Some reductions between theorems around ATR

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Some theorems

- any two well-orderings are strongly comparable, i.e., one is isomorphic to an initial segment of the other
- any tree with uncountably many paths has a perfect subtree
- in every open game, one of the players has a winning strategy

In reverse math, these are equivalent to arithmetic transfinite recursion (ATR) (Friedman '76, Steel '76).

These equivalences elide significant differences in their computational content.

A different, usually finer, lens

Instead of provability, one could study the computational content of theorems using computable reductions:

> *Given an instance X of a problem* P*, can we compute an instance Y of problem* Q *such that any solution to Y, together with X, computes a solution to X?*

Many proofs in reverse math can be directly translated into such reductions. Exceptions include proofs which invoke their premise multiple times.

Some proofs have complicated case divisions. In order to calibrate how hard those case divisions have to be, one could consider *uniform* reductions.

Reductions (and the lack thereof) reveal computational content in theorems and the relationships between them!

Theorems as problems

Many theorems have the form

$$(\forall X)[\varphi(X) \to \exists Y \theta(X, Y)].$$

These theorems can be regarded as *problems*, with

- *instances* being those X which satisfy φ;
- *solutions to* X being those Y such that $\theta(X, Y)$ holds.

Example:

- instances are pairs of well-orderings;
- solutions are isomorphisms from one well-ordering onto an initial segment of the other.

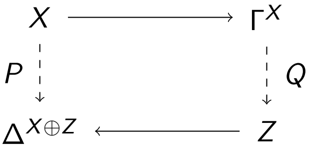
Weihrauch (uniform) and computable reducibility

Let P and Q be problems.

Definition

$P \leq_W Q$ if there are Turing functionals Γ (forward) and Δ (backward) such that for every P-instance X,

1. Γ^X is a Q-instance;
2. for every Q-solution Z to Γ^X, $\Delta^{X \oplus Z}$ is a P-solution to X.



$P \leq_c Q$ if every P-instance X computes a Q-instance Y such that for every Q-solution Z to Y, $X \oplus Z$ computes a P-solution to X.

A subset of what is known

$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: given $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which has a path, produce any path

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$: given $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which has a unique path, produce said path

PTT: given $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which has uncountably many paths, produce a perfect subtree

CWO: given a pair of well-orderings, produce an embedding from one onto an initial segment of the other

WCWO: given a pair of well-orderings, produce an embedding from one into the other

Marcone (to appear?) showed that

$$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PTT}$$
$$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{CWO},$$

and asked if WCWO $\equiv_W$ CWO. We show that the answer is yes.

An ATR-like problem (similar to Pauly's lim†)

ATR: given a well-ordering L and a set A, produce the jump hierarchy $\langle X_a \rangle_{a \in L}$ on L which starts with A

ATR is robust (wrt Weihrauch reducibility) in a few ways. For example, we can ask for a hierarchy constructed by iterating an arithmetical formula, not just the jump.

Theorem (G.)

CWO $\leq_W$ ATR $\leq_W$ WCWO, so CWO $\equiv_W$ WCWO.

ATR $\leq_W$ WCWO uses a theorem of Chen '76, which was used by Shore '93 to study the reverse math of versions of Fraïssé's conjecture.

Two-sided ATR

Being an ATR-instance (well-ordering) is Π^1_1, so its failure is witnessed by a real (infinite descending sequence). We can ask for some such real!

Definition

ATR_2: given a linear ordering L and a set A, produce either a jump hierarchy on L which starts with A, or an infinite L-descending sequence

$\langle Y_a \rangle_{a \in L}$ is a *jump hierarchy on L* if for all b, $Y_b = \bigoplus_{a <_L b} Y'_a$.

- There is a recursive ATR_2-instance with no hyperarithmetic solution, so $\mathrm{ATR}_2 \not\leq_c \mathrm{ATR}$.
- Since some ill-founded linear orderings support jump hierarchies, ATR_2 does not (obviously) decide whether L is well-founded!

The König duality theorem (Podewski, Steffens '76)

matching: a set of vertex-disjoint edges in a graph

cover: a set of vertices such that every edge has at least one endpoint in the cover

KDT: given a bipartite graph, produce a matching M and a cover C such that C contains exactly one vertex from each edge in M

In reverse math, ATR is equivalent to KDT (Aharoni, Magidor, Shore '92, Simpson '94)

Theorem (G.)

- ATR $\leq_W$ KDT;
- $\mathrm{ATR}_2 \leq_c$ KDT.
  The forward reduction is uniform and the backward reduction is uniform in the jump of the KDT-solution.

Conclusions

- Many theorems around ATR, including WCWO, are Weihrauch equivalent to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ or $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$
- Some could be strictly between $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$
  - $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{ATR}_2 \leq_c \mathsf{KDT} \leq_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$
- Some are strictly above $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$
  - open determinacy, two-sided perfect tree theorem (Pauly, Kihara to appear?)
- Some could be strictly below $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$
  - Fraïssé's conjecture restricted to well-orderings
- Are any incomparable with $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ or $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$?

Thank you!