## Interior operators in the Weihrauch lattice

Jun Le GOH





Organized by Department of Mathematics, Nazarbayev University and Mathematical Center, Akademgorodok

Jun Le GOH (NUS)

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## Weihrauch reducibility on problems

Goal: Classify the computational content of mathematical theorems

- Formalize mathematical statements as multivalued functions on represented spaces a.k.a. problems
- **2** Compare them using the preorder Weihrauch reducibility  $\leq_W$ :

roughly,  $f \leq_W g$  if there is a (uniformly) computable procedure for solving f, which queries g exactly once

Choose an element not in the range of p using pigeonhole:

Features of the quotient structure (i.e., Weihrauch degrees)

Rich structure: Medvedev degrees embed

Many algebraic operations:

Join, meet, parallel product, compositional product, ...

Many "benchmark" problems:

Problem	Instance	Solution(s)
id	A real	That real
$\mathrm{C}_{\mathbb{N}}$	$oldsymbol{ ho}:\mathbb{N} o\mathbb{N}$ not onto	$i \notin \operatorname{range}(p)$
WKL	Infinite tree in $2^{<\mathbb{N}}$	Infinite path
WWKL	Positive measure tree in $2^{<\mathbb{N}}$	Infinite path
lim	Convergent sequence in $\mathbb{N}^{\mathbb{N}}$	The limit
$\mathbf{C}_{\mathbb{N}^{\mathbb{N}}}$	Infinite tree in $\mathbb{N}^{<\mathbb{N}}$	Infinite path

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### Closure and interior operators

In a partial order  $(P, \leq)$ , a closure operator  $C : P \rightarrow P$  satisfies:

- *f* ≤ *C*(*f*)
- if  $f \leq g$ , then  $C(f) \leq C(g)$
- CC(f) = C(f)

An interior operator satisfies  $f \ge C(f)$  instead.

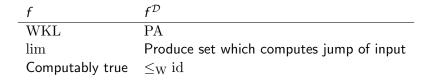
Computing C(f) helps us make sense of f. Ideally C(f) is well-behaved since it enjoys a closure property (unlike f).

Closure operator on Weihrauch degrees(Countable) parallelizationBrattka, Gherardi '11Finite parallelizationPauly '10DiamondNeumann, Pauly '18Unbounded finite parallelizationSoldà, Valenti '23

# I. Expanding the solution set

Upper Turing cone version of *f* (Brattka '21):

$$f^{\mathcal{D}}(p) = \{x \in \mathbb{N}^{\mathbb{N}} \mid x ext{ computes some } f ext{-solution to } p\}$$



## II. Dualizing a closure operator

The parallelization  $\hat{f}$  of  $f :\subseteq \mathbf{X} \Rightarrow \mathbf{Y}$  is defined by

$$\widehat{f}((x_n)_{n\in\mathbb{N}})=\big\{(y_n)_n\in\mathbf{Y}^{\mathbb{N}}\mid y_n\in f(x_n) ext{ for all } nig\}.$$

Stashing (Brattka '21) has same domain as  $\hat{f}$  but with solution sets

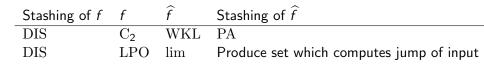
$$\{(y_n)_n \in \overline{\mathbf{Y}}^{\mathbb{N}} \mid y_n \in f(x_n) \text{ for some } n\}.$$

 $\overline{\mathbf{Y}}$  is the completion of  $\mathbf{Y}$  (Dzhafarov '19; Brattka, Gherardi '20).

# II. Dualizing a closure operator

Brattka '21:

$$\{(y_n)_n \in \mathbf{Y}^{\mathbb{N}} \mid y_n \in f(x_n) \text{ for some } n\}.$$



Theorem (Brattka '21)  
Stashing of 
$$\hat{f} \equiv_{W}$$
 Upper Turing cone version of  $\hat{f}$ .

For other problems few examples of stashings have been characterized.

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# III. Residual of a binary operator

Brattka, Pauly '18:

 $h \star g$  is the compositional product of h and g (intuitively, apply g then h).

Implication is the right co-residual of \*:

$$(h \rightarrow f) \equiv_{\mathrm{W}} \min_{\leq_{\mathrm{W}}} \{g \mid f \leq_{\mathrm{W}} h \star g\}$$

• 
$$(h \rightarrow f) \leq_{\mathrm{W}} f$$

• If 
$$f_0 \leq_{\mathrm{W}} f_1$$
, then  $(h 
ightarrow f_0) \leq_{\mathrm{W}} (h 
ightarrow f_1)$ 

#### Observation

If  $h \star h \equiv_{W} h$ , then  $f \mapsto (h \to f)$  is an interior operator.

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III. Residual of a binary operator

$$(h \rightarrow f) \equiv_{\mathrm{W}} \min_{\leq_{\mathrm{W}}} \{g \mid f \leq_{\mathrm{W}} h \star g\}$$

Examples of h such that  $h \star h \equiv_{W} h$  (Brattka, de Brecht, Pauly '12; Brattka, Gherardi, Hölzl '15, Soldà, Valenti '23, Brattka '23):

$$\mathrm{C}^{(n)}_{\mathbb{N}}, \hspace{0.2cm} \mathrm{K}^{(n)}_{\mathbb{N}}, \hspace{0.2cm} \mathrm{MLR}, \hspace{0.2cm} \mathrm{WWKL}, \hspace{0.2cm} \mathrm{WKL}, \hspace{0.2cm} \mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}, \hspace{0.2cm} \mathrm{C}_{\mathbb{N}^{\mathbb{N}}}.$$

Theorem (Brattka, Hendtlass, Kreuzer '17; Brattka, Pauly '18)

• 
$$C_{\mathbb{N}}^{(n)} \to WKL \equiv_W PA$$
 for every  $n \ge 1$ . (Open for  $n = 0$ .)

•  $C_{\mathbb{N}} \to WWKL \equiv_W MLR.$ 

 $\begin{array}{l} \mbox{Proposition (Dzhafarov, G., Hirschfeldt, Patey, Pauly '20)}\\ \label{eq:CN} C_{\mathbb{N}} \rightarrow RT_2^2 \equiv_W RT_2^2 \mbox{ with finite error.} \end{array}$ 

### IV. Max over restricted codomain

For some represented spaces  $\mathbf{X}$ , this max exists:

 $\max_{\leq_{\mathrm{W}}} \{ g \leq_{\mathrm{W}} f \mid g \text{ has codomain } X \}$ 

 $\mathbf{X} = \mathbb{N}$ : First-order part <sup>1</sup>f (Dzhafarov, Solomon, Yokoyama '23)

f	$^{1}f$	
lim	$\mathrm{C}_{\mathbb{N}}$	Brattka, Gherardi, Marcone '12
WKL, WWKL	$\mathrm{K}_{\mathbb{N}}$	Dzhafarov, Solomon, Yokoyama '23
MLR	id	Brattka, Pauly '18
$\mathrm{C}_{\mathbb{N}}  o h$	$\leq_{\mathrm{W}} \mathrm{id}$	"
Any $g^{\mathcal{D}}$	$\leq_{\mathrm{W}} \mathrm{id}$	
DS	$\Pi_1^1\text{-}\mathrm{Bound}$	G., Pauly, Valenti '21

 $\mathbf{X} = k$ : k-finitary part (Cipriani, Pauly '23)

### Question

For which other  $\mathbf{X}$  is this defined/useful?

## V. Max over single-valued functions

Deterministic part (G., Pauly, Valenti '21):

 $\operatorname{Det}(f) \equiv_{\operatorname{W}} \max_{\leq_{\operatorname{W}}} \{g \leq_{\operatorname{W}} f \mid g \text{ single-valued, codomain } \mathbb{N}^{\mathbb{N}}\}$ 

f	$\operatorname{Det}(f)$	
WKL	id	Gherardi, Marcone '09
List a countable closed $A\subseteq 2^{\mathbb{N}}$	$\lim$	Kihara, Marcone, Pauly '20
$\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$	$\mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$	"
DS	lim	G., Pauly, Valenti '21

Being single-valued is not a degree-theoretic property.

We say f is deterministic if it is equivalent to a single-valued problem with codomain  $\mathbb{N}^{\mathbb{N}}$ .

Max over single-valued functions with restricted codomain

#### G., Pauly, Valenti '21:

 $\operatorname{Det}_{\mathbf{X}}(f) \equiv_{\operatorname{W}} \max_{\leq_{\operatorname{W}}} \{g \leq_{\operatorname{W}} f \mid g \text{ single-valued, codomain } \mathbf{X}\}$ 

#### Proposition

<sup>1</sup>Det $(f) \equiv_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} \mathrm{Det}^{1}(f).$ 

It follows that  $f \mapsto {}^{1}\mathrm{Det}(f)$  is an interior operator. Furthermore:

$$^{1}\text{Det}(f) \equiv_{W} \text{Det}^{1}\text{Det}(f) \equiv_{W} ^{1}\text{Det}^{1}(f) \equiv_{W} \dots$$

#### Question

Is it possible to have  ${}^{1}\mathrm{Det}(f) <_{\mathrm{W}} \mathrm{Det}^{1}(f)$ ?

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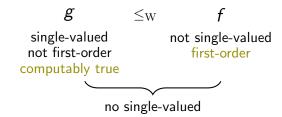
# Is it possible to have ${}^{1}\text{Det}(f) <_{W} \text{Det}^{1}(f)$ ?

Equivalently:

Is there a first-order problem f such that Det(f) is not first-order?

Suppose we had such an f and let g denote Det(f).

Since  $g \leq_W f$  and f is first-order, g is computably true, i.e.  $g(p) \leq_T p$  for all  $p \in dom(g)$ .



# First-order closure (G., Pauly, Valenti in preparation)

Dzhafarov, Solomon, Yokoyama '23 showed that every problem which is computably true is below some first-order problem.

#### Theorem

If g is computably true, then

$$\min_{\leq_{\mathrm{W}}} \{ f \geq_{\mathrm{W}} g \mid f \text{ is first-order} \}$$

exists and is represented by  $g^1$  : dom $(g) \rightrightarrows \mathbb{N}$ , defined by

$$g^1(p) = \{e \in \mathbb{N} \mid \Phi_e(p) \in g(p)\}.$$

#### Corollary

If g is computably true, single-valued with codomain  $\mathbb{N}^{\mathbb{N}}$ , and  $g^1$  is not deterministic, then  $\operatorname{Det}(g^1)$  is not first-order.

Proof.  $g \leq_W \text{Det}(g^1) <_W g^1$  so  $\text{Det}(g^1)$  is not first-order.

First-order closure (G., Pauly, Valenti in preparation)

#### Corollary

If g is computably true, single-valued with codomain  $\mathbb{N}^{\mathbb{N}}$ , and  $g^1$  is not deterministic, then  $\operatorname{Det}(g^1)$  is not first-order.

By diagonalization, we can construct a sequence of computable reals  $(g_n)_{n\in\mathbb{N}}$  such that if we define  $g:\mathbb{N}\to\mathbb{N}^\mathbb{N}$  by

$$g(n) = g_n,$$

then

$$g^1(n) = \{e \in \mathbb{N} \mid \Phi_e(n) = g_n\}$$

is not deterministic.

 $g^1$  is first-order but  $Det(g^1)$  is not.

Thanks!

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