Extensions of embeddings in the Σ_2^0 enumeration degrees

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Enumeration reducibility

Definition (various authors, 1950s)

For (nonempty) $A, B \subseteq \mathbb{N}$, we say that A is enumeration reducible to B ($A \leq_{\mathrm{e}} B$) if

every enumeration of B computes an enumeration of A.

Example

Let B be a maximal independent set of vertices in a computable graph. Then $B^c \leq_{\mathrm{e}} B$ (but $B \not\leq_{\mathrm{e}} B^c$ in general.)

The enumeration degrees (e-degrees) are defined from $\leq_{\mathrm{e}}$ in the same way that the Turing degrees are defined from $\leq_{\mathrm{T}}$.

They form an upper-semilattice under $\leq_{\mathrm{e}}$ and the usual effective join.

Beyond the Turing degrees: The enumeration degrees

The Turing degrees embed into the e-degrees in a natural way:

$$A \leq_{\mathrm{T}} B \quad \text{if and only if} \quad A \oplus A^c \leq_{\mathrm{e}} B \oplus B^c$$

so the map $A \mapsto A \oplus A^c$ induces an embedding.

Questions we can ask about a partial order

1. Is it linear?
2. Which finite partial orders embed into it?
3. Is it dense?
4. Given finite partial orders $\mathcal{P} \subseteq \mathcal{Q}$, can every embedding of $\mathcal{P}$ into it be extended to an embedding of $\mathcal{Q}$?
5. (!) Given a first-order sentence in the language of $\{\leq\}$, can we algorithmically decide if it is true?

For many degree structures, the answer to 5 is very much no (Slaman, Woodin 1997).

A central question

On the other hand, every finite partial order embeds into the e-degrees (corollary of Sacks 1963), so we can compute if a sentence of the form

$$\exists a_0 \exists a_1 \cdots \exists a_n \text{ (Boolean combination of } a_i \leq a_j\text{)}$$

holds in the e-degrees by checking whether the Boolean combination is consistent with the axioms of partial orders.

At what point does computability break down?

A countable substructure: The Σ^0_2 e-degrees

We are working on this question for the Σ^0_2 e-degrees.

Reasons to study the Σ^0_2 e-degrees:

1. They are analogous to the c.e. Turing degrees
2. Any nontrivial automorphism of the e-degrees must move some Σ^0_2 e-degree (Slaman, Soskova 2017)
3. They exhibit unusual order-theoretic phenomenon (next slide)

The Σ^0_2 e-degrees: Ahmad pairs

The Σ^0_2 e-degrees are dense (Cooper).

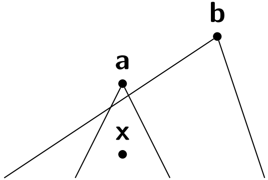
Compare: The c.e. Turing degrees are dense (Sacks density).

Question (Cooper)

Does the Σ^0_2 e-degrees satisfy the same first-order sentences as the c.e. Turing degrees?

Theorem (Ahmad 1989)

In the Σ^0_2 e-degrees, there are incomparable $\mathbf{a}$ and $\mathbf{b}$ such that if $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$, then $\mathbf{x} \leq_{\mathrm{e}} \mathbf{b}$.



Notice that **a** cannot be the join of two degrees below it.

On the other hand, in the c.e. Turing degrees, every degree is the join of two degrees below it (Sacks splitting).

There are no Ahmad triples

Theorem (Ahmad 1989)

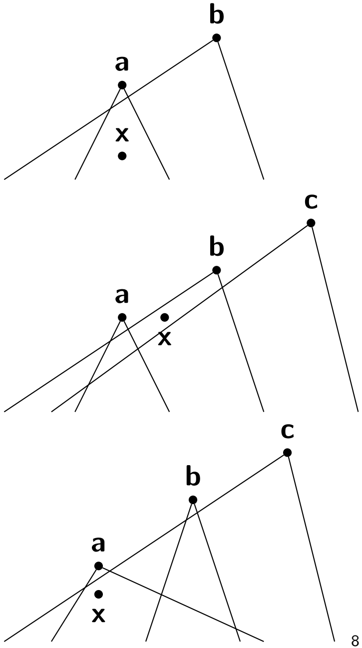
There are incomparable **a** and **b** such that if $\mathbf{x} <_e \mathbf{a}$, then $\mathbf{x} \leq_e \mathbf{b}$.

Theorem (G., Lempp, Ng, Soskova 2021)

For every incomparable **a**, **b**, **c**, there is some **x** such that:

- $\mathbf{x} <_e \mathbf{a}$ but $\mathbf{x} \not\leq_e \mathbf{b}$, OR
- $\mathbf{x} <_e \mathbf{b}$ but $\mathbf{x} \not\leq_e \mathbf{c}$.

(Actually we just need $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{c}$, so **a** and **c** could be comparable or even equal.)



Reformulating as extensions of embeddings

Theorem (Ahmad 1989)

There are incomparable **a** and **b** such that if $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$, then $\mathbf{x} \leq_{\mathrm{e}} \mathbf{b}$.

Reformulation:

> *Not every embedding of the antichain $\{a, b\}$ (into the Σ^0_2 e-degrees) can be extended to an embedding of $\{a, b, x\}$ where $x < a$ and $x \not\leq b$.*

Our result can be reformulated similarly:

> *Every embedding of the antichain $\{a, b, c\}$ can be extended to one of the following:*
> - *an embedding of $\{a, b, c, x\}$ where $x < a$ and $x \not\leq b$*
> - *an embedding of $\{a, b, c, x\}$ where $x < b$ and $x \not\leq c$.*

(Actually, there are four choices here because in the first ordering, we didn't specify the relationship between x and c, and in the second ordering, we didn't specify the relationship between x and a.)

"Disjunctive" results

Our result on no "Ahmad triples" generalizes the following

Theorem (Ahmad 1989)

There are no "symmetric Ahmad pairs", i.e., for every incomparable **a** and **b**, there is some **x** such that:

- $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$ but $\mathbf{x} \not\leq_{\mathrm{e}} \mathbf{b}$, or
- $\mathbf{x} <_{\mathrm{e}} \mathbf{b}$ but $\mathbf{x} \not\leq_{\mathrm{e}} \mathbf{a}$.

In other words, every embedding of the antichain $\{a, b\}$ can be extended to one of the following:

- an embedding of $\{a, b, x\}$ where $x < a$ and $x \not\leq b$
- an embedding of $\{a, b, x\}$ where $x < b$ and $x \not\leq a$.

What's the goal here?

We'd like to give an algorithm for deciding the **two-quantifier** first-order theory of the Σ^0_2 e-degrees. In terms of quantifier alternations, two quantifiers is the most we can hope to decide (Kent 2006).

Fact: Every two-quantifier sentence can be thought of as a disjunctive extension of embeddings problem.

Our focus (for now) is the following special case:

One-point extensions of antichains problem

Given a (finite) **antichain** $\mathcal{P}$ and **one-point** extensions $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ of $\mathcal{P}$, decide whether every embedding of $\mathcal{P}$ into the Σ^0_2 e-degrees extends to an embedding of some $\mathcal{Q}_i$.

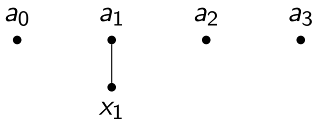
$n = 0$ was solved by Lempp, Slaman, Sorbi 2005. $n \geq 1$ is open.

One-point extensions of antichains problem

We'll restrict ourselves further to one-point extensions where the new element is **not above** any of the old elements.

Given an antichain $\mathcal{P} = \{a_0, \ldots, a_k\}$ and such one-point extensions $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ of $\mathcal{P}$, let x_i denote the new element added by $\mathcal{Q}_i$.

Suppose that in $\mathcal{Q}_i$, we have $x_i < a_i$ and $x_i \mid a_j$ for $j \neq i$.



We shall call such $\mathcal{Q}_i$ singleton extensions.

Singleton extensions and weak Ahmad bases

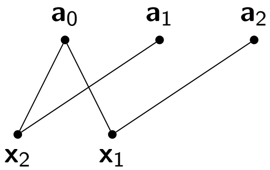
If $\mathcal{Q}_i$ is a singleton extension and $\mathbf{a}_0, \ldots, \mathbf{a}_k$ does not extend to an embedding of $\mathcal{Q}_i$ then:

Every $\mathbf{x} <_{\mathrm{e}} \mathbf{a}_i$ must be below some other $\mathbf{a}_j$.

If $\mathbf{a}_i$ has the above property we say that it is a weak Ahmad base.

- If **a** and **b** form an Ahmad pair, then **a** is a weak Ahmad base.
- Not all weak Ahmad bases come from Ahmad pairs (G., Lempp, Ng, Soskova 2021).

Right: Every $\mathbf{x} <_{\mathrm{e}} \mathbf{a}_0$ is below $\mathbf{a}_1$ or below $\mathbf{a}_2$, but $\mathbf{a}_0$ does not form an Ahmad pair with $\mathbf{a}_1$ or with $\mathbf{a}_2$, as witnessed by $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.



We want to understand which degrees can be weak Ahmad bases, because in general, we may want to construct an embedding of $\mathcal{P}$ which cannot be extended to any of $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$, not just a single $\mathcal{Q}_i$.

Theorem (G., Lempp, Ng, Soskova ongoing)

If **a** and **b** form an Ahmad pair, then **b** is not a weak Ahmad base.

This generalizes our result on "no Ahmad triple".

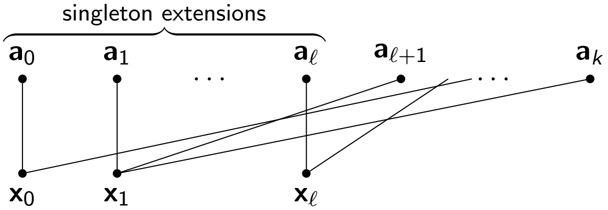
In fact, we believe we have proved the following stronger statement $(*)$:

> *If $\mathbf{a}_0, \ldots, \mathbf{a}_\ell$ are incomparable weak Ahmad bases, then for each $i \leq \ell$, there is some $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_i$ such that $\mathbf{x}_i \mid_{\mathrm{e}} \mathbf{a}_j$ for all $j \neq i$.*

(To derive the theorem from $(*)$, take $\ell = 1$, $\mathbf{a}_0 = \mathbf{a}$, $\mathbf{a}_1 = \mathbf{b}$.)

(∗): If $\mathbf{a}_0, \ldots, \mathbf{a}_\ell$ are incomparable weak Ahmad bases, then for each $i \leq \ell$, there is some $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_i$ such that $\mathbf{x}_i \mid_{\mathrm{e}} \mathbf{a}_j$ for all $j \neq i$.

Assuming $(*)$, we can derive a combinatorial condition on $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ which is **necessary** for there to be an embedding of $\{a_0, \ldots, a_k\}$ which does not extend to any of $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$.



We must have that $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\ell$ cannot be used to extend the embedding $\mathbf{a}_0, \ldots, \mathbf{a}_k$ to any of $\mathcal{Q}_{\ell+1}, \ldots, \mathcal{Q}_n$.

Our plan is to prove that this combinatorial condition is also **sufficient** for constructing an embedding of $\{a_0, \ldots, a_k\}$ which does not extend to any of $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$.

If true, this would give us an algorithm for deciding whether every embedding of a given finite antichain $\mathcal{P}$ extends to an embedding of some $\mathcal{Q}_i$, where $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ are one-point extensions of $\mathcal{P}$ where the new element is not above any old elements.

Thanks!