A theorem of Halin and hyperarithmetic analysis

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Theorem (Halin, 1965)

If a graph contains k many disjoint rays for every $k \in \omega$, then it contains infinitely many disjoint rays.

- Graph can be directed or undirected
- Rays are infinite paths indexed by ω
- Disjoint means vertex-disjoint

Compactness? Not quite; rays are not first-order objects.

Two formalizations in second-order arithmetic:

IRT: $\forall G(\forall k \exists X[X \text{ is a disjoint set of } k \text{ many rays in } G]$
$\rightarrow \exists X[X \text{ is an infinite disjoint set of rays in } G])$

WIRT: $\forall G(\exists (X_k)_k \forall k[X_k \text{ is a disjoint set of } k \text{ many rays in } G]$
$\rightarrow \exists X[X \text{ is an infinite disjoint set of rays in } G])$.

WIRT is provable in ACA_0

Theorem (essentially Andreae's proof of Halin's theorem)

Given a graph G and a sequence $(X_k)_k$ where each X_k is a set of k disjoint rays, $(G \oplus (X_k)_k)'$ uniformly computes an infinite disjoint set of rays.

Sketch.

At the beginning of stage n, we have constructed disjoint rays $R_0^n, \dots, R_{n-1}^n$ and committed to $R_0^n \upharpoonright n, \dots, R_{n-1}^n \upharpoonright 1$.

Fix a large set of disjoint rays S_j from the given sequence. Discard all S_j's which intersect our commitment.

For those R_i^n's which do not intersect too many S_j's, define $R_i^{n+1} = R_i^n$ and discard the S_j's intersecting them.

For those R_i^n's which intersect many S_j's, we put appropriate initial segments of them and of the S_j's into a finite graph. Apply Menger's theorem to reroute the initial segments of the R_i^n's. □

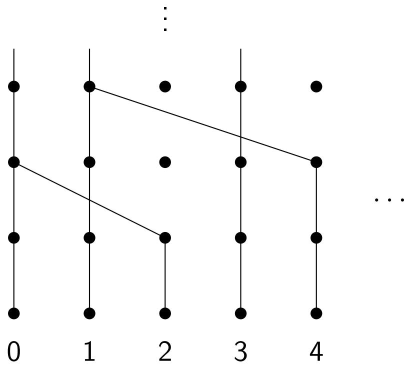
A priority argument shows that

Theorem

WIRT is not provable in RCA_0. TODO: Close this gap!

The above proof applies to a special case of WIRT:

> *Given a c.e. equivalence relation with infinitely many classes, compute an infinite independent set.*



Hyperarithmetic reduction

The hyperarithmetic sets are a natural extension of the arithmetic sets into the transfinite:

Theorem (Kleene)

For $X \subseteq \omega$, TFAE:

- X is computable in some jump hierarchy along a computable well-ordering;
- X is Δ^1_1-definable (without parameters).

If either (both) of the above conditions hold, we say that X is *hyperarithmetic*.

HYP denotes the set of all hyperarithmetic sets.

We can relativize the above to define $\mathrm{HYP}(Y)$, the set of all sets which are hyperarithmetic in Y.

ω-models of second-order arithmetic

For some systems in reverse math, their ω-models have nice computability-theoretic characterizations.

Theory	ω-**models**
RCA_0	closed under $\oplus$ and Turing reduction
ACA_0	closed under $\oplus$ and jump
	closed under $\oplus$ and hyp reduction

Every ω-model of ATR_0 is closed under $\oplus$ and hyp reduction, but HYP is not a model of ATR_0 because of pseudo-well-orderings! van Wesep (1977) showed that there is **no theory** whose ω-models are exactly those closed under $\oplus$ and hyp reduction.

Definition (Steel 1978, Montalbán 2006)

T is a *theory of hyp analysis* if:

1. every ω-model of T is closed under $\oplus$ and hyp reduction;
2. for every $Y \subseteq \omega$, $\mathrm{HYP}(Y)$ is a model of T.

Definition (Steel 1978, Montalbán 2006)

T is a *theory of hyp analysis* if:

1. every ω-model of T is closed under $\oplus$ and hyp reduction;
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Examples abound:

Σ^1_1-AC_0: $\forall n \exists Y \varphi(n, Y) \to \exists (Z_n)_n \forall n \varphi(n, Z_n)$,
where $\varphi(n, Y)$ is arithmetic.

Δ^1_1-CA_0: $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \psi(n))$,
where $\varphi(n)$ is Σ^1_1 and $\psi(n)$ is Π^1_1.

unique-Σ^1_1-AC_0: $\forall n \exists ! Y \varphi(n, Y) \to \exists (Z_n)_n \forall n \varphi(n, Z_n)$,
where $\varphi(n, Y)$ is arithmetic.

But all known examples **except one** (a theorem of Jullien on indecomposable scattered linear orderings, studied by Montalbán) are defined using notions from logic.

A new natural theory of hyperarithmetic analysis

IRT: $\forall G(\forall k \exists X[X$ is a disjoint set of k many rays in $G]$
$\to \exists X[X$ is an infinite disjoint set of rays in $G])$

WIRT: $\forall G(\exists (X_k)_k \forall k[X_k$ is a disjoint set of k many rays in $G]$
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Theorem (Barnes, G., Shore)

IRT is a theorem of hyperarithmetic analysis.

Proof that Σ^1_1-AC_0 implies IRT.

Given G satisfying the premise of IRT, use Σ^1_1-AC_0 to choose a sequence $(X_k)_k$ where each X_k is a set of k disjoint rays. Then apply WIRT, which is provable in ACA_0. □

This shows that for every $Y \subseteq \omega$, $\mathrm{HYP}(Y)$ is a model of IRT.

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Proof that $I\Sigma^1_1 +$ IRT implies unique-Σ^1_1-choice.

Unique-Σ^1_1-choice can be reformulated as:

> *Given a sequence $(T_n)_n$ of subtrees of $\omega^{<\omega}$, each of which has a unique path P_n, the sequence $(P_n)_n$ exists.*

Think of the sequence $(T_n)_n$ as a graph G. Using $I\Sigma^1_1$, we can show that G is an instance of IRT.

Apply IRT to G to obtain an infinite disjoint set of rays.
This gives us a sequence of infinitely many distinct P_n. ☹

Instead, apply IRT to the cumulative product of T_n's. ☺ □

This shows that every ω-model of (RCA_0+)IRT is closed under $\oplus$ and hyp reduction.

A Σ^1_1 axiom of finite choice

finite-Σ^1_1-AC_0: $\forall n \exists$ **finitely** many $Y \varphi(n, Y) \to \exists (Z_n)_n \forall n \varphi(n, Z_n)$,
where $\varphi(n, Y)$ is arithmetic.

Theorem (Barnes, G., Shore)

$I\Sigma^1_1$ + IRT implies finite-Σ^1_1-AC_0.

Both IRT and finite-Σ^1_1-AC_0 are related to another theorem of hyp analysis called ABW (studied by Friedman 1975 and Conidis 2012).

Separations? Steel (1978) used forcing with tagged trees to show that Δ^1_1-CA_0 does not imply Σ^1_1-AC_0. We add locks to his forcing to show that

Theorem (G.)

Δ^1_1-CA_0 does not imply finite-Σ^1_1-AC_0.

Thanks!

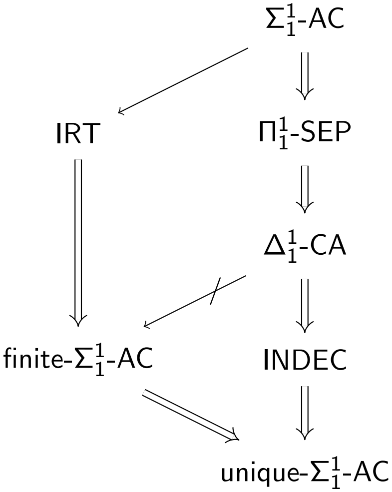

Figure: Partial zoo of theories of hyp analysis (assuming $I\Sigma^1_1$)