Compositions of multi-valued functions

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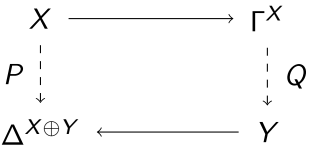
In order to answer such questions, we need to formalize statements such as

> *"P can be solved by invoking Q_0, and then invoking Q_1, ..., and finally invoking Q_{n-1}."*

Weihrauch reducibility among multi-valued functions

$P \leq_W Q$ if there are Turing functionals Γ and Δ such that for every P-instance X,

- Γ^X is a Q-instance;
- for every Q-solution Y to Γ^X, $\Delta^{X \oplus Y}$ is a P-solution to X.



This formalizes the statement "P can be solved by invoking Q once (uniformly)."

Composition

X is a $(Q_1 \circ Q_0)$-instance if

- X is a Q_0-instance;
- every Q_0-solution Y to X is itself a Q_1-instance.

Z is a $(Q_1 \circ Q_0)$-solution to X if there is Y such that

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- Z is a Q_1-solution to Y.

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$P \leq_W Q_1 \circ Q_0$ is too weak! We should at least allow ourselves to computably transform the Q_0-solution Y into some Q_1-instance.

$\bullet_\Theta$ (Dorais, Dzhafarov, Hirst, Mileti, Shafer '15)

X is a $(Q_1 \bullet_\Theta Q_0)$-instance if

- X is a Q_0-instance;
- for every Q_0-solution Y to X, $\Theta^{X \oplus Y}$ is a Q_1-instance.

A $(Q_1 \bullet_\Theta Q_0)$-solution to X is a pair (Y, Z) such that

- Y is a Q_0-solution to X;
- Z is a Q_1-solution to $\Theta^{X \oplus Y}$.

$$X \overset{Q_0}{\dashrightarrow} Y \longrightarrow \Theta^{X \oplus Y} \overset{Q_1}{\dashrightarrow} Z$$

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$P \leq_W Q_1 \bullet_\Theta Q_0$ is too weak! Θ cannot access the original P-instance, only the Q_0-instance computed from it.

Reduction games (Hirschfeldt, Jockusch '16)

$P \leq_{gW}^2 Q$ if there is a computable winning strategy Φ for Player II in the following game:

Player I	Player II

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Round 1	P-instance X_0	

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Round 1		
		P-solution Φ^{X_0} win!

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	Q-solution X_1		
Round 2			

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Round 2			
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	Q-solution X_2		
Round 3			

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Round 3			
			P-solution $\Phi^{X_0 \oplus X_1 \oplus X_2}$ win!

Fine print: Φ has to specify whether it is providing a P-solution or a Q-instance.

Compositional product (Brattka, Gherardi, Marcone '11)

The composition of multifunctions is weak, but we can take the sup of the composition of all Weihrauch equivalent multifunctions.

If $\mathbf{q_0}$ and $\mathbf{q_1}$ are Weihrauch degrees, define

$$\mathbf{q_1} \star \mathbf{q_0} = \sup\{Q_1 \circ Q_0 : Q_0 \equiv_W \mathbf{q_0}, Q_1 \equiv_W \mathbf{q_1}\}.$$

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Brattka and Pauly ('16) showed that:

- the above sup exists and is realized;
- $\star$ is associative;
- $\star$ is monotone in both components.

Composing a multi-valued function with itself

Theorem

For all P and Q, TFAE:

1. $P \leq_W Q \star Q$;
2. there is a strategy for II witnessing that $P \leq^2_{gW} Q$, which always wins in round 3;
3. there is a functional Θ such that $P \leq_W Q \bullet_\Theta \overline{Q}$, where $\overline{Q}$ is defined by $\overline{Q}(A, X) = Y$ whenever $Q(X) = Y$.

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Corollaries

For all Q, there is a functional Θ such that $Q \star Q \equiv_W Q \bullet_\Theta \overline{Q}$.

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For all Q,

$$\sup_\Lambda Q \bullet_\Lambda \overline{Q} \equiv_W Q \star Q \leq_{gW}^2 Q.$$

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For all Q which have computable instances,

$$\sup_\Lambda Q \bullet_\Lambda \overline{Q} \equiv_W Q \star Q \equiv_W \sup\{P : P \leq_{gW}^2 Q\}.$$

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Sketch of proof of theorem.

(1) $\Rightarrow$ (2): take $Q_0, Q_1 \leq_W Q$ such that $P \leq_W Q_1 \circ Q_0$, then use that to define a strategy.
(2) $\Rightarrow$ (3): encode the original P-instance in the $\overline{Q}$-instance. Θ follows the strategy.
(3) $\Rightarrow$ (1): straightforward. $\square$

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Theorem

For all P and Q, TFAE:

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Questions

- In (3), can we consider $Q \bullet_\Theta Q$ instead of $Q \bullet_\Theta \overline{Q}$?
- In (2), can we consider $P \leq^2_{gW} Q$, without the condition that the strategy wins in the last round?

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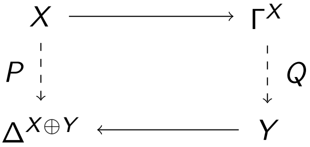
- In (2), can we consider $P \leq^2_{gW} Q$, without the condition that the strategy wins in the last round? Not unless Q has computable instances, but...

Back to Weihrauch reducibility

In a Weihrauch reduction from P to Q, given a P-instance, one must compute a Q-instance even if one could already solve said P-instance!



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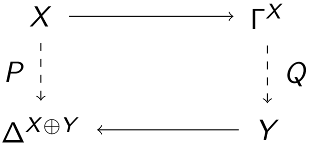

$P \leq^1_{gW} Q$ has more flexibility: given a P-instance, Player II can either choose to solve it directly, or compute a Q-instance for Player I to solve.

The $\equiv^1_{gW}$ lattice

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$\leq^1_{gW}$ is reflexive and transitive. The $\equiv^1_{gW}$-degrees form a lattice with the usual join and meet operations.

Q: How does the $\equiv^1_{gW}$-lattice compare with the Weihrauch lattice?

Summary

We studied three formalizations of “one can solve P by uniformly invoking Q twice in series”:

- the compositional product $Q \star Q$;
- the reduction game $P \leq^2_{gW} Q$;
- the step product $Q \bullet_\ominus Q$.

Our results:

- For those Q that arise from mathematical theorems, the first two are equivalent.
- The third is weaker than the first two, but can be made equivalent with a simple modification.