Interior operators in the Weihrauch lattice

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Weihrauch reducibility on problems

Goal: Classify the computational content of mathematical theorems

1. Formalize mathematical statements as multivalued functions on represented spaces a.k.a. problems
2. Compare them using the preorder Weihrauch reducibility $\leq_{\mathrm{W}}$:

> roughly, $f \leq_{\mathrm{W}} g$ if there is a (uniformly) computable procedure for solving f, which queries g exactly once

Choose an element not in the range of p using pigeonhole:

$$\begin{array}{ccc} p : \mathbb{N} \to \mathbb{N} \textbf{ not onto} & \longrightarrow & c(n) = \min(\mathbb{N} \setminus \mathrm{range}(p \upharpoonright n)) \textbf{ finite range} \\ \big\downarrow \mathrm{C}_{\mathbb{N}} & & \big\downarrow \text{pigeonhole} \\ c(\min X) \notin \mathrm{range}(p) & \xleftarrow[\text{can use } p]{} & \text{infinite monochromatic } X \subseteq \mathbb{N} \end{array}$$

Features of the quotient structure (i.e., Weihrauch degrees)

Rich structure: Medvedev degrees embed

Many algebraic operations:

Join, meet, parallel product, compositional product, ...

Many "benchmark" problems:

Problem	Instance	Solution(s)
id	A real	That real
$\mathrm{C}_{\mathbb{N}}$	$p : \mathbb{N} \to \mathbb{N}$ not onto	$i \notin \mathrm{range}(p)$
WKL	Infinite tree in $2^{<\mathbb{N}}$	Infinite path
WWKL	Positive measure tree in $2^{<\mathbb{N}}$	Infinite path
lim	Convergent sequence in $\mathbb{N}^{\mathbb{N}}$	The limit
$\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$	Infinite tree in $\mathbb{N}^{<\mathbb{N}}$	Infinite path

Closure and interior operators

In a partial order $(P, \leq)$, a closure operator $C : P \to P$ satisfies:

- $f \leq C(f)$
- if $f \leq g$, then $C(f) \leq C(g)$
- $CC(f) = C(f)$

An interior operator satisfies $f \geq C(f)$ instead.

Computing $C(f)$ helps us make sense of f.
Ideally $C(f)$ is well-behaved since it enjoys a closure property (unlike f).

Closure operator on Weihrauch degrees	
(Countable) parallelization	Brattka, Gherardi '11
Finite parallelization	Pauly '10
Diamond	Neumann, Pauly '18
Unbounded finite parallelization	Soldà, Valenti '23

I. Expanding the solution set

Upper Turing cone version of f (Brattka '21):

$$f^{\mathcal{D}}(p) = \{x \in \mathbb{N}^{\mathbb{N}} \mid x \text{ computes some } f\text{-solution to } p\}$$

f	$f^{\mathcal{D}}$
WKL	PA
$\lim$	Produce set which computes jump of input
Computably true	$\leq_{\mathrm{W}} \mathrm{id}$

II. Dualizing a closure operator

The parallelization $\widehat{f}$ of $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is defined by

$$\widehat{f}((x_n)_{n\in\mathbb{N}}) = \left\{(y_n)_n \in \mathbf{Y}^{\mathbb{N}} \;\middle|\; y_n \in f(x_n) \text{ for all } n\right\}.$$

Stashing (Brattka '21) has same domain as $\widehat{f}$ but with solution sets

$$\left\{(y_n)_n \in \overline{\mathbf{Y}}^{\mathbb{N}} \;\middle|\; y_n \in f(x_n) \text{ for some } n\right\}.$$

$\overline{\mathbf{Y}}$ is the completion of $\mathbf{Y}$ (Dzhafarov '19; Brattka, Gherardi '20).

II. Dualizing a closure operator

Brattka '21:

$$\left\{(y_n)_n \in \overline{\mathbf{Y}}^{\mathbb{N}} \;\middle|\; y_n \in f(x_n) \text{ for some } n\right\}.$$

Stashing of f	f	$\widehat{f}$	Stashing of $\widehat{f}$
DIS	C_2	WKL	PA
DIS	LPO	lim	Produce set which computes jump of input

Theorem (Brattka '21)

Stashing of $\widehat{f}$ $\equiv_{\mathrm{W}}$ Upper Turing cone version of $\widehat{f}$.

For other problems few examples of stashings have been characterized.

III. Residual of a binary operator

Brattka, Pauly '18:
$h \star g$ is the compositional product of h and g (intuitively, apply g then h).

Implication is the right co-residual of $\star$:

$$(h \to f) \equiv_{\mathrm{W}} \min_{\leq_{\mathrm{W}}}\{g \mid f \leq_{\mathrm{W}} h \star g\}$$

- $(h \to f) \leq_{\mathrm{W}} f$
- If $f_0 \leq_{\mathrm{W}} f_1$, then $(h \to f_0) \leq_{\mathrm{W}} (h \to f_1)$

Observation

If $h \star h \equiv_{\mathrm{W}} h$, then $f \mapsto (h \to f)$ is an interior operator.

III. Residual of a binary operator

$$(h \to f) \equiv_{\mathrm{W}} \min_{\leq_{\mathrm{W}}} \{g \mid f \leq_{\mathrm{W}} h \star g\}$$

Examples of h such that $h \star h \equiv_{\mathrm{W}} h$ (Brattka, de Brecht, Pauly '12; Brattka, Gherardi, Hölzl '15, Soldà, Valenti '23, Brattka '23):

$$\mathrm{C}_{\mathbb{N}}^{(n)}, \quad \mathrm{K}_{\mathbb{N}}^{(n)}, \quad \mathrm{MLR}, \quad \mathrm{WWKL}, \quad \mathrm{WKL}, \quad \mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}, \quad \mathrm{C}_{\mathbb{N}^{\mathbb{N}}}.$$

Theorem (Brattka, Hendtlass, Kreuzer '17; Brattka, Pauly '18)

- $\mathrm{C}_{\mathbb{N}}^{(n)} \to \mathrm{WKL} \equiv_{\mathrm{W}} \mathrm{PA}$ for every $n \geq 1$. (Open for $n = 0$.)
- $\mathrm{C}_{\mathbb{N}} \to \mathrm{WWKL} \equiv_{\mathrm{W}} \mathrm{MLR}$.

Proposition (Dzhafarov, G., Hirschfeldt, Patey, Pauly '20)

$\mathrm{C}_{\mathbb{N}} \to \mathrm{RT}^2_2 \equiv_{\mathrm{W}} \mathrm{RT}^2_2$ *with finite error.*

IV. Max over restricted codomain

For some represented spaces $\mathbf{X}$, this max exists:

$$\max_{\leq_W}\{g \leq_W f \mid g \text{ has codomain } \mathbf{X}\}$$

$\mathbf{X} = \mathbb{N}$: First-order part $^1 f$ (Dzhafarov, Solomon, Yokoyama '23)

f	$^1 f$	
$\lim$	$C_{\mathbb{N}}$	Brattka, Gherardi, Marcone '12
WKL, WWKL	$K_{\mathbb{N}}$	Dzhafarov, Solomon, Yokoyama '23
MLR	id	Brattka, Pauly '18
$C_{\mathbb{N}} \to h$	$\leq_W \mathrm{id}$	"
Any $g^{\mathcal{D}}$	$\leq_W \mathrm{id}$	
DS	Π^1_1-Bound	G., Pauly, Valenti '21

$\mathbf{X} = k$: k-finitary part (Cipriani, Pauly '23)

Question

For which other $\mathbf{X}$ is this defined/useful?

V. Max over single-valued functions

Deterministic part (G., Pauly, Valenti '21):

$$\mathrm{Det}(f) \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{g \leq_{\mathrm{W}} f \mid g \text{ single-valued, codomain } \mathbb{N}^{\mathbb{N}}\}$$

f	$\mathrm{Det}(f)$	
WKL	id	Gherardi, Marcone '09
List a countable closed $A \subseteq 2^{\mathbb{N}}$	lim	Kihara, Marcone, Pauly '20
$\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$	$\mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$	"
DS	lim	G., Pauly, Valenti '21

Being single-valued is not a degree-theoretic property.

We say f is deterministic if it is equivalent to a single-valued problem with codomain $\mathbb{N}^{\mathbb{N}}$.

Max over single-valued functions with restricted codomain

G., Pauly, Valenti '21:

$$\mathrm{Det}_{\mathbf{X}}(f) \equiv_{\mathrm{W}} \mathsf{max}_{\leq_{\mathrm{W}}}\{g \leq_{\mathrm{W}} f \mid g \text{ single-valued, codomain } \mathbf{X}\}$$

Proposition

${}^1\mathrm{Det}(f) \equiv_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} \mathrm{Det}^1(f)$.

It follows that $f \mapsto {}^1\mathrm{Det}(f)$ is an interior operator. Furthermore:

$${}^1\mathrm{Det}(f) \equiv_{\mathrm{W}} \mathrm{Det}^1\mathrm{Det}(f) \equiv_{\mathrm{W}} {}^1\mathrm{Det}^1(f) \equiv_{\mathrm{W}} \ldots$$

Question

Is it possible to have ${}^1\mathrm{Det}(f) <_{\mathrm{W}} \mathrm{Det}^1(f)$?

Is it possible to have ${}^1\mathrm{Det}(f) <_\mathrm{W} \mathrm{Det}^1(f)$?

Equivalently:

Is there a first-order problem f such that $\mathrm{Det}(f)$ is not first-order?

Suppose we had such an f and let g denote $\mathrm{Det}(f)$.

Since $g \leq_\mathrm{W} f$ and f is first-order, g is computably true, i.e. $g(p) \leq_\mathrm{T} p$ for all $p \in \mathrm{dom}(g)$.

g	$\leq_\mathrm{W}$	f
single-valued not first-order computably true		not single-valued first-order

no single-valued

First-order closure (G., Pauly, Valenti in preparation)

Dzhafarov, Solomon, Yokoyama '23 showed that every problem which is computably true is below some first-order problem.

Theorem

If g is computably true, then

$$\min_{\leq_W}\{f \geq_W g \mid f \text{ is first-order}\}$$

exists and is represented by $g^1 : \mathrm{dom}(g) \rightrightarrows \mathbb{N}$, *defined by*

$$g^1(p) = \{e \in \mathbb{N} \mid \Phi_e(p) \in g(p)\}.$$

Corollary

If g is computably true, single-valued with codomain $\mathbb{N}^\mathbb{N}$, and g^1 is not deterministic, then $\mathrm{Det}(g^1)$ is not first-order.

Proof. $g \leq_W \mathrm{Det}(g^1) <_W g^1$ so $\mathrm{Det}(g^1)$ is not first-order. □

First-order closure (G., Pauly, Valenti in preparation)

Corollary

If g is computably true, single-valued with codomain $\mathbb{N}^{\mathbb{N}}$, and g^1 is not deterministic, then $\mathrm{Det}(g^1)$ is not first-order.

By diagonalization, we can construct a sequence of computable reals $(g_n)_{n \in \mathbb{N}}$ such that if we define $g : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ by

$$g(n) = g_n,$$

then

$$g^1(n) = \{e \in \mathbb{N} \mid \Phi_e(n) = g_n\}$$

is not deterministic.

g^1 is first-order but $\mathrm{Det}(g^1)$ is not.

Thanks!

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