Inseparable pairs and recursion theory

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Inseparable pairs of sets

There are disjoint coanalytic sets of reals which cannot be separated by any Borel set. A well-known example:

- WF: well-founded trees on ω
- UB: trees on ω with a unique branch

Our results imply that more is true:

Theorem
Any (co)analytic set which separates WF and UB is complete (co)analytic.

The above follows from analytic determinacy, but we do not assume any determinacy.

We have similar results for a variety of pointclasses. To prove them, we turn to recursion theory.

Outline of talk

1. Σ^0_1 subsets of ω
2. Π^1_1 subsets of ω
   2.1 Segue into Turing jump hierarchies
3. Coanalytic subsets of ω^ω
4. Other pointclasses

Creative subsets of ω

Definition (Post, 1940s)

A Σ^0_1 set $C \subseteq \omega$ is *creative* if there is some recursive $f : \omega \to \omega$ such that for each $x \in \omega$:

$$W_x \text{ and } C \text{ are disjoint} \quad \Rightarrow \quad f(x) \notin C \cup W_x.$$

Here W_x is the x-th Σ^0_1 subset of ω. (We call x an *index* for W_x.)

Creativeness effectivizes "C is not Δ^0_1":

- W_x threatens to show that $\overline{C}$ is Σ^0_1;
- $f(x)$ is an effectively produced witness which defeats W_x.

Example

$\{x \in \omega : x \in W_x\}$ is creative, as witnessed by $f(x) = x$.

Creative $\Leftrightarrow$ complete

Definition (Post, 1940s)

A Σ^0_1 set $C \subseteq \omega$ is *creative* if there is some recursive $f : \omega \to \omega$ such that for each $x \in \omega$:

$$W_x \text{ and } C \text{ are disjoint} \quad \Rightarrow \quad f(x) \notin C \cup W_x.$$

Here W_x is the x-th Σ^0_1 subset of ω.

Definition

A Σ^0_1 set $C \subseteq \omega$ is *complete for* Σ^0_1 if for every Σ^0_1 set $D \subseteq \omega$, there is some recursive $f : \omega \to \omega$ such that $x \in D \Leftrightarrow f(x) \in C$.

Theorem (Myhill, 1955)

Creative $\Leftrightarrow$ complete for Σ^0_1.

($\Rightarrow$) This uses the recursion theorem.

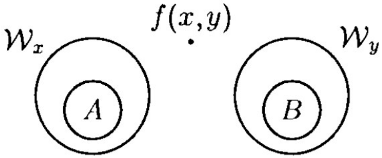
($\Leftarrow$) Creativeness is preserved under reduction.

Onward to pairs of subsets of ω

Definition (Kleene, 1950s)

A pair (A, B) of Σ^0_1 subsets of ω is *effectively inseparable* if there is some recursive $f : \omega^2 \to \omega$ such that for each $x, y \in \omega$:

$$W_x \supseteq A \text{ and } W_y \supseteq B \text{ are disjoint} \quad \Rightarrow \quad f(x, y) \notin W_x \cup W_y.$$



Effective inseparability effectivizes "(A, B) has no Δ^0_1 separating set":

- W_x and W_y together threaten to define a Δ^0_1 separating set;
- $f(x, y)$ is an effectively produced witness which defeats the pair (W_x, W_y).

Effectively inseparable $\Leftrightarrow$ complete for pairs

Just as creativeness is equivalent to completeness, effective inseparability is equivalent to the following notion:

Definition

A pair of disjoint Σ^0_1 sets (A, B) is *complete for Σ^0_1 pairs* if for every pair of disjoint Σ^0_1 sets (C, D), there is some recursive $f : \omega \to \omega$ such that

$$x \in C \;\Leftrightarrow\; f(x) \in A \qquad \text{and} \qquad x \in D \;\Leftrightarrow\; f(x) \in B.$$

Theorem (Smullyan, 1961)

Effectively inseparable $\Leftrightarrow$ complete for Σ^0_1 pairs.

($\Rightarrow$) This uses Smullyan's double recursion theorem.

($\Leftarrow$) Fix a pair of effectively inseparable sets. Effective inseparability is preserved under reduction.

Sets which separate complete pairs are complete

Corollary (folklore?)

Suppose that (A, B) is complete for Σ^0_1 pairs. If C is a Σ^0_1 set which separates A and B, then C is complete for Σ^0_1.
Similarly, if C is a Π^0_1 set which separates A and B, then C is complete for Π^0_1.

Proof of first statement.

By Smullyan's theorem, (A, B) is effectively inseparable.
This implies that C is creative: WLOG $A \subseteq C \subseteq \overline{B}$. Given $W_x \subseteq \overline{C}$, apply effective inseparability of (A, B) to the disjoint Σ^0_1 sets

$$C \supseteq A \quad \text{and} \quad W_x \cup B \supseteq B.$$

By Myhill's theorem, C is complete for Σ^0_1. □

I do not know a proof of this result which does not use the recursion theorem.

Π^1_1 subsets of ω

The goal now is to carry out the previous analysis in other settings. Let's start with Π^1_1 subsets of ω.

Definition (G.)

A Π^1_1 set $C \subseteq \omega$ is *creative for* Π^1_1 if there is some recursive $f : \omega \to \omega$ such that for each $x \in \omega$:

$$G_x \text{ and } C \text{ are disjoint} \quad \Rightarrow \quad f(x) \notin C \cup G_x.$$

Here G_x is the x-th Π^1_1 subset of ω.

A Π^1_1 set $C \subseteq \omega$ is *complete for* Π^1_1 if for every Π^1_1 set $D \subseteq \omega$, there is some recursive $f : \omega \to \omega$ such that $x \in D \Leftrightarrow f(x) \in C$.

Theorem (G.)

Creative for Π^1_1 $\Leftrightarrow$ complete for Π^1_1.

The main point here is that the Π^1_1 subsets of ω have a "good" parametrization.

Pairs of Π^1_1 subsets of ω

Similarly, one can define the notions:

- effectively inseparable for Π^1_1; and
- complete for Π^1_1 pairs

and prove that they are equivalent.

To prove ($\Uparrow$), one needs the existence of a pair of Π^1_1 subsets of ω which is effectively inseparable for Π^1_1.

Such a pair can be constructed using the fact that Π^1_1 subsets of ω have the reduction property.

The above equivalence is slightly inadequate for our intended applications. To see why, let's consider a pair of Π^1_1 sets which we hope to prove is complete.

A natural complete pair of Π^1_1 subsets of ω

wf: the set of indices for recursive well-founded trees on ω

hb: the set of indices for recursive trees on ω with some hyperarithmetic branch

Theorem (G.)

$(\mathrm{wf}, \mathrm{hb})$ is complete for Π^1_1 pairs.

Sketch.

Given disjoint Π^1_1 sets (A, B), fix reductions f and g from A and B to wf respectively. Given any $x \in \omega$, consider the recursive tree whose branches are strict-order-preserving embeddings from $T_{f(x)}$ to $T_{g(x)}$ (joined with Skolem functions).
If $x \in B$, then $T_{f(x)}$ is ill-founded and $T_{g(x)}$ is well-founded, so there is no strict-order-preserving embedding.
If $x \in A$, then $T_{f(x)}$ is well-founded and $T_{g(x)}$ is ill-founded, so there is a hyperarithmetic strict-order-preserving embedding. □

NB: This sketch does not address the case where $x \notin A \cup B$!

Semi-completeness for pairs

Definition

A pair (A, B) of Π^1_1 subsets of ω is *semi-complete for Π^1_1 pairs* if for any pair C, D of disjoint Π^1_1 subsets of ω, there is some total recursive function $f : \omega \to \omega$ such that

$$x \in C \Rightarrow f(x) \in A \qquad \text{and} \qquad x \in D \Rightarrow f(x) \in B.$$

The previous slide sketched a proof that $(\mathrm{wf}, \mathrm{hb})$ is semi-complete for Π^1_1 pairs. But this is equivalent to completeness:

Theorem (G.)

Let (A, B) be a pair of disjoint Π^1_1 subsets of ω. TFAE:

1. (A, B) is effectively inseparable for Π^1_1;
2. (A, B) is complete for Π^1_1;
3. (A, B) is semi-complete for Π^1_1.

The point behind (3) $\Rightarrow$ (1) is that a "semi-reduction" preserves effective inseparability. (Smullyan proved an analog of this for Σ^0_1.)

What about trees with a unique branch?

ub: the set of indices for recursive trees on ω which have a unique branch

We shall show that $(\mathrm{wf}, \mathrm{ub})$ is complete for Π^1_1 pairs.

$\mathrm{ub} \subset \mathrm{hb}$, so every separating set for $(\mathrm{wf}, \mathrm{hb})$ also separates $(\mathrm{wf}, \mathrm{ub})$. Therefore if $(\mathrm{wf}, \mathrm{ub})$ is inseparable, so is $(\mathrm{wf}, \mathrm{hb})$.

We will indirectly reduce $(\mathrm{wf}, \mathrm{hb})$ to $(\mathrm{wf}, \mathrm{ub})$ by considering Turing jump hierarchies along countable linear orderings.

Turing jump hierarchies are in fact my primary motivation for this work.

Turing jump hierarchies on countable linear orderings

Given any real, the (Turing) jump operator produces a more complicated real (wrt Turing reducibility).

Jump hierarchies can be obtained by iterating the jump operator along a countable well-ordering. At limit stages, one collects all previous results of the iteration using the (effective) join.

More generally, one can define what it means to be a jump hierarchy along any countable linear ordering. (There are a few ways to define this, but they are equivalent for our present purposes.)

There are jump hierarchies along ill-founded linear orderings because:

- "L supports some jump hierarchy" is a Σ^1_1 property of L
- "L is a well-ordering" is not Σ^1_1.

A question, answered by Harrington

Which linear orderings support a jump hierarchy? Specifically:

Question

Is the set of indices for recursive linear orderings which support a jump hierarchy complete for Σ^1_1?

The above question arose in my investigation of the computational strength of problems relating to jump hierarchies
(a bit on that later).

Theorem (Harrington 2017, unpublished)

Yes.

Harrington's proof was clever, ad hoc, and relied heavily on the recursion theorem.

The present work was directly inspired by Harrington's proof.

Well-orderings and linear orderings with hyperarithmetic descending sequences

In order to derive Harrington's result, we consider

- wo: the set of indices for recursive well-orderings;
- hds: the set of indices for recursive linear orderings with a hyperarithmetic descending sequence.

H. Friedman showed that

well-ordering ⇒ supports a jump hierarchy ⇒ no hyp desc. seq.

Theorem (G.)

$(\mathrm{wo}, \mathrm{hds})$ is complete for Π^1_1 pairs.

Proof.

The Kleene-Brouwer ordering reduces $(\mathrm{wf}, \mathrm{hb})$ to $(\mathrm{wo}, \mathrm{hds})$. □

Since any Σ^1_1 separating set for a complete Π^1_1 pair is complete for Σ^1_1, Harrington's result follows.

Returning to trees with a unique branch

Theorem (G.)

$(\mathrm{wf}, \mathrm{ub})$ is complete for Π^1_1 pairs.

Sketch.

We semi-reduce $(\mathrm{hds}, \mathrm{wo})$ to $(\mathrm{wf}, \mathrm{ub})$. Given an index for a recursive linear ordering L, consider the tree T of jump hierarchies along L (joined with minimal Skolem functions).

$$\begin{array}{rcl} L \text{ has a hyp desc. seq.} & \Rightarrow & T \text{ is well-founded (H. Friedman)} \\ L \text{ is well-ordered} & \Rightarrow & T \text{ has a unique branch.} \end{array}$$

□

Corollary (G.)

Any Σ^1_1 subset of ω which separates wf and ub is complete for Σ^1_1. (Likewise for Π^1_1.)

Applications of our results, I

Before moving on to subsets of ω^ω, we mention other applications of our results for Π^1_1 subsets of ω.

Our results imply that the set of indices for recursive linear orderings which are isomorphic to an ordinal in some ω-model of Kripke-Platek set theory is complete for Σ^1_1.

This answers a question of Knight, Turetsky, Weisshaar (ta).

Applications of our results, II

Our results have been used to calibrate the (uniform) computational strength of problems concerning jump hierarchies (Anglès d'Auriac, Kihara 2021; G., Pauly, Valenti 2021).

Example problems (stated informally):

- Given a well-ordering, produce the jump hierarchy along it
- Given a linear ordering, produce either a jump hierarchy along it or a descending sequence.
- Given a linear ordering, produce a jump hierarchy if it supports a jump hierarchy; otherwise produce a descending sequence.

All of these problems are analogs of the Arithmetical Transfinite Recursion principle from reverse mathematics.

How about coanalytic subsets of ω^ω?

WF: the set of well-founded trees on ω

UB: the set of trees on ω with a unique branch

Theorem (Saint Raymond, 2007)

For any pair (A, B) of disjoint coanalytic subsets of ω^ω, there is a continuous function $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(\mathrm{WF}) = A$ and $f^{-1}(\mathrm{UB}) = B$.

If we relativize the definitions of creativeness, completeness, and effective inseparability appropriately, we can obtain an effective version of the above:

Theorem (G.)

Suppose $\gamma \in \omega^\omega$. For any pair (A, B) of disjoint $\Pi^1_1(\gamma)$ subsets of ω^ω, there is a γ-recursive function $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(\mathrm{WF}) = A$ and $f^{-1}(\mathrm{UB}) = B$.

Using our result (or Saint Raymond's), we obtain completeness results for subsets of ω^ω, including the theorem on slide 1.

How about other pointclasses?

Let $\mathcal{X}$ denote either ω or ω^ω. Suppose $\Gamma \restriction \mathcal{X}$ is $\mathcal{X}$-parametrized in a Λ-recursive way. Let G_x denote the x-th subset of $\mathcal{X}$ in Γ. Example definition:

Definition (G.)

$C \subseteq \mathcal{X}$ is Λ-*creative for* Γ if it lies in Γ and there is a Λ-recursive function $f : \mathcal{X} \to \mathcal{X}$ such that for every $x \in \mathcal{X}$, if G_x and C are disjoint, then $f(x) \notin C \cup G_x$.

$$\begin{aligned} \Lambda\text{-creative for } \Gamma \quad &\Leftrightarrow \quad \Lambda\text{-complete for } \Gamma \\ \Lambda\text{-effectively inseparable for } \Gamma \quad &\Leftrightarrow \quad \Lambda\text{-complete for } \Gamma \text{ pairs} \end{aligned}$$

hold assuming:

1. the recursion theorem for Λ-recursive functions
2. the class of Λ-recursive functions is closed under composition
3. Γ satisfies the reduction property
4. Γ is closed under preimages by Λ-recursive functions
5. other mild closure properties.

Summary

- We develop a theory of effectively inseparable sets for a variety of pointclasses, along the lines of the classical development for Σ^0_1 subsets of ω.
- This provides bridges between different notions of completeness for pairs and completeness for separating sets.
- We construct natural pairs of effectively inseparable Π^1_1 subsets of ω, such as $(\mathrm{wo}, \mathrm{hds})$ and $(\mathrm{wf}, \mathrm{ub})$.
- With appropriate relativizations, we can do the same for coanalytic subsets of ω^ω.
- Applications of our results:
  - completeness of various sets
  - an effective version of a theorem from descriptive set theory
  - calibrating the uniform computational strength of problems.

Thanks!