Q}(Y) = \mathscr{Q}(Z^\circ)$.*

Proof. The proof is similar to Theorem 4.4 in [4].

Since $\mathscr{L}$ is locally free on $\mathcal{O}$, $\mathscr{Q}$ is locally free on Y. Hence

$$\operatorname{depth} \mathscr{Q}_y = \operatorname{depth} \mathscr{O}_{Y,y}$$

for any $y \in Y$. Let $\partial Z^\circ = Y - Z^\circ$. Let $H^i_{\partial Z^\circ}(\mathscr{Q})$ (resp. $\mathscr{H}^i_{\partial Z^\circ}(\mathscr{Q})$) be the cohomology group (resp. cohomology sheaf) of Y with coefficient in $\mathscr{Q}$ and support in ∂Z°. By Lemma A.8 in Appendix A, $\operatorname{codim}(Y, \partial Z^\circ) \geq 2$. By Lemma A.6(ii), Y is a normal scheme so it satisfies Serre's (S2) condition. Therefore

$$\operatorname{depth}_y \mathscr{Q}_y = \operatorname{depth} \mathscr{O}_{Y,y} \geq \min \{ \dim \mathscr{O}_{Y,y}, 2 \} = 2$$

for all $y \in \partial Z^\circ$. By a vanishing theorem of Grothendieck (see [11, Theorem 3.8]),

$$\mathscr{H}^i_{\partial Z^\circ}(\mathscr{Q}) = 0 \quad \text{for} \quad i = 0, 1.$$

By Proposition 1.11 in [11], a spectral sequence argument implies $H^0(Y,\mathscr{Q})\simeq H^0(Z^\circ,\mathscr{Q})$ as required. $\square$

We continue with the proof of Proposition 5.1. By the above lemma

$$B=\mathscr{B}(\overline{\mathcal{O}'})=(\mathscr{Q}(Y))^{K_{\mathbb{C}}}=(\mathscr{Q}(Z^\circ))^{K_{\mathbb{C}}}=\mathscr{B}(\mathcal{O}').$$

By (19) we have

$$\mathscr{B}(\mathcal{O}')=\operatorname{Ind}_{K'_{x'}}^{K'_{\mathbb{C}}}(\chi_x\circ\alpha)=(i_{\mathcal{O}'})_*\mathscr{L}'(\overline{\mathcal{O}'}).$$

Hence $\mathscr{B}(\overline{\mathcal{O}'})=(i_{\mathcal{O}'})_*\mathscr{L}'(\overline{\mathcal{O}'})$. Since both $\mathscr{B}$ and $(i_{\mathcal{O}'})_*\mathscr{L}'$ are quasi-coherent sheaves over the affine scheme $\overline{\mathcal{O}'}$, $\mathscr{B}=(i_{\mathcal{O}'})_*\mathscr{L}'$ and this completes the proof of Proposition 5.1. $\square$

5.2. *Proof of Theorem F.* By (21), $A|_K=\varsigma|_{\widetilde{K}}\otimes V_{\rho^*}|_{\widetilde{K}}$ as K-modules. Therefore

$$\begin{aligned}
\varsigma|_{\widetilde{K}'}^{-1}\otimes\rho'|_{\widetilde{K}'} &= \varsigma|_{\widetilde{K}'}^{-1}\otimes\operatorname{Gr}(\rho')|_{\widetilde{K}'}=(\varsigma|_{\widetilde{K}}\otimes V_{\rho^*}|_{\widetilde{K}}\otimes\mathcal{H})^K && \text{(by Proposition 2.1)}\\
&= (A|_K\otimes\mathcal{H})^K=B|_{K'} && \text{(by Proposition 2.1 again)}\\
&= (i_{\mathcal{O}'})_*\mathscr{L}'(\overline{\mathcal{O}'}) && \text{(by Proposition 5.1)}\\
&= \mathscr{L}'(\mathcal{O}')=\operatorname{Ind}_{K'_{x'}}^{K'_{\mathbb{C}}}\left((\varsigma|_{\widetilde{K}}\otimes\chi_x)\circ\alpha\right).
\end{aligned}$$

Twisting the above equation by $\varsigma|_{\widetilde{K}'}$ proves Theorem F. $\square$

6. Admissible data

In this section we will show that the theta lift of an admissible data is still an admissible data. We continue to assume that (G,G') is an irreducible type I dual pair in the stable range where G is the smaller member.

Let $\mathcal{O}$ be a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^*$ as in (1). Let $\mathcal{O}'=\theta(\mathcal{O})$. Let $w\in W$ such that $x=\phi(w)\in\mathcal{O}$ and $x'=\phi'(w)\in\mathcal{O}'$. Let $\alpha\colon K'_{x'}\to K_x$ be the map defined by Proposition 4.1. Proposition G follows from the next proposition.

Proposition 6.1. *Suppose (G,G') is in the stable range where G is the smaller member. Let χ_x be an admissible representation of $\widetilde{K}_x$ as defined in Section 1.7. We set*

$$\chi_{x'}:=\varsigma|_{\widetilde{K}'_{x'}}\otimes(\varsigma|_{\widetilde{K}_x}\otimes\chi_x)\circ\alpha.$$

Then $\chi_{x'}$ is an admissible representation of $\widetilde{K}'_{x'}$.

Proof. We have to verify that

$$\chi_{x'}(\exp(X'))=\det(\operatorname{Ad}^*(\exp(X'/2))|_{(\mathfrak{k}'/\mathfrak{k}'_{x'})^*})\qquad\forall X'\in\mathfrak{k}'_{x'}.$$

Since χ_x is admissible, it reduces to the following lemma after taking square of above equation.

Lemma 6.2. *As $\mathfrak{k}'_{x'}$-modules,*

$$\textstyle\bigwedge^{\mathrm{top}}(\mathfrak{k}'/\mathfrak{k}'_{x'})\simeq(\bigwedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{k}_x)\circ\alpha)\otimes\varsigma|_{K'}^{-2}\otimes(\varsigma|_K^{-2}\circ\alpha).$$

Proof. Let $E=K_{\mathbb{C}}K'_{\mathbb{C}}w$, $F=\phi^{-1}(x)$, and $F'=\phi^{-1}(x')$. Let $S_w:=\operatorname{Stab}_{K_{\mathbb{C}}\times K'_{\mathbb{C}}}(w)=\{\,(\alpha(k'),k')\mid k'\in K'_{x'}\,\}\simeq K'_{x'}$. Let T_wF' denote the tangent space of F' at w etc. We have following two exact sequences of S_w-modules:

$$0\longrightarrow\mathrm{T}_wF'\longrightarrow\mathrm{T}_wE\longrightarrow\mathrm{T}_{x'}\mathcal{O}'\longrightarrow 0$$

and

$$0\longrightarrow\mathrm{T}_wF\longrightarrow\mathrm{T}_wE\longrightarrow\mathrm{T}_x\mathcal{O}\longrightarrow 0\ .$$

Here S_w acts on $\mathrm{T}_x\mathcal{O}$ (resp. $\mathrm{T}_{x'}\mathcal{O}'$) via the projection $S_w\to K_x$ (resp. $S_w\xrightarrow{\sim}K'_{x'}$). Since $S_w\simeq K'_{x'}$, the above are also exact sequences of $K'_{x'}$-modules.

By Proposition 4.1(i) $\mathrm{T}_w F' \simeq \mathfrak{k}$. The $\mathfrak{k}'_{x'}$-action on $\bigwedge^{\mathrm{top}} \mathfrak{k}$ is trivial since $\mathfrak{k}$ is reductive. Therefore

$$\bigwedge^{\mathrm{top}} \mathrm{T}_{x'}\mathcal{O}' \simeq \bigwedge^{\mathrm{top}} \mathrm{T}_w E \simeq \bigwedge^{\mathrm{top}} \mathrm{T}_x \mathcal{O} \otimes \bigwedge^{\mathrm{top}} \mathrm{T}_w F \tag{22}$$

as $\mathfrak{k}'_{x'}$-modules. Since we are in the stable range, $\phi : W \to \mathfrak{p}^*$ is a submersion at every point $w \in W$. We have following exact sequence of $K'_{x'}$-modules:

$$0 \longrightarrow \mathrm{T}_w F \longrightarrow \mathrm{T}_w W \longrightarrow \mathrm{T}_x \mathfrak{p}^* \longrightarrow 0.$$

Since $\mathrm{T}_x\mathfrak{p}^* \simeq \mathfrak{p}^*$ and $\mathfrak{k}'_{x'}$ acts trivially on $\bigwedge^{\mathrm{top}} \mathfrak{p}^*$, we have

$$\bigwedge^{\mathrm{top}} \mathrm{T}_w W \simeq \bigwedge^{\mathrm{top}} \mathrm{T}_w F. \tag{23}$$

Combining (22), (23), $\mathrm{T}_w W \simeq W$, $\mathrm{T}_x\mathcal{O} \simeq \mathfrak{k}/\mathfrak{k}_x$ and $\mathrm{T}_{x'}\mathcal{O}' \simeq \mathfrak{k}'/\mathfrak{k}'_{x'}$, we have

$$\bigwedge^{\mathrm{top}} (\mathfrak{k}'/\mathfrak{k}'_{x'}) = (\bigwedge^{\mathrm{top}} \mathfrak{k}/\mathfrak{k}_x \circ \alpha) \otimes \bigwedge^{\mathrm{top}} W. \tag{24}$$

We view $u \in \mathrm{U}_{\mathbb{C}}$ as a linear transformation on W. By our choice of oscillator representation,

$$\varsigma^{-2}(u) = \det(u|_W).$$

Hence the action of $k' \in K'_{x'}$ on $\bigwedge^{\mathrm{top}} W$ is

$$\det((k', \alpha(k'))|_W) = \varsigma|_{K'}^{-2}(k') \otimes (\varsigma|_K^{-2} \circ \alpha(k')).$$

Putting this into (24) proves the lemma and Proposition 6.1. □

Appendix A. The Geometry of theta lifts of nilpotent orbits

A.1. **The Fock model.** We retain the notation in Section 1.1 where $(W_{\mathbb{R}}, \langle , \rangle)$ is a symplectic space and we have fixed a maximal compact subgroup $\mathrm{U} \subset \mathrm{Sp}(W_{\mathbb{R}})$. It is well known that there are two oscillator representations. We will specify our choice of oscillator representation in this paper by describing its Fock model.

First we fix a square root of -1, say i. The centralizer of U in $\mathrm{Sp}(W_{\mathbb{R}})$ is isomorphic to $\mathrm{U}(1)$. For an element J in the centralizer such that $J^2 = -1$, we define a complex structure on $W_{\mathbb{R}}$ such that $i \cdot v := Jv$ for all $v \in W_{\mathbb{R}}$. We denote the corresponding complex vector space by W. Then $\langle v_1, v_2 \rangle_H := \langle Jv_1, v_2 \rangle + i \langle v_1, v_2 \rangle$ defines a Hermitian form on W. There are two choices of J and we choose the one such that $\langle , \rangle_H$ is positive definite. Now U is the unitary group $\mathrm{U}(W, \langle , \rangle_H)$. Its complexification is $\mathrm{U}_{\mathbb{C}} = \mathrm{GL}(W)$ and the covering group of $\mathrm{U}_{\mathbb{C}}$ is $\widetilde{\mathrm{U}}_{\mathbb{C}} = \{ (g, z) \in \mathrm{GL}(W) \times \mathbb{C}^\times \mid \det(g) = z^2 \}$. We identify $\widetilde{\mathrm{U}}$ with the inverse image of U in $\widetilde{\mathrm{U}}_{\mathbb{C}}$. We fix the oscillator representation ω such that its Fock module or $(\mathfrak{g}, \widetilde{\mathrm{U}})$-module $\mathscr{Y}$ is isomorphic to $\mathbb{C}[W]$ such that $(\omega(\tilde{g})f)(v) = z^{-1} f(g^{-1}v)$ for all $\tilde{g} = (g, z) \in \widetilde{\mathrm{U}}_{\mathbb{C}}$ and $f \in \mathbb{C}[W]$. In particular the minimal $\widetilde{\mathrm{U}}$-type is one dimensional consisting of constant functions on W and $\widetilde{\mathrm{U}}$ acts on it via the character $\varsigma(\tilde{g}) = z^{-1}$ where $\tilde{g} = (g, z) \in \widetilde{\mathrm{U}}$.

A.2. **The moment maps.** Let (G, G') denote a type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$ as in Table 1. Let $\mathfrak{sp} = \mathrm{Lie}(\mathrm{Sp}(W_{\mathbb{R}}))_{\mathbb{C}}$ and $\mathfrak{sp}^{(1,1)} = \mathrm{Lie}(\mathrm{U})_{\mathbb{C}}$. Under the adjoint action of U, $\mathfrak{sp} = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(0,2)}$. Here $\mathfrak{sp}^{(2,0)}$ is an abelian Lie subalgebra acting on $\mathbb{C}[W]$ via multiplication by degree two polynomials. In particular, we have $\mathcal{S}(\mathfrak{sp}^{(2,0)}) \to \mathbb{C}[W]$ by $p \mapsto p \cdot 1$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition of G. The composition $\mathfrak{p} \hookrightarrow \mathfrak{sp} \twoheadrightarrow \mathfrak{sp}^{(2,0)}$ induces an algebra homomorphism $\mathcal{S}(\mathfrak{p}) \to \mathcal{S}(\mathfrak{sp}^{(2,0)})$. Composing the two maps gives $\phi^* : \mathcal{S}(\mathfrak{p}) \to \mathbb{C}[W]$ which defines $\phi \colon W \to \mathfrak{p}^* = \mathrm{Spec}\,(\mathcal{S}(\mathfrak{p}))$. Similarly we define $\phi' \colon W \to \mathfrak{p}'^* = \mathrm{Spec}\,(\mathcal{S}(\mathfrak{p}'))$. Hence we have

$$\mathfrak{p}^* \xleftarrow{\ \phi\ } W \xrightarrow{\ \phi'\ } \mathfrak{p}'^*.$$

The maps ϕ and ϕ' are called the *moment maps*.

We describe explicitly the moment maps in Table 2 below. Here J_{2p} is the skew symmetric $2p$ by $2p$ matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

G	G'	W $w \in W$	$\mathfrak{p}^*$ $\phi(w)$	$\mathfrak{p}'^*$ $\phi'(w)$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	$M_{p,n} \times M_{q,n}$ (A,B)	$\mathrm{Sym}_n \times \mathrm{Sym}_n$ (A^TA, B^TB)	$M_{p,q}$ AB^T
$\mathrm{U}(n_1,n_2)$	$\mathrm{U}(p,q)$	$M_{p,n_1} \times M_{p,n_2} \times M_{q,n_1} \times M_{q,n_2}$ (A,B,C,D)	$M_{n_1,n_2} \times M_{n_2,n_1}$ (A^TB, D^TC)	$M_{p,q} \times M_{q,p}$ (AC^T, DB^T)
$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p,q)$	$M_{2p,n} \times M_{2q,n}$ (A,B)	$\mathrm{Alt}_n \times \mathrm{Alt}_n$ $(A^TJ_{2p}A, B^TJ_{2q}B)$	$M_{2p,2q}$ AB^T
$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{O}(p,\mathbb{C})$	$M_{p,2n}$ A	Sym_{2n} A^TA	Alt_p $AJ_{2n}A^T$

TABLE 2. Moment maps for non-compact dual pairs.

The following fact is true for every reductive dual pairs, not necessarily in the stable range. The moment map factors through the affine quotient:

$$\begin{tikzcd} W \arrow[r, two heads] \arrow[rr, bend left, "\phi'"] & W/K_{\mathbb{C}} \arrow[r, hook, "i_{W/K_{\mathbb{C}}}"] & \mathfrak{p}'^*. \end{tikzcd} \tag{25}$$

By the First Fundamental Theorem of classical invariant theory, $\mathbb{C}[W]^{K_{\mathbb{C}}}$ is a quotient of $\mathcal{S}(\mathfrak{p}')$, i.e. $i_{W/K_{\mathbb{C}}}$ is a closed embedding. For every $K_{\mathbb{C}}$-invariant closed subset $E \subseteq W$, its image in $W/K_{\mathbb{C}}$ is closed by Corollary 4.6 in [32]. This implies that $\phi'(E)$ is closed in $\mathfrak{p}'^*$. Hence for every $K_{\mathbb{C}}$-invariant closed subset $S \subseteq \mathfrak{p}^*$, $\theta(S) := \phi'(\phi^{-1}(S))$ is a $K'_{\mathbb{C}}$-invariant closed subset of $\mathfrak{p}'^*$.

A.3. We recall the nilpotent cone $N(\mathfrak{p}^*) = \left\{ x \in \mathfrak{p}^* \;\middle|\; 0 \in \overline{K_{\mathbb{C}} \cdot x} \right\}$. Let $\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ be the set of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$. We define $N(\mathfrak{p}'^*)$ and $\mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ in the same way.

We summarize some results in [31], [6] and [29].

Theorem A.1. *Let (G, G') be a reductive dual pair in stable range where G is the smaller member as in Table 1.*

(i) For any nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}^*$, there is a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$ in $\mathfrak{p}'^*$ such that

$$\phi'(\phi^{-1}(\overline{\mathcal{O}})) = \overline{\mathcal{O}'}.$$

This defines an injective map $\theta \colon \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*) \to \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ given by $\mathcal{O} \mapsto \mathcal{O}'$. This map is called the theta lifting of nilpotent orbits.

(ii) Theta lifting of nilpotent orbits preserves closure relation, i.e. if $\mathcal{O}_0 \subset \overline{\mathcal{O}}$ then $\theta(\mathcal{O}_0) \subset \overline{\theta(\mathcal{O})}$. □

We refer to Table 2 where W is written as a product of matrix spaces. Let W° be the open dense subset of elements in W whose every component has full rank. Before we discuss the finer structures of orbits, we state following lemma.

Lemma A.2. *Let (G, G') be a reductive dual pair in the stable range as in Table 1.*

(i) We have $\phi'^{-1}(\phi'(W^\circ)) = W^\circ$.

(ii) For any $x' \in \phi'(W^\circ)$, $\phi'^{-1}(x') = \phi'^{-1}(x') \cap W^\circ$ is a single $K_{\mathbb{C}}$-orbit where $K_{\mathbb{C}}$ acts freely.

(iii) For any $x \in \phi(W^\circ)$, $\phi^{-1}(x) \cap W^\circ$ is a single $K'_{\mathbb{C}}$-orbit.

(iv) We have one-to-one correspondences of the following sets of orbits

$$\begin{array}{rcccl} \{\, K_{\mathbb{C}}\text{-orbits in } \phi(W^\circ)\,\} & \leftrightarrow & \{\, K_{\mathbb{C}} \times K'_{\mathbb{C}}\text{-orbits in } W^\circ \,\} & \leftrightarrow & \{\, K'_{\mathbb{C}}\text{-orbits in } \phi'(W^\circ)\,\} \\ \phi(C) & \leftarrow\!\shortmid & C & \mapsto & \phi'(C) \\ \mathcal{O} & \mapsto & \phi^{-1}(\mathcal{O}) \cap W^\circ & & \\ & & \phi'^{-1}(\mathcal{O}') = \phi'^{-1}(\mathcal{O}') \cap W^\circ & \leftarrow\!\shortmid & \mathcal{O}'. \end{array}$$

Proof. The proof for each dual pair is similar so we will give the proof for the first pair in Table 1 and leave the other cases to the reader.

Consider $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$, $W = M_{p,n} \times M_{q,n}$, $\mathfrak{p}'^* = M_{p,q}$ and $p, q \geq 2n$. For $(A, B) \in M_{p,n} \times M_{q,n} = W$, $\phi'(A, B) = AB^T$ has rank n if and only if A and B have rank n. This proves (i) and the equality in (ii).

Let $x' \in \phi'(W^\circ)$. Let $(A, B), (A', B') \in \phi'^{-1}(x') \cap W^\circ$. We have

$$AB^T = \phi'(A, B) = x' = \phi'(A', B') = A'(B')^T. \tag{26}$$

Here x', A, B, A', B' are all rank n matrices. Since the column space of A (resp. A') is same as the column space of x', we may assume that $A = A'$ by the action of $K_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. If we interpret $A : \mathbb{C}^n \to \mathbb{C}^p$ as an injective linear map, it is clear that (26) implies $B^T = B'^T$. This proves that (A, B) and (A', B') are in the same $K_{\mathbb{C}}$-orbit. Hence $\phi'^{-1}(x') \cap W^\circ = \phi'^{-1}(x')$ is a single $K_{\mathbb{C}}$-orbit.

Next suppose $k \in K_{\mathbb{C}}$ stabilizes (A, B). Hence $k \cdot A = Ak^{-1} = A$. Since A is an injective map, k is the identity element. This shows that the $K_{\mathbb{C}}$-action is faithful. This proves (ii).

Let $x \in \phi(W^\circ)$. Let $(A, B), (A', B') \in \phi^{-1}(x) \cap W^\circ$. We have

$$(A^T A, B^T B) = \phi(A, B) = x = \phi(A', B') = (A'^T A', B'^T B').$$

Since $\operatorname{Ker} A = \operatorname{Ker} A' = 0$ and $A^T A = A'^T A'$, there is an $o \in \mathrm{O}(p, \mathbb{C})$ such that $A = oA'$ by Witt's theorem (for example, see Theorem 3.7.1 in [16]). The same argument applies to B and B'. Hence $\phi^{-1}(x) \cap W^\circ$ is a single orbit of $K'_{\mathbb{C}} = \mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})$. This proves (iii).

Part (iv) follows from (i), (ii) and (iii). □

Theorem A.3. *Let (G, G') be a reductive dual pair in the stable range where G is the smaller member as in Table 1. Let $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$. Set $\mathcal{O}' = \phi'(\phi^{-1}(\mathcal{O}) \cap W^\circ)$. Then*

(i) $\phi'^{-1}(\mathcal{O}') = \phi'^{-1}(\mathcal{O}') \cap W^\circ = \phi^{-1}(\mathcal{O}) \cap W^\circ = \phi'^{-1}(\mathcal{O}') \cap \phi^{-1}(\overline{\mathcal{O}})$ is a $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit.
(ii) $\mathcal{O}'$ is a $K'_{\mathbb{C}}$-orbit.
(iii) $\phi(\phi'^{-1}(\mathcal{O}')) = \mathcal{O}$.
(iv) $\mathcal{O}' = \theta(\mathcal{O})$.

Using Table 4 in [6], one may calculate the above orbits in $\phi^{-1}(\mathcal{O})$ and $\phi'(\phi^{-1}(\mathcal{O}))$ explicitly and verify the theorem directly. However we will sketch a simpler proof below.

Sketch of the proof of Theorem A.3. Parts (i) to (iii) are direct consequences of Lemma A.2.

By Theorem 2.5 in [29], $\phi^{-1}(\overline{\mathcal{O}})$ has a unique open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\mathcal{D}$. Since $\phi^{-1}(\mathcal{O}) \cap W^\circ$ is open and nonempty in $\phi^{-1}(\overline{\mathcal{O}})$, it is equal to $\mathcal{D}$ and $\phi^{-1}(\overline{\mathcal{O}}) = \overline{\mathcal{D}}$. Hence $\overline{\mathcal{O}'} \supseteq \phi'(\overline{\mathcal{D}}) = \phi'(\phi^{-1}(\overline{\mathcal{O}})) \supseteq \mathcal{O}'$ so $\overline{\mathcal{O}'} = \theta(\overline{\mathcal{O}})$. This proves (iv). □

A.4. Proof of Proposition 4.1. Suppose $x = \phi(w)$ and $x' = \phi'(w)$ as in Proposition 4.1. Then $w \in W^\circ$ by Theorem A.3(i). Fix a $k' \in K'_{x'}$. Then $(k')^{-1} \cdot w \in \phi'^{-1}(x')$. By Lemma A.2(ii) there is a unique $k \in K_{\mathbb{C}}$ such that $k \cdot w = (k')^{-1} \cdot w$. Since $x = \phi((k')^{-1} \cdot w) = \phi(k \cdot w) = k \cdot \phi(w) = k \cdot x$, we have $k \in K_x$. We define $\alpha(k') = k$. It is straightforward to check that α is a group homomorphism and (17) holds.

Now we prove that α is surjective. Fix a $k \in K_x$. Since $k \cdot w \in \phi^{-1}(x) \cap W^\circ$, there is an element $k' \in K'_{\mathbb{C}}$, such that $k' \cdot w = k \cdot w$ by Lemma A.2(iii). It is clear that that $k' \in K'_{x'}$ so $k \cdot w = \alpha(k')^{-1} \cdot w$. Since the $K_{\mathbb{C}}$-action on $\phi'^{-1}(x')$ is free, we have $\alpha(k')^{-1} = k$. This proves that α is surjective. □

A.5. We discuss the scheme theoretical properties of the moment maps.

Let $R = W - W^\circ$ be the set of matrices without full rank. Let $\mathcal{N} = \phi^{-1}(0) \cap W^\circ$ and $\partial\mathcal{N} := \overline{\mathcal{N}} - \mathcal{N}$. By Theorem A.3, $\mathcal{N}$ is a single $K'_{\mathbb{C}}$-orbit, $\overline{\mathcal{N}} = \phi^{-1}(0)$ and $\partial\mathcal{N} = R \cap \overline{\mathcal{N}}$. We state some well known geometric properties of the null fiber $\overline{\mathcal{N}}$.

Theorem A.4 ([7] [19] [29])**.** *Let (G, G') be a real reductive dual pair in the stable range as in Table 1.*

(i) We have $\mathbb{C}[W] = \mathcal{H} \otimes \mathcal{S}(\mathfrak{p})$ where $\mathcal{H}$ is the space of $K'_{\mathbb{C}}$-harmonic. In particular, the map $\phi\colon W \to \mathfrak{p}^*$ is a faithfully flat morphism. All the fibers of ϕ have the same dimension (see for example, the discussion on page 239 in [32]*).*

(ii) The scheme theoretical fiber $W \times_{\mathfrak{p}^*} \{ 0 \}$ is reduced, i.e. $\overline{\mathcal{N}} = W \times_{\mathfrak{p}^*} \{ 0 \}$.

(iii) If the dual pair is not (††) in Section 1.6, then $\overline{\mathcal{N}}$ is normal and $\partial\mathcal{N}$ has codimension at least 2 in $\overline{\mathcal{N}}$. □

We state Proposition 11.3.13(ii) in [8] which we will need later in the proof of Lemma A.6.

Proposition A.5. *Suppose $f\colon X_1 \to X_2$ is a finitely presented flat morphism of schemes. Let $x_1 \in X$ and $x_2 = f(x_1) \in X_2$. Then x_1 is reduced (resp. normal) in X_1 if*

(i) x_2 is reduced (resp. normal) in X_2 and

(ii) x_1 is reduced (resp. normal) in $X_1 \times_{X_2} \{ x_2 \}$. □

Let $\mathcal{O}$ be a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^*$. Let $Z := W \times_{\mathfrak{p}^*} \overline{\mathcal{O}}$ (resp. $Y := W \times_{\mathfrak{p}^*} \mathcal{O}$) be the scheme theoretic inverse image of $\overline{\mathcal{O}}$ (resp. $\mathcal{O}$).

Lemma A.6. *(i) The schemes Z and Y are reduced.*

(ii) Suppose the dual pair is not (††). Then Y is normal. If $\overline{\mathcal{O}}$ is normal, then Z is normal.

By the above lemma, we can also view $Z = \phi^{-1}(\overline{\mathcal{O}})$ and $Y = \phi^{-1}(\mathcal{O})$ as the set theoretical inverse images.

Proof. Our base field is $\mathbb{C}$ so geometrically reduced (resp. geometrically normal) is equivalent to reduced (resp. normal).

(i) Let E_r (resp. E_n) be the set of elements in W which is geometrically reduced (resp. geometrically normal) in the fiber of $\phi(w)$, i.e.

$$E_r := \{ w \in W \mid w \text{ is geometrically reduced in } W \times_{\mathfrak{p}^*} \phi(w) \} .$$

Since $\phi\colon W \to \mathfrak{p}^*$ is faithfully flat, E_r (resp. E_n) is open in W by Theorem 12.1.1(vii) (resp. Theorem 12.1.6(iv)) in [8]. By Theorem A.4(ii) and (iii), $\overline{\mathcal{N}} \subseteq E_r$ (resp. $\overline{\mathcal{N}} \subseteq E_n$).

We claim that $Z \subseteq E_r$ (resp. $Z \subseteq E_n$). We only prove $Z \subseteq E_r$. The proof of $Z \subseteq E_n$ is the same. Since E_r is open and Z is closed, it suffices to prove that E_r contains every closed point $z \in Z$. Indeed let $z \in Z$ be a closed point. Since $\phi\colon W \to \mathfrak{p}^*$ is an affine quotient map, it maps a $K_{\mathbb{C}}$-invariant closed subset in W to a closed subset in $\mathfrak{p}^*$ (see Corollary 4.6 in [32]). Therefore

$$\phi(\overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z}) = \overline{\phi(K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z)} = \overline{K_{\mathbb{C}} \phi(z)} = \overline{\mathcal{O}} \ni 0.$$

Hence $\emptyset \neq \overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z} \cap \overline{\mathcal{N}} \subset \overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z} \cap E_r$. The subset E_r is open so $(K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z) \cap E_r \neq \emptyset$. Finally E_r contains z because it is $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-invariant. This proves our claim.

We note that $\phi|_Z\colon W \times_{\mathfrak{p}^*} \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ and $\phi|_Y\colon W \times_{\mathfrak{p}^*} \mathcal{O} \to \mathcal{O}$ are faithfully flat. Since $\overline{\mathcal{O}}$ is reduced and $Y \subseteq Z \subseteq E_r$, applying Proposition A.5 to $X_1 = Z$ (resp. $X_1 = Y$) and $X_2 = \overline{\mathcal{O}}$ proves that Z (resp. Y) is reduced. This gives (i).

Since $\mathcal{O}$ is smooth, it is normal. By assumption $\overline{\mathcal{O}}$ is normal. The proof of (ii) follows a similar argument as that of (i). This completes the proof of Lemma A.6. □

We state a consequence of Lemma A.6.

Lemma A.7. *We have* $\overline{\mathcal{O}'} = Z/K_{\mathbb{C}}$*, or equivalently,* $\mathbb{C}[\overline{\mathcal{O}'}] = (\mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}])^{K_{\mathbb{C}}}$.

Proof. By Lemma A.6(i), $\mathbb{C}[Z] = (\mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}])$ is reduced. By (25) and [37], ϕ' is an affine quotient map onto its image so $\mathbb{C}[Z]^{K_{\mathbb{C}}} = \mathbb{C}[\overline{\mathcal{O}'}]$. Also see Proposition 3(2) in [31]. This proves lemma. ☐

By Theorem 2.5 in [29], $Y = \phi^{-1}(\mathcal{O})$ contains an open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit Z°. Let $\partial Z^{\circ} = Y - Z^{\circ}$. By Theorem A.3(i), $\partial Z^{\circ} = R \cap Y$ where $R = W - W^{\circ}$ is the set of elements without full rank.

Lemma A.8. *Suppose the dual pair* (G, G') *is in the stable range where* G *is the smaller member and we exclude the dual pairs* (††)*. Then* $\operatorname{codim}(Y, \partial Z^{\circ}) \geq 2$.

Proof. If $\partial Z^{\circ} = \emptyset$, then there is nothing to prove. Now suppose $\partial Z^{\circ} \neq \emptyset$. Let $C = K_{\mathbb{C}} \times K'_{\mathbb{C}}$ and let $C_0 = K_0 \times K'_0$ be its connected component. Since C may not be connected, Z°, Y and Z may not be irreducible. We decompose $Z^{\circ} = \bigsqcup_{j \in J} Z^{\circ}_j$ in to C_0-orbits. Each Z°_j is irreducible. Since C/C_0 permutes $\{\, Z^{\circ}_j \mid j \in J \,\}$, $\dim Z^{\circ} = \dim Z^{\circ}_j$ for all $j \in J$. Let $Z_j = \overline{Z^{\circ}_j}$ in W and let $Y_j = Z_j \cap Y$. Then Z_j and Y_j are irreducible and C_0-invariant. In fact $Z = \bigcup_{j \in J} Z_j$ and $Y = \bigcup_{j \in J} Y_j$ are the decompositions of Z and Y respectively into irreducible components.

Let $\overline{\mathcal{N}} = \phi^{-1}(0)$ be the closed null cone and let $\partial \mathcal{N} = \overline{\mathcal{N}} - \mathcal{N}$. In the stable range, it is known that $\partial\mathcal{N} = R \cap \overline{\mathcal{N}}$. Furthermore, if the dual pair is not (††), then $\operatorname{codim}(\overline{\mathcal{N}}, \partial\mathcal{N}) = \dim \overline{\mathcal{N}} - \dim \partial \mathcal{N} \geq 2$.

Consider $\phi|_Z \colon Z \to \overline{\mathcal{O}}$. By Theorem A.4 (i), we have

$$\dim Y = \dim Z = \dim \overline{\mathcal{O}} + \dim \overline{\mathcal{N}}.$$

We claim that $\dim \partial Z^{\circ} \leq \dim Y - 2$ which will prove the lemma. It suffices to show that for any closed point $z \in \partial Z^{\circ}$, $\dim_z \partial Z^{\circ} \leq \dim Y - 2$. Here $\dim_v V$ denotes the Krull dimension of the local ring $\mathcal{O}_{V,v}$ at a point v in a variety V.

We consider the morphism $\phi|_{R \cap Z} \colon R \cap Z \to \overline{\mathcal{O}}$. By the semi-continuity of fiber dimension, the set $E = \{\, w \in R \cap Z \mid \dim_w \phi|^{-1}_{R \cap Z}(w) \leq \dim \partial \mathcal{N} \,\}$ is open. For $z \in \partial Z^{\circ}$, let $S = K_{\mathbb{C}} K'_{\mathbb{C}} z$ be the orbit of z. Since $\overline{S}$ is $K'_{\mathbb{C}}$-invariant and closed, the discussion after (25) shows that $\phi(\overline{S})$ is closed. In fact $\phi(\overline{S}) = \overline{\mathcal{O}}$, because $\phi(\overline{S})$ is closed and contains $\phi(\partial Z^{\circ}) = \mathcal{O}$. This implies $0 \in \phi(\overline{S})$ and $\emptyset \neq \overline{S} \cap \overline{\mathcal{N}} \subseteq R \cap \overline{\mathcal{N}} \subseteq \partial \mathcal{N} \subseteq E$. Therefore $z \in E$, i.e. $\dim_z \phi|^{-1}_{R \cap Z}(\phi(z)) \leq \dim \partial \mathcal{N}$. Hence

$$\begin{aligned} \dim_z \partial Z^{\circ} &\leq \dim_z R \cap Z \leq \dim_{\phi(z)} \overline{\mathcal{O}} + \dim_z \phi|^{-1}_{R \cap Z}(\phi(z)) \\ &\leq \dim \overline{\mathcal{O}} + \dim \partial \mathcal{N} \leq \dim \overline{\mathcal{O}} + \dim \overline{\mathcal{N}} - 2 = \dim Y - 2. \end{aligned}$$

This proves the claim and the lemma. ☐

Appendix B. Invariants of contragredient representations

In this section, we state some known facts about the invariants of contragredient representations. Since the proofs are not easily available elsewhere, we supply them as well.

Let G be a real reductive group with complexified Lie algebra g and let K be a maximal compact subgroup of G. Let (ϱ, V) be a (g, K)-module of finite length. Let (ϱ^*, V^*) be its contragredient representation where $V^* = \operatorname{Hom}(V, \mathbb{C})_{\mathsf{K}-\text{finite}}$.

We recall the variety $\mathrm{V}_{\mathbb{C}}(V)$ associated to the annihilator ideal $\operatorname{Ann} V = \operatorname{Ann}_{\mathcal{U}(\mathsf{g})} V$. It is a subvariety in the nilpotent cone of g^* cut out by the graded ideal $\operatorname{Gr}(\operatorname{Ann} V)$.

Proposition B.1. *We have* $\mathrm{V}_{\mathbb{C}}(V^*) = \mathrm{V}_{\mathbb{C}}(V)$.

Proof. Let ι be the anti-involution on $\mathcal{U}(\mathsf{g})$ such that $\iota(X) = -X$ and $\iota(XY) = YX$ for all $X, Y \in \mathsf{g}$. Passing to the graded module $\mathbb{C}[\mathsf{g}^*] = S(\mathsf{g}) = \operatorname{Gr}\mathcal{U}(\mathsf{g})$, $\operatorname{Gr}\iota$ is given by pre-composing the map on g^* defined by $\mathsf{g}^* \ni \lambda \mapsto -\lambda$. Then $\iota(\operatorname{Ann} V) = \operatorname{Ann} V^*$ and $\mathrm{V}_{\mathbb{C}}(V^*) = -\mathrm{V}_{\mathbb{C}}(V)$. On the other hand, $\mathrm{V}_{\mathbb{C}}(V)$ is a union of nilpotent $\mathsf{G}_{\mathbb{C}}$-orbits so $\mathrm{V}_{\mathbb{C}}(V) = -\mathrm{V}_{\mathbb{C}}(V)$. This proves the proposition. □

B.1. Let G and K as in Table 1. Let $\widetilde{G}$ and $\widetilde{K}$ be their respective inverse images in $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$. We relate the associated cycles of an irreducible Harish-Chandra module of $\widetilde{G}$ and its contragredient module. By Proposition 4.I.8 in [26], Theorem 2.4 in [35] and [22], there is an automorphism $C \in \operatorname{Aut}(\widetilde{G})$ such that for all semisimple $g \in \widetilde{G}$, $C(g)$ is conjugate to g^{-1} in $\widetilde{G}$. By replacing C with $\operatorname{Ad}(\tilde{g}) \circ C$ for some $\tilde{g} \in \widetilde{G}$ if necessary, we may further assume that C stabilizes $\widetilde{K}$ and a Cartan subgroup of $\widetilde{K}$. Hence $\operatorname{Ad}_C(\mathfrak{k}) = \mathfrak{k}$ and $\operatorname{Ad}_C(\mathfrak{p}) = \mathfrak{p}$. We call C a *dualizing automorphism.* If $\widetilde{G}$ is the trivial double cover of a connected real algebraic group, then we may choose C to be the Chevalley involution [1].

Let (ϱ, V) be an irreducible $(\mathfrak{g}, \widetilde{K})$-module. We define a representation (ϱ^C, V^C) where $V^C = V$, $\varrho^C(k) = \varrho(C(k))$ for all $k \in \widetilde{K}$ and $\varrho^C(X) = \varrho(\operatorname{Ad}_C(X))$ for all $X \in \mathcal{U}(\mathfrak{g})$. Then (ϱ^C, V^C) is isomorphic to the contragredient representation (ϱ^*, V^*) (c.f. Corollary 1.2 in [1] and Theorem 3.1 in [35]).

If $\mathcal{O}$ is a nilpotent $\widetilde{K}_{\mathbb{C}}$-orbit in $\mathfrak{p}^*$ generated by x, then $\operatorname{Ad}_C^*(\mathcal{O})$ is a nilpotent $\widetilde{K}_{\mathbb{C}}$-orbit in $\mathfrak{p}^*$ generated by $y := \operatorname{Ad}_C^*(x)$. We recall that $\widetilde{K}_x$ is the stabilizer of x in $\widetilde{K}_{\mathbb{C}}$. Then $\widetilde{K}_y = C(\widetilde{K}_x)$. If χ_x is a $\widetilde{K}_x$-module (resp. $\widetilde{K}_x$-character), then $\chi_x \circ C$ is a $\widetilde{K}_y$-module (resp. $\widetilde{K}_y$-character).

Proposition B.2. *We have*

(i) $\mathrm{AV}(\varrho^*) = \operatorname{Ad}_C^*(\mathrm{AV}(\varrho))$ *and*
(ii) $\mathrm{AC}(\varrho^*) = \operatorname{Ad}_C^*(\mathrm{AC}(\varrho))$.
(iii) *Suppose $x \in \mathfrak{p}^*$ generates an open orbit in $\mathrm{AV}(\varrho)$. Let χ_x be the isotropy character of ϱ at x. Then $\chi_x \circ C$ is the isotropy character of ϱ^* at $\operatorname{Ad}_C^*(x)$.*

Proof. Let $\{V_j\}_{j\in\mathbb{N}}$ be an good filtration of (ϱ, V). Then $\{V_j\}_{j\in\mathbb{N}}$ is also a good filtration of (ϱ^C, V^C) since $C(\widetilde{K}) = \widetilde{K}$ and $\operatorname{Ad}_C(\mathfrak{g}) = \mathfrak{g}$. Therefore the $(\mathcal{S}(\mathfrak{p}), \widetilde{K}_{\mathbb{C}})$ action on $\operatorname{Gr} V^C$ is given by pre-composing C, i.e. $\operatorname{Gr} V^C = \operatorname{Gr} V \circ C$. This proves the lemma. □

B.2. Let (ρ, V) be an irreducible $(\mathfrak{g}, \widetilde{K})$-module which is a quotient of $\mathscr{Y}$. Let $\{\, V_j = \mathcal{U}_j(\mathfrak{g}) V'_\tau \,\}_{j\in\mathbb{N}}$ be the filtration generated by lowest degree $\widetilde{K}$-type V_τ. For a regular semisimple element k in a Cartan subgroup of $\widetilde{K}$, one can show that $C(k)$ is $\widetilde{K}$-conjugate to k^{-1}. This implies that $\tau \circ C|_{\widetilde{K}} \simeq \tau^*$. We fix a $(\mathfrak{g}, \widetilde{K})$-module isomorphism between V^C and V^*. Since V_τ has multiplicity 1 in V, V_{τ^*} has multiplicity 1 in V^* too. We set $V_j^C := V_j$ and $V_j^* := \mathcal{U}_j(\mathfrak{g}) V_{\tau^*}$. Therefore the filtration $\left\{\, V_j^C \,\right\}_{j\in\mathbb{N}}$ defined on V^C is the same as the filtration $\left\{\, V_j^* \,\right\}_{j\in\mathbb{N}}$ defined on V^*.

References

[1] J. Adams, *The real Chevalley Involution* (2012). Preprint.

[2] J. Bernstein, *Modules over the ring of differential operators; the study of fundamental solutions of equations with constant coefficients*, Functional Analysis and its Applications **5** (1972), no. 2, 1–16.

[3] R. Brylinski, *Dixmier Algebras for Classical Complex Nilpotent Orbits via Kraft-Procesi Models I*, The orbit method in geometry and physics: in honor of A.A. Kirillov, Birkhäuser, Boston-Basel Berlin, 2003.

[4] E. Cline, B. Parshall, and L. Scott, *A Mackey imprimitivity theory for algebraic groups*, Math. Z. **182** (1983), 447–471.

[5] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda, *Dual pairs and Kostant-Sekiguchi correspondence. I*, J. Algebra **250** (2002), 408–426.

[6] ———, *Dual pairs and Kostant-Sekiguchi correspondence. II. Classification of nilpotent elements*, Central European Journal of Mathematics **3** (2005), no. 3, 430–474.

[7] O. Debarre and T.-T. Tuong, *Representations of $SO(k, \mathbb{C})$ on harmonic polynomials on a null cone*, Proc. Am. Math. Soc. **112** (1991), no. 1, 31–44.

[8] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique(rédigés avec la collaboration de Jean Dieudonné) IV: Étude locale des schémas et des morphismes de schémas (Troisième partie)*, Publ. Math., Inst. Hautes Étud. Sci. **28** (1966).

[9] T. J. Enright, R. Howe, and N. R. Wallach, *A classification of unitary highest weight modules*, Representation theory of reductive groups (Park City, Utah, 1982), Progr. Math., vol. 40, Birkhäuser, Boston, MA, 1983, pp. 97–147.

[10] R. Gomez and C.-b. Zhu, *Theta lifting of generalized Whittaker models associated to nilpotent orbits* (2013). To appear in GAFA.

[11] R. Hartshorne, *Local cohomology: a seminar given by A. Grothendieck, Harvard University, Fall 1961*, Lecture Notes in Mathematics, vol. 41, Springer, 1967.

[12] ———, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag, 1983.

[13] H. He, *Theta Correspondence in Semistable Range I: Construction and Irreducibility*, Commun. Contemp. Math. **2** (2000), no. 02, 255–283.

[14] R. Howe, *Reciprocity laws in the theory of dual pairs*, Representation theory of reductive groups (Park City, Utah, 1982), Progr. Math., vol. 40, Birkhäuser, Boston, MA, 1983, pp. 159–175.

[15] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), no. 3, 535–552.

[16] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Piatetski-Shapiro, Ilya (ed.) et al., The Schur lectures (1992). Ramat-Gan: Bar-Ilan University, Isr. Math. Conf. Proc. 8, (1995), 1–182.

[17] H. P. Jakobsen and M. Vergne, *Restrictions and expansions of holomorphic representations*, J. Func. A. **34** (1979), no. 1, 29–53.

[18] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), no. 1, 1–47.

[19] B. Kostant, *Lie group representations on polynomial rings*, Am. J. Math. **85** (1963), 327–404.

[20] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[21] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), no. 2, 237-255.

[22] Y. Lin, B. Sun, and S. Tan, *MVW-extensions of quaternionic classical groups*, Math. Z., posted on 2013, 1-9, DOI 10.1007/s00209-013-1246-6.

[23] H. Y. Loke, *Howe quotients of unitary characters and unitary lowest weight modules. With an appendix by S. T. Lee*, Represent. Theory **10** (2006), 21–47.

[24] H. Y. Loke, J.-J. Ma, and U.-L. Tang, *Associated cycles of local theta lifts of unitary characters and unitary lowest weight modules*, arXiv:1207.6451 (2012).

[25] J.-J. Ma, *Two topics in local theta correspondence*, National University of Singapore, 2012.

[26] C. Moeglin, M. F. Vigneras, and J. L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer Verlag, Berlin, 1987.

[27] C. Moeglin and J. L. Waldspurger, *Modèles de Whittaker dégénéres pour des groupes p-adiques. (Degenerate Whittaker models of p-adic groups)*, Math. Z. **196** (1987), 427–452 (French).

[28] K. Nishiyama, H. Ochiai, and K. Taniguchi, *Bernstein degree and associated cycles of Harish-Chandra modules-Hermitian symmetric case*, Astérisque **273** (2001), 13–80.

[29] K. Nishiyama, H. Ochiai, and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, Trans. AMS **358** (2006), no. 6, 2713–2734.

[30] K. Nishiyama and C.-B. Zhu, *Theta lifting of unitary lowest weight modules and their associated cycles*, Duke Math. J. **125** (2004), no. 3, 415–465.

[31] T. Ohta, *The closure of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. **43** (1991), no. 2, 161–211.

[32] V. Popov and E. Vinberg, *Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory, Encyclopaedia of Mathematical Sciences, vol. 55, Springer, 1994.

[33] V. Protsak and T. Przebinda, *On the occurrence of admissible representations in the real Howe correspondence in stable range*, Manuscr. Math. **126** (2008), no. 2, 135–141.

[34] T. Przebinda, *Characters, dual pairs, and unitary representations*, Duke Math. J. **69** (1993), no. 3, 547–592.
[35] B. Sun, *A Note on MVW-involutions*, Fifth International Congress of Chinese Mathematicians, AMS and IP, 2012, pp. 305–314.
[36] D. A. Vogan, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, 1991, pp. 315–388.
[37] H. Weyl, *The classical groups: their invariants and representations*, Princeton Landmarks in Mathematics., vol. 1, Princeton University Press., 1997.
[38] H. Yamashita, *Cayley transform and generalized Whittaker models for irreducible highest weight modules*, Astérisque **273** (2001), 81–138.
[39] L. Yang, *On quantization of spherical nilpotent orbits of* $\mathfrak{g}$*-height 2*, HKUST Thesis (2010).
[40] ———, *On the quantization of spherical nilpotent orbits*, Trans. Amer. Math. Soc **365** (2013), 6499-6515.

Department of Mathematics, National University of Singapore, Block S17, 10, Lower Kent Ridge Road, Singapore 119076
E-mail address: matlhy@nus.edu.sg

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, Israel
E-mail address: jiajun@math.bgu.ac.il