Computing descending sequences in linear orderings

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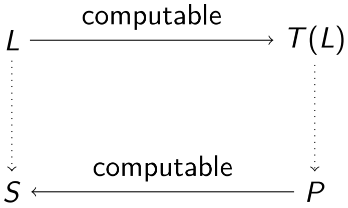
How hard is it to compute an infinite descending sequence in an ill-founded linear ordering L?

This problem reduces to the problem of computing a path on an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$:

- Given L, compute the tree $T(L)$ of all finite descending sequences in L;
- Every path P on $T(L)$ computes an infinite descending sequence S in L.

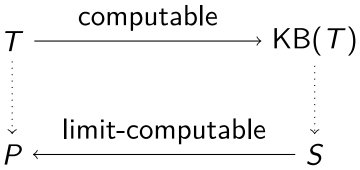


Going in the opposite direction

- Given a tree T, we can compute its Kleene-Brouwer ordering $\mathrm{KB}(T)$, defined by $\sigma \leq_{\mathrm{KB}} \tau$ iff σ extends τ or is lexicographically below τ.
- $\mathrm{KB}(T)$ is ill-founded if and only if T is ill-founded.
- Given a descending sequence $(\sigma_i)_{i\in\mathbb{N}}$ in $\mathrm{KB}(T)$,

$$P(n) = \lim_{i\to\infty} \sigma_i(n)$$

is a path on T.



Can we do better?

Weihrauch reducibility: represented spaces

If each object in a space X can be "encoded" as a real, then we can make it into a represented space, thereby transferring notions of computability from $\mathbb{N}^\mathbb{N}$ to X.

Formally, a represented space is a pair (X, δ) where $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$ is a (possibly partial) surjection.

Each element of X is named by some (possibly multiple) $p \in \mathbb{N}^\mathbb{N}$ via δ.

Examples:

$$\mathbb{N}^\mathbb{N}, \quad \mathbb{N}, \quad \mathrm{LO}, \quad \mathrm{Tr}, \quad \Pi^1_1(\mathbb{N}), \quad \Sigma^1_1(\mathrm{LO})$$

All of the above spaces can be represented in standard ways.

Weihrauch reducibility: problems

Examples of problems

DS: given an ill-founded linear ordering, produce any infinite descending sequence

$C_{\mathbb{N}^\mathbb{N}}$: given an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$, produce any path

lim: given a convergent sequence of reals, produce its limit

Formally, a problem $f :\subseteq X \rightrightarrows Y$ is a (possibly partial) multivalued function between represented spaces.

We also think of a problem as a set of instance-solution pairs:

- If $x \in \mathrm{dom}(f)$ then we say that x is an f-instance.
- For each f-instance x, the set of f-solutions to x is $f(x) \subseteq Y$.

Definition

A problem f is Weihrauch reducible to a problem g ($f \leq_W g$) if there are computable functions $\Phi, \Psi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that:

- if p is a name for an f-instance, then $\Phi(p)$ is a name for a g-instance;
- if p is a name for an f-instance and q is a name for a g-solution to $\Phi(p)$, then $\Psi(p, q)$ is a name for an f-solution to p.

$$
\begin{array}{ccc}
f\text{-instance } p & \xrightarrow{\Phi(\cdot)} & g\text{-instance } \Phi(p) \\
\vdots & & \vdots \\
f\text{-solution } \Psi(p, q) & \xleftarrow{\Psi(p, \cdot)} & g\text{-solution } q
\end{array}
$$

We call Φ and Ψ forward and backward functionals respectively.

Relationships between $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and DS

Our first reduction shows that $\mathsf{DS} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Our second reduction is not a Weihrauch reduction from $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to DS because our paths are obtained by applying the limit to descending sequences in $\mathrm{KB}(T)$. Nonetheless:

Proposition

$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \lim * \mathsf{DS}$.

$f * g$ is the compositional product (Brattka, Gherardi, Marcone), which captures what can be achieved by first applying g, followed by some computation, and then applying f.

Question (which we will answer)

Do we have $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{DS}$? Equivalently, does $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{DS}$?

Our results: $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{DS}$ (and more)

We show that DS is quite weak in terms of uniform computational strength.

Theorem (G., Pauly, Valenti)

A single-valued problem is Weihrauch reducible to DS if and only if it is Weihrauch reducible to lim, i.e.,

$$\max_{\leq_{\mathrm{W}}}\{f_0 :\subseteq Z \to \mathbb{N}^\mathbb{N} \mid f_0 \leq_{\mathrm{W}} \mathsf{DS}\} \equiv_{\mathrm{W}} \lim.$$

Corollary

The following problems are not Weihrauch reducible to DS: LPO', ADS, $\lim'$, $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Open question

Is KL (König's lemma) Weihrauch reducible to DS?

Our results: first-order part of DS

Our techniques characterize the problems which have codomain $\mathbb{N}$ and are reducible to DS:

Definition

Let Π^1_1-Bound $:\subseteq \Pi^1_1(\mathbb{N}) \rightrightarrows \mathbb{N}$ be the following problem: given a Π^1_1-code for a finite subset of $\mathbb{N}$, produce a bound.

Theorem (G., Pauly, Valenti)

$\max_{\leq_W}\{f_0 :\subseteq Z \rightrightarrows \mathbb{N} \mid f_0 \leq_W \mathsf{DS}\} \equiv_W \Pi^1_1\text{-Bound}.$

Dzhafarov, Solomon, Yokoyama (ta) were the first to define and study the first-order part ${}^1f$ of an arbitrary problem f:

$$^1f \equiv_W \max_{\leq_W}\{f_0 :\subseteq Z \rightrightarrows \mathbb{N} \mid f_0 \leq_W f\}.$$

Proof that ${}^1\mathsf{DS} \leq_{\mathrm{W}} \Pi^1_1$-Bound

Suppose $f :\subseteq Z \rightrightarrows \mathbb{N}$ reduces to DS. Given an f-instance p, we can find an f-solution to p as follows.

1. At stage s, we can compute a finite piece L_s of the DS-instance defined by the forward functional.
2. List all descending sequences in L_s on which the backward functional converges (and hence gives a potential f-solution).
3. If such descending sequences exist, we can guess an f-solution by picking the L_s-rightmost descending sequence F_s.
4. The set of s such that F_s is undefined or not extendible is $\Pi^{1,p}_1$.
5. Apply Π^1_1-Bound to obtain an s such that F_s is extendible. This yields an f-solution to p.

Hence $f \leq_{\mathrm{W}} \Pi^1_1$-Bound.

Π^1_1-Bound and Σ^1_1 choice principles

Σ^1_1-$\mathsf{C}_{\mathbb{N}}$: given a Σ^1_1-code for a nonempty subset of $\mathbb{N}$, produce an element of the set.

Σ^1_1-$\mathsf{C}_{\mathbb{N}}^{\mathrm{cof}}$: given a Σ^1_1-code for a cofinite subset of $\mathbb{N}$, produce an element of the set.

It is easy to see that Σ^1_1-$\mathsf{C}_{\mathbb{N}}^{\mathrm{cof}}$ is Weihrauch equivalent to Π^1_1-Bound.

Theorem (Angles d'Auriac, Kihara ta)

$\widehat{\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}}^{\mathrm{cof}}} <_{\mathrm{W}} \widehat{\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}}}$, hence Σ^1_1-$\mathsf{C}_{\mathbb{N}}^{\mathrm{cof}} <_{\mathrm{W}} \Sigma^1_1$-$\mathsf{C}_{\mathbb{N}}$.

It is easy to see that the first-order part of $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is Σ^1_1-$\mathsf{C}_{\mathbb{N}}$, so this theorem and our results imply that DS and $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ can be separated by considering their first-order parts.

In fact our results imply that there is a single-valued problem with codomain 2 which separates DS and $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$, namely LPO$'$.

More about $\widehat{\Sigma^1_1\text{-C}_\mathbb{N}}$

Kihara, Marcone, Pauly asked if $\widehat{\Sigma^1_1\text{-C}_\mathbb{N}} <_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Theorem (Angles d'Auriac, Kihara ta)

$\widehat{\Sigma^1_1\text{-C}_\mathbb{N}} <_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. In fact $\mathsf{ATR}_2 \not\leq_W \widehat{\Sigma^1_1\text{-C}_\mathbb{N}}$.

AK proved the above separation using a pair of inseparable Π^1_1 sets.

We will extend their techniques to prove a stronger result about a strengthening of DS.

Definition (G.)

Let $\mathsf{ATR}_2 : \mathrm{LO} \rightrightarrows \mathbb{N}^\mathbb{N}$ be the following problem: given a linear ordering L, produce either an infinite descending sequence in L or a jump hierarchy on L (with a bit indicating which type of solution we produce).

In fact it suffices to consider ATR_2 restricted to computable linear orderings.

ATR_2 and a strengthening of DS

Our earlier results imply that $\mathsf{ATR}_2 \not\leq_{\mathrm{W}} \mathsf{DS}$, but much more is true:

Definition

Let Σ^1_1-DS $:\subseteq \Sigma^1_1(\mathrm{LO}) \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the following problem: given a Σ^1_1-code for an ill-founded linear ordering, produce an infinite descending sequence.

Theorem (G., Pauly, Valenti)

$\mathsf{ATR}_2 \not\leq_{\mathrm{W}} \Sigma^1_1$-DS, hence Σ^1_1-DS $<_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

This means that $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ does not reduce to DS even if we allow the forward functional to be Σ^1_1 rather than computable.

On the other hand, we saw earlier that if we allow the backward functional to be limit-computable, then $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ reduces to DS.

ATR_2 and inseparable Π^1_1 sets

We can think of ATR_2 as a "union" of DS and

JH: given a linear ordering which supports a jump hierarchy, produce a jump hierarchy.

It is well known that the set of indices of ill-founded linear orderings is Σ^1_1-complete.

Harrington (unpublished) showed that the set of indices of linear orderings which support a jump hierarchy is also Σ^1_1-complete.

Theorem (G., generalizing Harrington's proof)

Any Σ^1_1 set which separates wf and hds is Σ^1_1-complete.
(wf and hds are the set of indices for well-founded linear orderings and linear orderings with hyp descending sequences respectively.)

This generalizes Harrington's result because if a computable linear ordering has a hyp descending sequence, then it does not support a jump hierarchy (Friedman).

ATR_2 and inseparable Π^1_1 sets

Theorem (G., generalizing Harrington's proof)

Any Σ^1_1 set which separates wf and hds is Σ^1_1-complete.
(wf and hds are the set of indices for well-founded linear orderings and linear orderings with hyp descending sequences respectively.)

By Σ^1_1-separation,

Corollary

wf and hds cannot be separated by disjoint Σ^1_1 sets.

Angles d'Auriac, Kihara used the corollary to prove that $\mathsf{ATR}_2 \not\leq_{\mathrm{W}} \widehat{\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}}}$.

We will use the corollary to prove that $\mathsf{ATR}_2 \not\leq_{\mathrm{W}} \Sigma^1_1\text{-}\mathsf{DS}$.

Proof that $\mathsf{ATR}_2 \not\leq_{\mathrm{W}} \Sigma^1_1\text{-}\mathsf{DS}$ (G., Pauly, Valenti)

Suppose that $\mathsf{ATR}_2 \leq_{\mathrm{W}} \Sigma^1_1\text{-}\mathsf{DS}$. For each computable linear ordering L_e, the forward functional produces a Σ^1_1-code $\Phi(L_e)$ for an ill-founded linear ordering.

For the same e, the backward functional may produce either descending sequences in L_e or jump hierarchies on L_e, depending on which descending sequence in $\Sigma^1_1\text{-}\mathsf{DS}(\Phi(L_e))$ is given. However, descending sequences are sufficiently homogeneous so

Lemma

For each e, either $\mathsf{DS}(L_e)$ or $\mathsf{JH}(L_e)$ is Muchnik reducible to $\Sigma^1_1\text{-}\mathsf{DS}(\Phi(L_e))$.

Then we have disjoint Σ^1_1 sets which separate wf and hds:

$$\mathrm{wf} \subseteq \{e \in \mathbb{N} : \mathsf{DS}(L_e) \text{ is not Muchnik reducible to } \Sigma^1_1\text{-}\mathsf{DS}(\Phi(L_e))\}$$
$$\mathrm{hds} \subseteq \{e \in \mathbb{N} : \mathsf{JH}(L_e) \text{ is not Muchnik reducible to } \Sigma^1_1\text{-}\mathsf{DS}(\Phi(L_e))\}.$$

This contradicts the corollary on the previous slide. Thank you!