Π_1^0 classes relative to an enumeration oracle

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Enumeration reducibility: computing with positive information

Definition (Friedberg, Rogers 1959)

For $A, B \subseteq \omega$, we say A is e-reducible to B ($A \leq_{\mathrm{e}} B$) if there is a c.e. set W such that

$$n \in A \qquad \Leftrightarrow \qquad \exists \text{ finite } D(\langle n, D\rangle \in W \wedge D \subseteq B).$$

We think of W as an e-operator Γ, with $\Gamma(B) = A$.

A function f on ω is an enumeration of A if the range of f is A.

Theorem (Selman 1971)

(For nonempty A,) $A \leq_{\mathrm{e}} B$ if and only if every enumeration of B computes some enumeration of A.

Facts about $\leq_{\mathrm{e}}$ and e-operators

A set is c.e. if and only if it is e-reducible to $\emptyset$.

A and $\overline{A}$ need not be comparable under $\leq_{\mathrm{e}}$.

A is c.e. in B if and only if $A \leq_{\mathrm{e}} B \oplus \overline{B}$.

$A \leq_{\mathrm{T}} B$ if and only if $A \oplus \overline{A} \leq_{\mathrm{e}} B \oplus \overline{B}$.

$\leq_{\mathrm{e}}$ is reflexive and transitive so we can define the structure of the e-degrees from $\leq_{\mathrm{e}}$ in the same way that the structure of the Turing degrees is defined from $\leq_{\mathrm{T}}$.

The Turing degrees embed into the e-degrees via $A \mapsto A \oplus \overline{A}$.

If Γ is an e-operator and $B \subseteq C$, then $\Gamma(B) \subseteq \Gamma(C)$.

Some motivations for studying enumeration reducibility

It is a natural way for modeling computation with partial functions.

It forms a broader framework for measuring the relative complexity of mathematical objects (e.g., Miller 2004).

Unlike the Turing degrees, the e-degrees have several “natural” subclasses which are definable using a “natural” first-order formula in the language of partial orders (e.g., Kalimullin 2003).

Any nontrivial automorphism of the e-degrees induces a nontrivial automorphism of the Turing degrees (Cai, Ganchev, Lempp, Miller, Soskova 2016).

Subclasses of the e-degrees

The total degrees are the image of the Turing degrees under the embedding $A \mapsto A \oplus \overline{A}$. Equivalently, they are the degrees with a representative A which satisfies $\overline{A} \leq_{\mathrm{e}} A$.

The continuous degrees are, roughly speaking, the degrees of continuous functions on $[0, 1]$.

The cototal degrees are those with a representative A which satisfies $A \leq_{\mathrm{e}} \overline{A}$.

Theorem (Miller 2004, Andrews, Ganchev, Kuyper, Lempp, Miller, Soskova, Soskova 2019)

$$\text{Total} \quad \subsetneq \quad \text{continuous} \quad \subsetneq \quad \text{cototal} \quad \subsetneq \quad \text{all e-degrees.}$$

Defining subclasses by lifting notions from the Turing degrees

Given a property true of all Turing degrees, we may:

- Relativize to enumeration degrees, often by replacing "c.e. in" with "$\leq_e$".
- Then, consider the subclass of all e-degrees which satisfy the (relativized) property.

The resulting subclass of e-degrees contains the total degrees (strictly, usually).

We study several such subclasses which are defined by considering properties of Π^0_1 classes and the relation "PA above".

Definition (Miller, Soskova)

For $X \subseteq \omega$, a $\Sigma_1^0\langle X\rangle$-class $U \subseteq 2^\omega$ is a union of cones $[\sigma]$, where the $\sigma \in 2^{<\omega}$ come from some $W \leq_{\mathrm{e}} X$, i.e.,

$$U = \{Y \in 2^\omega : (\exists \sigma \in W)(\sigma \prec Y)\}.$$

A $\Pi_1^0\langle X\rangle$-class is one whose complement is a $\Sigma_1^0\langle X\rangle$-class.

Examples:

- Every $\Pi_1^0(X)$-class is a $\Pi_1^0\langle X \oplus \overline{X}\rangle$-class.
- If $A, B \leq_{\mathrm{e}} X$, then the set of separators of A and B form a $\Pi_1^0\langle X\rangle$-class. We call this a separating $\Pi_1^0\langle X\rangle$-class.

Technicality: We view elements of a $\Pi_1^0\langle X\rangle$-class as total objects.

Degrees with a universal class

Definition

Let $P\langle X\rangle$ be a nonempty $\Pi^0_1\langle X\rangle$-class. $P\langle X\rangle$ is a universal class for X if for every nonempty $\Pi^0_1\langle X\rangle$-class $Q\langle X\rangle$, there is some Turing functional Φ such that for every $A \in P\langle X\rangle$, we have $\Phi^A \in Q\langle X\rangle$.

If X is total, then X has a universal class, e.g., the class of DNC_2^X functions.

Furthermore:

Theorem (Ganchev, Kalimullin, Miller, Soskova 2020)

Every continuous degree has a universal class.

Are there other degrees which have a universal class?

Another way to have a universal class: Low for PA

Recall that B is said to have PA degree if B computes a member of every nonempty Π^0_1 class.

Definition

X is low for PA if whenever B has PA degree, then B computes a member of every nonempty $\Pi^0_1\langle X\rangle$-class P.

Theorem (GKMS)

- X is low for PA if and only if every nonempty $\Pi^0_1\langle X\rangle$-class contains a nonempty Π^0_1 class.
- If X is low for PA, then DNC_2 is a universal class for X.
- 1-generics are low for PA.

Theorem (Miller, Soskova)

If X is low for PA, then X is **not** continuous (unless X is c.e.)

Another way to have a universal class: Reduction property

Definition

X has the reduction property if for all pairs of sets $A, B \leq_{\mathrm{e}} X$, there are sets $A_0, B_0 \leq_{\mathrm{e}} X$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \emptyset$, and $A_0 \cup B_0 = A \cup B$.

If X is total, then it is easy to see that it has the reduction property.

Theorem (GKMS)

If X has the reduction property, then it has a universal class.

The reduction property (for e-ideals) was first studied by Kalimullin and Puzarenko (2004): Their results (and others) imply that the reduction property is incomparable with being low for PA, or being continuous.

A related notion? Universal functions

Definition

X has a universal function if there is a partial function U with $G_U \leq_{\mathrm{e}} X$ such that if φ is a partial function with $G_\varphi \leq_{\mathrm{e}} X$, then $\varphi = \lambda x.U(e,x)$ for some $e \in \omega$.

Theorem (GKMS)

If X has a universal class, then it has a universal function. The converse is false.

To prove the implication, we prove that X has a universal function if and only if there is a $\Pi^0_1\langle X\rangle$-class P which is universal for separating $\Pi^0_1\langle X\rangle$-classes, i.e., for every **separating** $\Pi^0_1\langle X\rangle$-class Q, there is some Φ such that for every $A \in P$, we have $\Phi^A \in Q$.

Generics which do not have a universal class

We work with subtrees of

$$f^{<\omega} := \{\sigma \in \omega^{<\omega} : (\forall n < |\sigma|)[\sigma(n) < 2^n]\}.$$

A forcing condition is a pair $\langle T, \varepsilon\rangle$, where:

- T is a finite tree where all leaves have the same height $|T|$
- ε is a rational number in $(0, 1)$.

$\langle S, \delta\rangle$ extends $\langle T, \varepsilon\rangle$ if:

- S adds no new strings of length $\leq |T|$
- Every $\sigma \in S$ with $|T| \leq |\sigma| < |S|$ has many immediate successors in S, specifically at least $1 - \varepsilon$ in proportion
- $\delta \leq \varepsilon$

A generic object will be an infinite subtree G of $f^{<\omega}$ with no leaves.

A generic object will be an infinite subtree G of $f^{<\omega}$ with no leaves.

We view its complement $A_G := f^{<\omega} \backslash G$ as an enumeration oracle. Then $[G]$ is a $\Pi^0_1\langle A_G \rangle$-class.

Later we will consider, for certain $\sigma \in G$, the subtree

$$G \backslash [\sigma]^{\preceq} := G - \{\tau : \tau \succeq \sigma\}.$$

Note

$$G \backslash [\sigma]^{\preceq} \subseteq G$$

$$A_{G \backslash [\sigma]^{\preceq}} \supseteq A_G$$

and so for any e-operator Γ,

$$\Gamma(A_{G \backslash [\sigma]^{\preceq}}) \supseteq \Gamma(A_G).$$

Therefore for any $\Pi^0_1\langle \cdot \rangle$-class P,

$$P\langle A_{G \backslash [\sigma]^{\preceq}} \rangle \subseteq P\langle A_G \rangle.$$

Lemma (combinatorial)

If $\langle S_0, \delta_0 \rangle$ and $\langle S_1, \delta_1 \rangle$ both extend $\langle T, \varepsilon/2 \rangle$, then $\langle S_0 \cap S_1, \varepsilon \rangle$ is a condition which extends $\langle T, \varepsilon \rangle$.

Lemma

If $G_\varphi \leq_{\mathrm{e}} A_G$, then $\{n : \varphi(n) = 0\}$ and $\{n : \varphi(n) = 1\}$ are separated by a pair of disjoint c.e. sets.

Sketch of proof.

Fix e-operators Γ_0 and Γ_1 such that $\Gamma_i(A_G) = \{n : \varphi(n) = i\}$.

Fix a condition $\langle T, \varepsilon \rangle$ in the generic filter which forces that $\Gamma_0(A_G)$ and $\Gamma_1(A_G)$ are disjoint.

There is a condition $\langle T', \varepsilon' \rangle$ in the generic filter which extends $\langle T, \varepsilon \rangle$ and satisfies $\varepsilon' \leq \varepsilon/2$. Then define

$$C_i = \{n : \exists \langle S, \delta \rangle \leq \langle T', \varepsilon' \rangle (n \in \Gamma_i(A_S))\},$$

where A_S is the set of strings in $f^{<\omega}$ of height $\leq |S|$ which are not in S. □

Theorem (GKMS)

A_G has a universal function.

Proof.

Define $U(\langle e, i\rangle, x) = y$ if

1. $\langle x, y\rangle \in \Gamma_e(A_G)$, and
2. there is a level n and a stage s such that for every $\sigma \in 2^{<\omega}$ of length n, either
   - we see that σ is not an initial segment of a DNC_2 function, or
   - $\Phi^{\sigma}_{i,s}(x) \downarrow = y$.

U is a partial function and $G_U \leq_{\mathrm{e}} A_G$.

Suppose $G_\varphi = \Gamma_e(A_G)$. Then there is some i such that if X is a DNC_2 function, Φ^X_i is total and separates $\{n : \varphi(n) = 0\}$ and $\{n : \varphi(n) = 1\}$. (Such i exists by the previous lemma.)

By compactness of 2^ω, we have $\varphi = \lambda x.U(\langle e, i\rangle, x)$. □

Our next goal:

Theorem (GKMS)

A_G has no universal class, i.e.,
for every nonempty $\Pi^0_1\langle A_G\rangle$-class $P\langle A_G\rangle$,
there is a nonempty $\Pi^0_1\langle A_G\rangle$-class $Q\langle A_G\rangle$ such that
for every Turing functional Φ, there is some $X \in P\langle A_G\rangle$ with
$\Phi^X \notin Q\langle A_G\rangle$.

Tension: If, in the construction of G, we omit certain strings in order to construct a "small" $Q\langle A_G\rangle$, then we might make $P\langle A_G\rangle$ smaller too, making it harder to find $X \in P\langle A_G\rangle$.

Solution: Decouple by choosing $Q\langle A_G\rangle$ to be in an appropriate cone $[\sigma]^{\preceq}$ of G, such that $P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ is always nonempty.

Suppose $P\langle A_G\rangle$ is nonempty. Fix a condition $\langle T, \varepsilon\rangle$ in the generic filter which forces this.

Extend T to a tall tree S in a maximal way, i.e., by including every extension of every leaf in T.

Fix a leaf σ of S. Then $\langle S, \varepsilon\rangle$ forces:

- σ is extendible in G (so $[G] \cap [\sigma]$ is a nonempty $\Pi^0_1\langle A_G\rangle$-class)
- $P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ is nonempty (because if $\langle R, \delta\rangle$ extends $\langle S, \varepsilon\rangle$, then $\langle R\setminus[\sigma]^{\preceq}, \delta\rangle$ is a condition which extends $\langle T, \varepsilon\rangle$.)

By genericity, we can find such $\langle S, \varepsilon\rangle$ in the generic filter.

Suppose, towards a contradiction, that there is a Turing functional Φ such that for every X in $P\langle A_G\rangle$, we have $\Phi^X \in [G] \cap [\sigma]$.

From before:

- $\langle S, \varepsilon\rangle$ forces "σ is extendible in G and $P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ is nonempty"
- For every X in $P\langle A_G\rangle$, we have $\Phi^X \in [G] \cap [\sigma]$.

By making an extension, we can decide $\Phi^X(|\sigma|)$ (somewhat):

Lemma

There is some $\langle R, \delta\rangle \leq \langle S, \varepsilon\rangle$ and an immediate successor τ of σ such that:

- *$\langle R, \delta\rangle$ forces that $\{X : \Phi^X \succ \tau\} \cap P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ is nonempty*
- *R contains every immediate successor of σ.*

To prove the above, we use:

Lemma (easy generalization of combinatorial lemma)

If $\langle S_1, \delta_1\rangle, \ldots, \langle S_m, \delta_m\rangle$ all extend $\langle T, \varepsilon/m\rangle$, then $\langle S_1 \cap \cdots \cap S_m, \varepsilon\rangle$ extends $\langle T, \varepsilon\rangle$.

From before:

- For every X in $P\langle A_G\rangle$, we have $\Phi^X \in [G] \cap [\sigma]$
- $\langle R, \delta\rangle \leq \langle S, \varepsilon\rangle$ and $\langle R, \delta\rangle$ contains every immediate successor of σ, including τ
- $\langle R, \delta\rangle$ forces that $\{X : \Phi^X \succ \tau\} \cap P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ is nonempty.

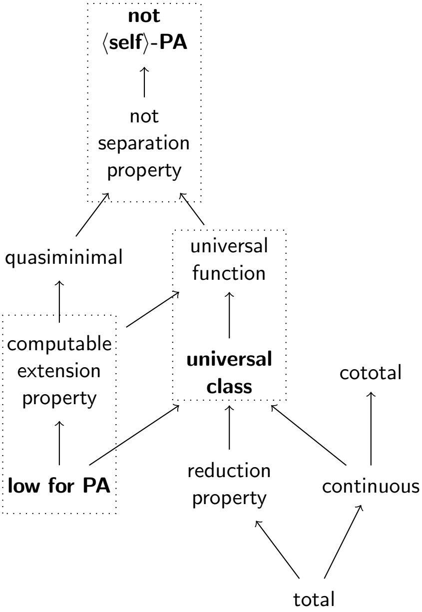
Now we diagonalize: Define $R' = R\setminus[\tau]^{\preceq}$. Then:

- $\langle R', \delta\rangle$ is a condition extending $\langle S, \varepsilon\rangle$
- $\langle R', \delta\rangle$ still forces that $\{X : \Phi^X \succ \tau\} \cap P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ is nonempty (because $P\langle A_{G\setminus[\sigma]^{\preceq}}\rangle$ isn't affected by $G \cap [\sigma]^{\preceq}$)
- $\langle R', \delta\rangle$ forces that τ is **not** extendible in G.

By genericity one can find such $\langle R', \delta\rangle$ in the generic filter, contradiction.

This completes the proof that A_G has no universal class.

Other subclasses we studied



Arrows indicate inclusion. No other inclusions hold.

In each box, the two subclasses are closely related by our work:

- the one in bold is defined by quantifying over all $\Pi^0_1\langle X\rangle$-classes, while
- the other can be characterized by quantifying over only separating $\Pi^0_1\langle X\rangle$-classes.

Two open questions

1. Are the subclasses on the previous slide first-order definable? (Some are known to be; most are not known to be.)

2. Does the uniformity in the definition of universal class matter?

   (Recall: $P\langle X\rangle$ is a universal class for X if
   for every nonempty $\Pi^0_1\langle X\rangle$-class $Q\langle X\rangle$,
   there is some Turing functional Φ such that
   for every $A \in P\langle X\rangle$, we have $\Phi^A \in Q\langle X\rangle$.)

Thanks!